Some Inequalities Of Ostrowski Type And Applications∗

Zheng Liu†

Received 19 April 2006

Abstract

Generalizations of Ostrowski type inequality for functions of Lipschitzian type are established. Applications for cumulative distribution functions are given.

1 Introduction

The following Ostrowski inequality ([5] or [4, p.468]) is well known:

\[ \left| f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \left[ \frac{1}{4} + \frac{(x-(a+b)/2)^2}{(b-a)^2} \right] (b-a)M, \quad x \in [a,b], \quad (1) \]

where \( f : [a,b] \to \mathbb{R} \) is a differentiable function such that \( |f'(x)| \leq M \), for every \( x \in [a,b] \).

In Theorem 3.1 of [2], Cheng has generalized the Ostrowski inequality (1) in the following form.

THEOREM 1. Let \( f : I \to \mathbb{R} \), where \( I \subset \mathbb{R} \) is an interval, be a mapping differentiable in the interior Int \( I \) of \( I \), and let \( a, b \in \text{Int} I \), \( a < b \). If \( f' \) is integrable and \( \gamma \leq f'(t) \leq \Gamma, \forall t \in [a,b] \) and some constants \( \gamma, \Gamma \in \mathbb{R} \), then we have

\[ \left| \frac{1}{2} [(x-a)f(a) + (b-a)f(x) + (b-x)f(b)] - \int_a^b f(t) \, dt \right| \leq \frac{1}{4} \left[ \left( x - \frac{a+b}{2} \right)^2 + \frac{(b-a)^2}{4} \right] (\Gamma - \gamma), \quad (2) \]

for all \( x \in [a,b] \).

From Theorem 2 in [6], we may provide new estimations of the left part of (2) as follows:

∗Mathematics Subject Classifications: 26D15
†Institute of Applied Mathematics, School of Science, Liaoning University of Science and Technology, Anshan 114044, Liaoning, P. R. China
THEOREM 2. Let the assumptions of Theorem 1 hold. Then for all \( x \in [a, b] \), we have

\[
\left| \frac{1}{2}[(x - a)f(a) + (b - a)f(x) + (b - x)f(b)] - \int_a^b f(t) \, dt \right| \\
\leq \frac{b - a}{2} \left[ \frac{b - a}{2} + \left| x - \frac{a + b}{2} \right| \right] (S - \gamma)
\] (3)

and

\[
\left| \frac{1}{2}[(x - a)f(a) + (b - a)f(x) + (b - x)f(b)] - \int_a^b f(t) \, dt \right| \\
\leq \frac{b - a}{2} \left[ \frac{b - a}{2} + \left| x - \frac{a + b}{2} \right| \right] (\Gamma - S),
\] (4)

where \( S = (f(b) - f(a))/(b - a) \).

In this paper, we shall generalize Theorem 1 and Theorem 2 to functions of some larger classes. For convenience, we define functions of Lipschitzian type as follows:

DEFINITION 1. The function \( f : [a, b] \to \mathbb{R} \) is said to be \( L \)-Lipschitzian on \( [a, b] \) if for some \( L > 0 \) and all \( x, y \in [a, b] \),

\[ |f(x) - f(y)| \leq L|y - y| \]

DEFINITION 2. The function \( f : [a, b] \to \mathbb{R} \) is said to be \((l, L)\)-Lipschitzian on \( [a, b] \) if

\[ l(x_2 - x_1) \leq f(x_2) - f(x_1) \leq L(x_2 - x_1) \] for \( a \leq x_1 \leq x_2 \leq b \),

where \( l, L \in \mathbb{R} \) with \( l < L \).

We will need the following well-known results.

LEMMA 1. (see e.g. (3.3) in [3]) Let \( h, g : [a, b] \to \mathbb{R} \) be such that \( h \) is Riemann-integrable on \([a, b]\) and \( g \) is \( L \)-Lipschitzian on \([a, b]\). Then

\[
\left| \int_a^b h(t) \, dg(t) \right| \leq L \int_a^b |h(t)| \, dt.
\] (5)

LEMMA 2. (see e.g. (2.3) in [3]) Let \( h, g : [a, b] \to \mathbb{R} \) be such that \( h \) is continuous on \([a, b]\) and \( g \) is of bounded variation on \([a, b]\). Then

\[
\left| \int_a^b h(t) \, dg(t) \right| \leq \max_{t \in [a, b]} |h(t)| V_a^b(g).
\] (6)

The purpose of this paper is to generalize Theorem 1 and Theorem 2 to functions which are \( L \)-Lipschitzian and \((l, L)\)-Lipschitzian respectively. Applications for cumulative distribution functions are given.
2 Main Results

Our main results are as follows.

THEOREM 3. Let \( f : [a, b] \rightarrow \mathbb{R} \) be \((l, L)\)-Lipschitzian on \([a, b]\). Then for all \( x \in [a, b] \), we have

\[
\left| \frac{1}{2} [(x - a)f(a) + (b - a)f(x) + (b - x)f(b)] - \int_a^b f(t) \, dt \right| \\
\leq \frac{1}{4} \left( \frac{x - a + b}{2} \right)^2 + \frac{(b - a)^2}{4} (L - l),
\]

(7)

\[
\left| \frac{1}{2} [(x - a)f(a) + (b - a)f(x) + (b - x)f(b)] - \int_a^b f(t) \, dt \right| \\
\leq \frac{b - a}{2} \left( \frac{b - a}{2} + \left| x - \frac{a + b}{2} \right| \right) (S - l),
\]

(8)

and

\[
\left| \frac{1}{2} [(x - a)f(a) + (b - a)f(x) + (b - x)f(b)] - \int_a^b f(t) \, dt \right| \\
\leq \frac{b - a}{2} \left( \frac{b - a}{2} + \left| x - \frac{a + b}{2} \right| \right) (L - S),
\]

(9)

where \( S = (f(b) - f(a))/(b - a) \).

PROOF. Let us define the function

\[
p(x, t) := \begin{cases} 
  t - \frac{a + x}{2}, & t \in [a, x], \\
  t - \frac{a + b}{2}, & t \in (x, b]. 
\end{cases}
\]

Put

\[
g(t) := f(t) - \frac{L + l}{2} t.
\]

(10)

It is easy to find that the function \( g : [a, b] \rightarrow \mathbb{R} \) is \( M \)-Lipschitzian on \([a, b]\) with \( M = \frac{L - l}{2} \). So, the Riemann-Stieltjes integral \( \int_a^b p(x, t) \, dg(t) \) exists. Using the integration by parts formula for Riemann-Stieltjes integral, we have

\[
\int_a^b p(x, t) \, dg(t) = \int_a^x (t - \frac{a + x}{2}) \, dg(t) + \int_x^b (t - \frac{x + b}{2}) \, dg(t)
\]

\[
= \frac{1}{2} [(x - a)g(a) + (b - a)g(x) + (b - x)g(b)] - \int_a^b g(t) \, dt.
\]

(11)

From (5) of the Lemma 1 we have

\[
\left| \frac{1}{2} [(x - a)g(a) + (b - a)g(x) + (b - x)g(b)] - \int_a^b g(t) \, dt \right| \leq \frac{L - l}{2} \int_a^b |p(x, t)| \, dt.
\]

(12)
It is not difficult to find that
\[
\int_a^b |p(x,t)| \, dt = \frac{(x-a)^2 + (b-x)^2}{4} = \frac{1}{2}[(x-a)^2 + (b-a)^2],
\] (13)
and so from (12) and (13) we get
\[
\left| \frac{1}{2}[(x-a)g(a) + (b-a)g(x) + (b-x)g(b)] - \int_a^b g(t) \, dt \right| \leq \frac{L - l}{4} \left( (x - \frac{a+b}{2})^2 + \frac{(b-a)^2}{4} \right). \tag{14}
\]
Consequently, the inequality (7) follows from substituting (10) to the left hand side of the inequality (14).

Now we proceed to prove the inequalities (8) and (9). Put
\[
g_1(t) := f(t) - lt \quad \text{and} \quad g_2(t) := f(t) - Lt. \tag{15}
\]
It is easy to find that both \(g_1, g_2 : [a, b] \rightarrow \mathbb{R}\) are functions of bounded variation on \([a, b]\) with
\[
V_a^b(g_1) = f(b) - f(a) - l(b-a) \quad \text{and} \quad V_a^b(g_2) = L(b-a) - [f(b) - f(a)]. \tag{16}
\]
So, the Riemann-Stieltjes integrals \(\int_a^b p(x,t) \, dg_1(t)\) and \(\int_a^b p(x,t) \, dg_2(t)\) exist. Using the integration by parts formula for Riemann-Stieltjes integral, we have
\[
\int_a^b p(x,t) \, dg_1(t) = \frac{1}{2}[(x-a)g_1(a) + (b-a)g_1(x) + (b-x)g_1(b)] - \int_a^b g_1(t) \, dt \tag{17}
\]
and
\[
\int_a^b p(x,t) \, dg_2(t) = \frac{1}{2}[(x-a)g_2(a) + (b-a)g_2(x) + (b-x)g_2(b)] - \int_a^b g_2(t) \, dt. \tag{18}
\]
Then by (6) of the Lemma 2 we can deduce that
\[
\left| \frac{1}{2}[(x-a)g_1(a) + (b-a)g_1(x) + (b-x)g_1(b)] - \int_a^b g_1(t) \, dt \right| \leq \max_{t \in [a,b]} |p(x,t)| V_a^b(g_1)
\]
and
\[
\left| \frac{1}{2}[(x-a)g_2(a) + (b-a)g_2(x) + (b-x)g_2(b)] - \int_a^b g_2(t) \, dt \right| \leq \max_{t \in [a,b]} |p(x,t)| V_a^b(g_2).
\]
Notice that
\[
\max_{t \in [a,b]} |p(x,t)| = \max \left\{ \frac{x-a}{2}, \frac{b-x}{2} \right\} = \frac{1}{2} \left[ \frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right]
\]
and from (16), we get
\[
\left| \frac{1}{2}[(x-a)g_1(a) + (b-a)g_1(x) + (b-x)g_1(b)] - \int_a^b g_1(t) \, dt \right| \\
\leq \frac{b-a}{2} \left[ \frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right] (S-l)
\]
(19)
\[
\left| \frac{1}{2}[(x-a)g_2(a) + (b-a)g_2(x) + (b-x)g_2(b)] - \int_a^b g_2(t) \, dt \right| \\
\leq \frac{b-a}{2} \left[ \frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right] (L-S),
\]
(20)
where \( S = (f(b) - f(a))/(b-a) \).

Consequently, inequalities (8) and (9) follow from substituting (15) to the left hand sides of (19) and (20), respectively.

**COROLLARY 1.** Under the assumptions of Theorem 3, we get trapezoid inequalities
\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \frac{b-a}{8} (L-l),
\]
\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \frac{b-a}{2} (S-l)
\]
and
\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \frac{b-a}{4} (L-S).
\]

**PROOF.** We set \( x = a \) or \( x = b \) in the above theorem.

**COROLLARY 2.** Under the assumptions of Theorem 3, we get simple three point inequalities (i.e., the average of a mid-point and trapezoid type rules)
\[
\left| \frac{f(a) + 2f(\frac{a+b}{2}) + f(b)}{4} - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \frac{b-a}{16} (L-l),
\]
\[
\left| \frac{f(a) + 2f(\frac{a+b}{2}) + f(b)}{4} - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \frac{b-a}{4} (S-l)
\]
and
\[
\left| \frac{f(a) + 2f(\frac{a+b}{2}) + f(b)}{4} - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \frac{b-a}{4} (L-S).
\]

**PROOF.** We set \( x = \frac{a+b}{2} \) in the above theorem.
REMARK 1. It is clear that Theorem 3 can be regarded as a generalization of Theorem 1 and Theorem 2.

THEOREM 4. Let $f : [a, b] \to \mathbb{R}$ be $L$-Lipschitzian on $[a, b]$. Then for all $x \in [a, b]$, we have

$$
\left| \frac{1}{2}[(x-a)f(a) + (b-a)f(x) + (b-x)f(b)] - \int_a^b f(t) \, dt \right| 
\leq \frac{L}{2} \left[ (x-a)^2 + \frac{(b-a)^2}{4} \right]
$$

and

$$
\left| \frac{1}{2}[(x-a)f(a) + (b-a)f(x) + (b-x)f(b)] - \int_a^b f(t) \, dt \right| 
\leq \frac{b-a}{2} \left[ \frac{b-a}{2} + |x - \frac{a+b}{2}| \right] (L - |S|),
$$

where $S = (f(b) - f(a))/(b-a)$.

PROOF. Inequality (21) is obtained from (7) and $l = -L$. Also, by taking $l = -L$ in (8) we get

$$
\left| \frac{1}{2}[(x-a)f(a) + (b-a)f(x) + (b-x)f(b)] - \int_a^b f(t) \, dt \right| 
\leq \frac{b-a}{2} \left[ \frac{b-a}{2} + |x - \frac{a+b}{2}| \right] (S + L).
$$

Consequently, the inequality (22) follows from (23) and (9) by considering the fact that $\min\{S + L, L - S\} = L - |S|$.

3 Applications

Now we consider some applications for cumulative distribution functions.

Let $X$ be a random variable having the probability density function $f : [a, b] \to \mathbb{R}_+$ and the cumulative distribution function $F(x) = \Pr(X \leq x)$, i.e.,

$$
F(x) = \int_a^x f(t) \, dt, \quad x \in [a, b].
$$

$E(X)$ is the expectation of $X$. Then we have the following inequality.

THEOREM 5. With the above assumptions and if there exist constants $M, m$ such that $0 \leq m \leq f(t) \leq M$ for all $t \in [a, b]$, then we have the inequalities

$$
\left| \Pr(X \leq x) - \frac{x - E(X)}{b-a} - \frac{b - E(X)}{b-a} \right| 
\leq \frac{b-a}{2} \left[ \left( \frac{x - \frac{a+b}{2}}{b-a} \right)^2 + \frac{1}{4} \right] (M - m),
$$

where $S = (f(b) - f(a))/(b-a)$.
\[
P_r(X \leq x) \leq \frac{x - E(X) - b - E(X)}{b - a} \leq \left( \frac{1}{b - a} - m \right) \left[ \frac{b - a}{2} + \left| x - \frac{a + b}{2} \right| \right]
\]
(25)

and
\[
P_r(X \leq x) = \frac{x - E(X) - b - E(X)}{b - a} \leq \left( M - \frac{1}{b - a} \right) \left[ \frac{b - a}{2} + \left| x - \frac{a + b}{2} \right| \right].
\]
(26)

**Proof.** It is easy to find that the function \( F(x) = \int_a^x f(t) \, dt \) is \((m, M)\)-Lipschitzian on \([a, b]\). So, by Theorem 3 we get
\[
\left| \frac{1}{2} \left( x - a \right) F(a) + (b - a) F(x) + (b - x) F(b) - \int_a^b F(t) \, dt \right| \leq \frac{(b - a)^2}{4} \left[ \left( \frac{x - \frac{a + b}{2}}{b - a} \right)^2 + \frac{1}{4} \right] (M - m),
\]
\[
\left| \frac{1}{2} \left( x - a \right) F(a) + (b - a) F(x) + (b - x) F(b) - \int_a^b F(t) \, dt \right| \leq \frac{b - a}{2} \left[ \frac{b - a}{2} + \left| x - \frac{a + b}{2} \right| \right] (S - m)
\]
and
\[
\left| \frac{1}{2} \left( x - a \right) F(a) + (b - a) F(x) + (b - x) F(b) - \int_a^b F(t) \, dt \right| \leq \frac{b - a}{2} \left[ \frac{b - a}{2} + \left| x - \frac{a + b}{2} \right| \right] (M - S),
\]

where \( S = \frac{F(b) - F(a)}{b - a} \). As \( F(a) = 0, F(b) = 1 \), and
\[
\int_a^b F(t) \, dt = b - E(X),
\]
then we can easily deduce inequalities (24), (25) and (26).

**Corollary 3.** Under the assumptions of Theorem 5, we have
\[
\left| E(X) - \frac{a + b}{2} \right| \leq \left( \frac{b - a}{8} \right) (M - m), \quad (27)
\]
\[
\left| E(X) - \frac{a + b}{2} \right| \leq \left( \frac{b - a}{2} \right) \left( \frac{1}{b - a} - m \right), \quad (28)
\]
and
\[
\left| E(X) - \frac{a + b}{2} \right| \leq \left( \frac{b - a}{2} \right) \left( M - \frac{1}{b - a} \right). \quad (29)
\]
**PROOF.** We set \( x = a \) or \( x = b \) in (24)-(26) to get (27)-(29).

**REMARK 2.** It should be noted that the inequality (27) improves the inequality (5.4) in [1].

**COROLLARY 4.** Under the assumptions of Theorem 5, we have

\[
\left| P_r \left( X \leq \frac{a+b}{2} \right) - \frac{1}{2} \right| \leq \frac{3(b-a)}{8}(M-m), \quad (30)
\]

\[
\left| P_r \left( X \leq \frac{a+b}{2} \right) - \frac{1}{2} \right| \leq \frac{3}{2}[1-m(b-a)] \quad (31)
\]

and

\[
\left| P_r \left( X \leq \frac{a+b}{2} \right) - \frac{1}{2} \right| \leq \frac{3}{2}[M(b-a) - 1]. \quad (32)
\]

**PROOF.** Set \( x = \frac{a+b}{2} \) in (24)-(26), we get

\[
\left| P_r \left( X \leq \frac{a+b}{2} \right) - \frac{2}{b-a} \left[ \frac{a+3b}{4} - E(X) \right] \right| \leq \frac{b-a}{8}(M-m), \quad (33)
\]

\[
\left| P_r \left( X \leq \frac{a+b}{2} \right) - \frac{2}{b-a} \left[ \frac{a+3b}{4} - E(X) \right] \right| \leq \frac{b-a}{2} \left( \frac{1}{b-a} - m \right), \quad (34)
\]

and

\[
\left| P_r \left( X \leq \frac{a+b}{2} \right) - \frac{2}{b-a} \left[ \frac{a+3b}{4} - E(X) \right] \right| \leq \frac{b-a}{2} \left( M - \frac{1}{b-a} \right). \quad (35)
\]

Using the triangle inequality, we get

\[
\left| P_r \left( X \leq \frac{a+b}{2} \right) - \frac{1}{2} \right| = \left| P_r \left( X \leq \frac{a+b}{2} \right) - \frac{1}{2} \right| + \frac{2}{b-a} \left[ \frac{a+3b}{4} - E(X) \right] - \frac{2}{b-a} \left[ \frac{a+3b}{4} - E(X) \right] \leq \left| P_r \left( X \leq \frac{a+b}{2} \right) - \frac{2}{b-a} \left[ \frac{a+3b}{4} - E(X) \right] \right| + \frac{2}{b-a} \left| E(X) - \frac{a+b}{2} \right|,
\]

and then inequalities (30)-(32) follow from (27)-(29) and (33)-(35).

**Acknowledgment.** The author wishes to thank the referee for his helpful suggestions.

**References**


