On the certain subsets of the space of metrics

S. ČERETKOVÁ, J. FULIER and J. T. TÓTH

Abstract. In this note we look at certain subsets of the metric space of metrics for an arbitrary given set \( X \) and show that in terms of cardinality these can be very large while being extremely small in the topological point of view.

Introduction

Let \( X \) be a given non-void set. Denote by \( \mathcal{M} \) the set of all metrics on \( X \) endowed with the metric:

\[
d^*(d_1, d_2) = \min\{1, \sup_{x, y \in X} \{|d_1(x, y) - d_2(x, y)|\} \text{ for } d_1, d_2 \in \mathcal{M}\}.
\]

First of all recall some basic definitions and notations.

Suppose \( \alpha > 0 \) and put

\[
\mathcal{H}_\alpha = \{d \in \mathcal{M} : \forall_{x, y \in X} d(x, y) \geq \alpha\} \text{ and } \mathcal{H} = \bigcup_{\alpha > 0} \mathcal{H}_\alpha.
\]

Results shown in [2] include \( \mathcal{M} \) is a non-complete Baire space and \( \mathcal{H} \) is an open and dense subset of \( \mathcal{M} \), thus \( \mathcal{M} \cap \mathcal{H} \) is nowhere dense in \( \mathcal{M} \). Other results on the metric space of metrics may be found in [2], [3] and [4].

Let \( \mathcal{A} \) and \( \mathcal{B} \) denote the set of all metrics on \( X \) that are unbounded and bounded, respectively. It is proved in [2] (Theorem 5) that \( \mathcal{A}, \mathcal{B} \) are non-empty, open subsets of the Baire space \( (\mathcal{M}, d^*) \) (of [2], Theorem 3) provided \( X \) is infinite. Thereby \( \mathcal{A}, \mathcal{B} \) are sets of the 2-nd cathegory in \( \mathcal{M} \), if \( X \) is finite, then \( \mathcal{B} = \mathcal{M} \) and \( \mathcal{A} = \emptyset \).

Now define the mapping

\[
f: \mathcal{M} \to [0, 1], \quad g: \mathcal{M} \to [0, \infty) \text{ and } h: \mathcal{B} \to (0, +\infty)
\]

as follows:

\[
f(d) = \sup_{x, y \in X} \frac{d(x, y)}{1 + d(x, y)} \text{ where } d \in \mathcal{M},
\]

\[
g(d) = \inf_{x, y \in X} d(x, y) \text{ where } d \in \mathcal{M}, \text{ and}
\]

\[
h(d) = \sup_{x, y \in X} d(x, y) \text{ where } d \in \mathcal{B}.
\]
Obviously $f^{-1}([1]) = \mathcal{A}$ and $g^{-1}([0]) = \mathcal{M} \setminus \mathcal{H}$.

It is purpose of this paper to establish how large sets $f^{-1}([t])$, $g^{-1}([t])$, are given.

In what follows if $\mathcal{U} \subset \mathcal{M}$, then $\mathcal{U}$ is considered as a metric space with the metric $d^*|_{\mathcal{U} \times \mathcal{U}}$ (a metric subspace of $\mathcal{M}$).

**Main results**

Let $\varphi(t) = \frac{t}{1 + t}$ for $t \in [0, +\infty)$. Then $\varphi$ is increasing and continuous function on $[0, +\infty)$. Therefore $f(d) = \sup_{x, y \in X} \varphi(d(x, y))$ for $d \in \mathcal{M}$. The natural question arises whether $f$ is continuous on $\mathcal{M}$, too. The answer of this question is positive. We have

**Lemma.** The function $f, g$ are uniformly continuous on $\mathcal{M}$ and the function $h$ is uniformly continuous on $\mathcal{B}$.

**Proof.** Let $0 < \varepsilon < 1$ and $d_1, d_2 \in \mathcal{M}$ such that $d^*(d_1, d_2) < \varepsilon$. We show

$$|f(d_1) - f(d_2)| \leq d^*(d_1, d_2), \quad |g(d_1) - g(d_2)| \leq d^*(d_1, d_2).$$

We can simply count

$$\varphi(d_1(x, y)) \leq \varphi(d_2(x, y)) + |\varphi(d_1(x, y)) - \varphi(d_2(x, y))|$$

$$\leq \varphi(d_2(x, y)) + d^*(d_1, d_2)$$

because

$$\frac{|d_1(x, y) - d_2(x, y)|}{(1 + d_1(x, y)) (1 + d_2(x, y))} \leq d^*(d_1, d_2).$$

Taking supremum in the previous inequality we obtain $f(d_1) \leq f(d_2) + d^*(d_1, d_2)$, therefore $f(d_1) - f(d_2) \leq d^*(d_1, d_2)$. From symetrics we have $f(d_2) - f(d_1) \leq d^*(d_1, d_2)$ and $|f(d_1) - f(d_2)| \leq d^*(d_1, d_2)$. From this we see that the function $f$ is uniformly continuous on $\mathcal{M}$. Obviously for $x, y \in X$

$$|d_1(x, y) - d_2(x, y)| \geq d_1(x, y) - d_2(x, y) \geq g(d_1) - g(d_2).$$

Then

(1) \hspace{1cm} $g(d_1) - g(d_2) \leq \inf_{x, y \in X} |d_1(x, y) - d_2(x, y)| \leq d^*(d_1, d_2).$

According the inequality $|d_1(x, y) - d_2(x, y)| \geq d_2(x, y) - d_1(x, y)$, similarly to the previous we get

(2) \hspace{1cm} $g(d_2) - g(d_1) \leq d^*(d_1, d_2).$
Then we required inequality follows from (1) and (2).

Analogously \( \left| h(d_1) - h(d_2) \right| \leq d^*(d_1, d_2) \). \( \blacksquare \)

**Remark 1.** The function \( h \) can be continuously continued on \( M \). Because the set \( B \) is closed in \( M \), the Hausdorff’s function (see [1], p. 382) is continuous continuation of the function \( h \) on \( M \).

**Remark 2.** Because \( A \cup B = M \), \( A \cap B = \emptyset \) and the set \( B \) is closed in \( M \), according the lemma of Uryshon there exists a function \( G: M \to [0, 1] \) such that \( G \) is continuous on \( M \) and \( G(A) = \{0\}, G(B) = \{1\} \). For this reason \( G(M) = \{0, 1\} \).

Space \((M, d^*)\) is Bair’s space, e.g. every non-empty open subset of the set \( M \) is of the 2-nd category in \( M \). The set \( A \) is non-void and open subset in \( M \), then the set \( f^{-1}(\{1\}) = A \) is of the 2-nd category in \( M \). One may ask: Is there any \( t \in (0, 1) \) such that the set \( f^{-1}(\{t\}) \) is of the 2-nd category in \( M \)? Similarly for \( g^{-1}(\{t\}) \) and \( h^{-1}(\{t\}) \). This question is answered in the next theorem.

**Theorem 1.** We have

(i) For arbitrary \( t \in (0, 1) \) the set \( f^{-1}(\{t\}) \) is nowhere dense in \( M \).

(ii) For arbitrary \( t \in [0, +\infty) \) the set \( g^{-1}(\{t\}) \) is nowhere dense in \( M \).

(iii) For arbitrary \( t \in [0, +\infty) \) the set \( h^{-1}(\{t\}) \) is nowhere dense in \( M \).

**Proof.** (i) Let \( 0 < t < 1 \). According to lemma the set \( f^{-1}(\{t\}) \) is closed in \( M \). Therefore it is sufficient to prove that the set \( M \setminus f^{-1}(\{t\}) \) is dense in \( M \). We will use inequality

\[
\frac{t_2}{1 + t_2} \geq \frac{t_1}{1 + t_1} + \frac{t_2 - t_1}{(1 + t_2)^2} \quad \text{for} \quad 0 \leq t_1 \leq t_2
\]

(it is equivalent to \( (t_2 - t_1)^2 \geq 0 \)).

Let \( d \in f^{-1}(\{t\}) \) and \( 0 < \varepsilon < 1 \). Clearly \( d \in B \) and there exists a \( K \in R^+ \) such that

\[
d(x, y) \leq K \quad \text{for every} \quad x, y \in X.
\]

Choose \( d' \in M \) as follows

\[
d'(x, y) = \begin{cases} 
   d(x, y) + \frac{\varepsilon}{2}, & \text{if} \quad x, y \in X, x \neq y \\
   0, & \text{if} \quad x = y.
\end{cases}
\]

Then \( d^*(d, d') < \varepsilon \). We show that \( d' \in M \setminus f^{-1}(\{t\}) \). From (3) and (4) for \( x, y \in X(x \neq y) \) and \( t_1 = d(x, y), t_2 = d'(x, y) \) we have

\[
\varphi(d'(x, y)) \geq \varphi(d(x, y)) + \frac{\frac{\varepsilon}{2}}{(1 + d'(x, y))^2} \geq \varphi(d(x, y)) + \frac{\frac{\varepsilon}{2}}{(1 + K)^2}.
\]
Then \( f(d') > f(d) \), so \( d' \notin f^{-1}(\{t\}) \).

(ii) According to lemma the set \( g^{-1}(\{t\}) \) is closed in \( B \). It is enough to show that the set \( B \setminus g^{-1}(\{t\}) \) is dense in \( B \). Let \( d \in g^{-1}(\{t\}) \) and \( 0 < \varepsilon < 1 \). Define \( d' \) on \( X \) as follows:

\[
\begin{align*}
d'(x, y) &= \begin{cases} 
d(x, y) + \frac{\varepsilon}{2}, & \text{if } x \neq y \\
0, & \text{if } x = y
\end{cases}
\end{align*}
\]

Evidently \( g(d') = t + \frac{\varepsilon}{2} \), therefore \( g' \in B \setminus g^{-1}(\{t\}) \) and \( d^*(d, d') < \varepsilon \).

(iii) We can prove similarly like (ii). \( \blacksquare \)

From the above Theorem 1 we can see that the sets \( f^{-1}(\{t\}) \), \( g^{-1}(\{t\}) \), \( h^{-1}(\{t\}) \) are small from the topological point of view but on the other hand we show, that the cardinality of them is equal to the cardinality of the set \( M \).

In [2] is proved: \( \text{card}(M) = c \) if \( X \) is a finite set having at least two elements and \( \text{card}(M) = 2^{\text{card}(X)} \) if \( X \) is infinite set (\( c \) denotes the cardinality of the set of all real numbers).

**Theorem 2.** Let \( X \) be an infinite set. Then we have:

1. \( \text{card}(f^{-1}(\{t\})) = 2^{\text{card}(X)} \) for \( t \in (0, 1] \)
2. \( \text{card}(g^{-1}(\{t\})) = 2^{\text{card}(X)} \) for \( t \in [0, +\infty) \)
3. \( \text{card}(h^{-1}(\{t\})) = 2^{\text{card}(X)} \) for \( t \in (0, +\infty) \).

**Proof.** 1. Let \( 0 < t < 1 \) and \( 0 < \varepsilon < \frac{1}{2} \cdot \frac{1}{1-t} \). Let \( B \subseteq X \) for which \( \text{card}(B) \geq 2 \). We define the metric on \( X \) as follows:

\[
\sigma_B(x, y) = \begin{cases} 
0, & \text{if } x = y \\
\frac{t}{1-t}, & \text{if } x, y \in B; x \neq y \\
\frac{t}{1-t} - \varepsilon, & \text{if } x \notin B \text{ or } y \notin B, x \neq y
\end{cases}
\]

It is to easy to verify that \( \sigma_B \) is a metric and that \( \sigma_B \neq \sigma'_B \), if \( B \neq B' \). Evidently \( f(\sigma_B) = t \). There are \( 2^{\text{card}(X)} \) many choices for \( B \) so we can see

\( 2^{\text{card}(X)} \leq \text{card}(f^{-1}(\{t\})) \leq \text{card}(M) \leq 2^{\text{card}(X)} \).

We get by the Cantor–Bernstein theorem that \( \text{card}(f^{-1}(\{t\})) = 2^{\text{card}(X)} \).

Let now \( t = 1 \) and \( X_0 = \{x_1 < x_2 < \cdots < x_n < \cdots \} \subset X \). Define the function \( d_B : X \times X \to R \):

\[
\begin{align*}
d_B(x_n, x_m) &= |n - m| \quad \text{for } n, m = 1, 2, \ldots \\
d_B(x, x_n) &= d_B(x_n, x) = n \quad \text{for } x \notin X_0 \\
d_B(x, y) &= d_B(y, x) = 1 \quad \text{for } x, y \notin X_0, x \neq y \\
d_B(x, x) &= 0 \quad \text{for } x \in X.
\end{align*}
\]
(The same function was used in [2], Theorem 5.) It can be easily verified that $d_B(x_n, x_1) \to \infty (n \to \infty)$, hence $f(d_B) = 1$. Thereby we have $2^{\text{card}(X)}$ possibilities for choosing of $B$, we get that $\text{card}(f^{-1}({\{t\}})) = 2^{\text{card}(X)}$.

2. For $t = 0$ it has been proved in [4] (Theorem 1), that $\text{card}(g^{-1}({\{t\}})) = 2^{\text{card}(X)}$. Let $t > 0$. Let $B \subset X$ is so, that $\text{card}(B) \geq 2$. Define $\rho_B$ on $X$ as follows:

\[
\rho_B(x, y) = \begin{cases} 
0, & \text{for } x = y \\
t, & \text{for } x, y \in B, x \neq y \\
t + 1, & \text{otherwise.}
\end{cases}
\]

Then $\rho_B \in \mathcal{M}$ and $g(\rho_B) = t$.

3. Let $t > 0$ and $0 < \zeta < \frac{t}{2}$. Then the function $\tau_B$ defined on $X$ by this way

\[
\tau_B(x, y) = \begin{cases} 
0, & \text{for } x = y \\
t, & \text{for } x, y \in B, x \neq y \\
t - \zeta, & \text{otherwise},
\end{cases}
\]

is a metric on $X$ and $h(\tau_B) = t$. ■

References


University of Education,
Department of Mathematics,
Farská 19, 949 74 Nitra, Slovakia
E-mail: toth@unitra.sk