An Upper Bound Of A Function With Two Independent Variables

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Abstract
An upper bound for a function with two independent variables is obtained.

1 Introduction

The following terminology was explicitly introduced in [7], formally published in [6], immediately studied or cited by [2, 3, 9, 10, 11]: A function $f$ is said to be logarithmically completely monotonic on an interval $I$ if $f$ has derivatives of all orders on $I$ and its logarithm $\ln f$ satisfies $(-1)^k[\ln f(x)]^{(k)} \geq 0$ for all $k \in \mathbb{N}$ on $I$. Recently, it is pointed out that this notion has appeared in [1] without definition.

For our own convenience, let $\mathcal{L}[I]$ stand for the set of all logarithmically completely monotonic functions on $I$. Among other things, it is proved in [2, 6, 7, 13] that a logarithmically completely monotonic function is always completely monotonic, that is, $\mathcal{L}[I] \subset \mathcal{C}[I]$, but not conversely, where $\mathcal{C}[I]$ denotes the set of all completely monotonic functions on $I$. Further, it is shown in [2] that $\mathcal{S} \setminus \{0\} \subset \mathcal{L}[[0, \infty)] \subset \mathcal{C}[[0, \infty)]$, where $\mathcal{S}$ denotes the set of all Stieltjes transforms. In [2, Theorem 1.1] and [3, 9] it is pointed out that the logarithmically completely monotonic functions on $(0, \infty)$ can be characterized as the infinitely divisible completely monotonic functions investigated by Horn in [4, Theorem 4.4]. In [8], among other things, the following basic property of the logarithmically completely monotonic functions is obtained: If $h'(x) \in \mathcal{C}[I]$ and $f(x) \in \mathcal{L}[h(I)]$, then $f(h(x)) \in \mathcal{L}[I]$. For more related information, please refer to [5] and the references therein.

Let $\Gamma$ denote the classical Euler’s gamma function. Let

$$
\tau(s, t) = \frac{1}{s} \left[ t - (t + s + 1) \left( \frac{t}{t+1} \right)^{s+1} \right]
$$

(1)

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for \((s, t) \in (0, \infty) \times (0, \infty)\), and let \(\tau_0 = \tau(s_0, t_0)\) be the maximum of \(\tau(s, t)\) on the set \(\mathbb{N} \times (0, \infty)\). In [7, 8], it was proved that \(\tau(s, t) > 0\) by using the well known Bernoulli’s inequality and, for any given real number \(\alpha\) satisfying \(\alpha \leq \frac{1}{1 + \tau_0}\), the function \(\frac{(x+1)^\alpha}{[t(x+1)]^\alpha} \in \mathcal{L}([-1, \infty])\).

It is clear that \(\lim_{t \to 0^+} \tau(s, t) = 0\) for any \(s \in (0, \infty)\). Now it is natural to ask for the maximum of \(\tau(s, t)\) on \((0, \infty) \times (0, \infty)\). To the best of our knowledge, it is not easy and trivial to give an upper bound for \(\tau(s, t)\) in \((0, \infty) \times (0, \infty)\). However, by using a novel approach, an endeavor was made in [12] and an upper bound of \(\tau(s, t)\) was obtained: \(\tau(s, t) < 1\).

The numerical calculation of \(\tau(s, t)\) can be carried out by the well known software MATHEMATICA easily. However, it is believed that an accurate upper bound or the maximum of \(\tau(s, t)\) cannot be found by numerical method, since the domain of \((s, t)\) is an infinite region. A plot below and a numerical computation by the MATHEMATICA version 5.2 reveals that the maximum of the function \(\tau(s, t)\) should be less than \(\frac{3}{10}\).

In this short note, as a subsequence of [12], we shall give a more accurate upper bound for the function \(\tau(s, t)\) on \((0, \infty) \times (0, \infty)\). Our main result is

THEOREM 1. For \((s, t) \in (0, \infty) \times (0, \infty)\), we have \(0 < \tau(s, t) < \frac{3}{10}\).

REMARK 1. The proof of Theorem 1 is dependent on an improved upper bound of the function \(\Psi(x) = \frac{1}{x}(1 - \frac{1+x}{e^x})\) defined for \(x \in (0, \infty)\) by (9) below. It is noted that the upper bound for the function \(\Psi(x)\) can be further improved by numerical method, theoretically or practically. Since \(\Psi'(x) = \frac{1+x+x^2-e^x}{xe^x}\), it is easy to obtain by the MATHEMATICA version 5.2 numerically the unique root \(x_0 = 1.79328213290076\cdots\) of equation \(e^x = 1 + x + x^2\) and \(\Psi(x) \leq 0.2984256075256390\cdots\) for \(x \in (0, \infty)\). So, we would like to pose an open problem: Can one find a best possible upper bound or show that the maximum is less than \(\frac{3}{10}\) for the function \(\tau(s, t)\) on the domain \((0, \infty) \times (0, \infty)\) by non-numerical method? Here we wish to obtain a “nice” upper bound for the considered function.
2 Proof

Let \( s = \mu t \) for \( \mu \in (0, \infty) \) and \( t \in (0, \infty) \). Then we have

\[
\tau(\mu t, t) = \frac{1}{\mu} \left[ 1 - \frac{(\mu + 1)t + 1}{1 + t} \left( \frac{t}{1 + t} \right)^{\mu t} \right] \approx \frac{1}{\mu} [1 - q_\mu(t)],
\]

\[
q_\mu'(t) = \mu \left( \frac{t}{1 + t} \right)^{\mu t} \frac{2 + t + \mu t + (1 + t)(1 + t + \mu t) \ln \frac{t}{1 + t}}{(1 + t)^2} \approx \frac{\mu p_\mu(t)}{(1 + t)^2} \left( \frac{t}{1 + t} \right)^{\mu t}.
\]

Let

\[
\phi_\mu(x) = \ln(1 + x) - \frac{x}{1 + x} - \frac{x^2}{(1 + x)(x + \mu + 1)}
\]

for \( x \geq 0 \). Then we have

\[
\phi_\mu'(x) = \frac{x(x^2 + \mu x + \mu^2 - 1)}{(x + \mu + 1)^2(1 + x)^2} \approx \frac{x g_\mu(x)}{(x + \mu + 1)^2(1 + x)^2}.
\]

It is clear that if \( \mu \geq 1 \) then \( g_\mu(x) > 0 \) in \( (0, \infty) \). As a result, we have \( \phi_\mu'(x) > 0 \), and then \( \phi_\mu(x) \) is strictly increasing in \( (0, \infty) \). Since \( \phi_\mu(0) = 0 \), it follows that \( \phi_\mu(x) > 0 \) in \( (0, \infty) \) for any given \( \mu > 0 \).

When \( 0 < \mu < 1 \), the function \( g_\mu(x) \) has a unique positive zero point \( x_0 = \left( \sqrt{4 - 3\mu^2} - \mu \right)/2 \), and then \( g_\mu(x) \) is negative in \( (0, x_0) \) and positive in \( (x_0, \infty) \). This means that the function \( \phi_\mu'(x) \) is negative in \( (0, x_0) \) and positive in \( (x_0, \infty) \), that is, \( \phi_\mu(x) \) is strictly decreasing in \( (0, x_0) \) and strictly increasing in \( (x_0, \infty) \). Since \( \phi_\mu(0) = 0 \), we have \( \phi_\mu(x) < 0 \) in \( (0, x_0) \). Since \( \lim_{x \rightarrow \infty} \phi_\mu(x) = \infty \), then there exists a unique point \( x_1 \in (x_0, \infty) \), which is dependent on \( \mu \), such that \( \phi_\mu(x) < 0 \) in \( (0, x_1) \) and \( \phi_\mu(x) > 0 \) in \( (x_1, \infty) \).

Let \( x = \frac{1}{\mu} \). Then \( \phi_\mu(x) > 0 \) is equivalent to

\[
p_\mu(t) = 2 + (\mu + 1)t + (1 + t)[(\mu + 1)t + 1] \ln \frac{t}{1 + t} < 0,
\]

and \( \phi_\mu(x) < 0 \) is equivalent to

\[
p_\mu(t) = 2 + (\mu + 1)t + (1 + t)[(\mu + 1)t + 1] \ln \frac{t}{1 + t} > 0.
\]
Therefore, we have the following conclusions:

1. If $\mu \geq 1$, we have $p_\mu(t) < 0$, then $q_\mu'(t) < 0$ in $(0, \infty)$, and $q_\mu(t)$ is strictly decreasing in $(0, \infty)$, thus $q_\mu(t) > \lim_{t \to \infty} q_\mu(t) = \frac{1 + \mu}{e^\mu}$.

2. If $0 < \mu < 1$, we have $p_\mu(t) > 0$ in $(0, x_1)$ and $p_\mu(t) < 0$ in $(x_1, \infty)$. These are equivalent to $q_\mu'(t) > 0$ in $(0, x_1)$ and $q_\mu'(t) < 0$ in $(x_1, \infty)$. Hence $q_\mu(t)$ is strictly increasing in $(0, x_1)$ and $q_\mu(t)$ is strictly decreasing in $(x_1, \infty)$. Therefore, we have

$$q_\mu(t) > \min \left\{ \lim_{t \to 0} q_\mu(t), \lim_{t \to \infty} q_\mu(t) \right\} = \min \left\{ 1, \frac{1 + \mu}{e^\mu} \right\} = \frac{1 + \mu}{e^\mu}.$$ (8)

These tell us that $q_\mu(t) > \frac{1 + \mu}{e^\mu}$ for any $t \in (0, \infty)$ and $\mu \in (0, \infty)$. Then

$$\tau(\mu t, t) < \frac{1}{\mu} \left( 1 - \frac{1 + \mu}{e^\mu} \right) = \Psi(\mu)$$ (9)

for $t \in (0, \infty)$ and $\mu \in (0, \infty)$.

In order to prove Theorem 1, it is sufficient to show $\Psi(\mu) < \frac{3}{10}$, which is equivalent to $g(\mu) = 3\mu e^\mu - 10e^\mu + 10 + 10 > 0$.

Easy calculation gives $g'(\mu) = 3\mu e^\mu - 7e^\mu + 10$ and $g''(\mu) = e^\mu(3\mu - 4)$. Hence, the function $g'(\mu)$ is decreasing in $(0, 4/3)$ and increasing in $(4/3, \infty)$. This means that the function $g'(\mu)$ attains its minimum at the point $\mu = 4/3$ and $g'(4/3) = -1.38100368 \ldots < 0$. Since $g'(0) = 3$ and $\lim_{\mu \to \infty} g'(\mu) = \infty$, the function $g'(\mu)$ has two zero points $\mu_1 \in (0, 4/3)$ and $\mu_2 \in (4/3, \infty)$. It is clear that $g'(\mu) > 0$ and $g(\mu)$ is increasing for $\mu \notin (\mu_1, \mu_2)$. Since $g(0) = 0$, it is easily concluded that $\mu_1$ is a point of local maximum and $\mu_2$ is a point of local minimum for the function $g(\mu)$. An elementary reasoning now yields that it is sufficient to prove that $g(\mu_2) \geq 0$ if we wish to prove what we want. For this purpose, we calculate

$$g(\mu_2) = 3\mu_2 e^{\mu_2} - 7e^{\mu_2} + 10 - 3e^{\mu_2} + 10\mu_2 = -3e^{\mu_2} + 10\mu_2;$$ (10)

since $g'(\mu_2) = 3\mu_2 e^{\mu_2} - 7e^{\mu_2} + 10 = 0$.

We now consider the function $h(\mu) = 10\mu - 3e^\mu$ such that $h(\mu_2) = g(\mu_2)$. It is clear that $h'(\mu) = 10 - 3e^\mu$ and $h''(\mu) = -3e^\mu < 0$. It is not difficult to see that the function $h(\mu)$ has a maximum at the point $\nu = \ln \frac{10}{3}$ and it has two zero points $\nu_1$ and $\nu_2$ such that $h(\mu) > 0$ for $\mu \in (\nu_1, \nu_2)$. Now it is sufficient to prove that $\mu_2 \in (\nu_1, \nu_2)$. A simple verification will show that $\nu_1 < 1 < \nu_2 < \frac{\ln 10}{3} < \nu_2$. This completes the proof.

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