On a class of differential equations connected with number-theoretic polynomials

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Abstract. In this paper we consider the special class of differential equations of second order. For this class we find a general solution which is strictly connected with some number-theoretic polynomials such as Dickson, Chebyshev, Pell and Fibonacci.

1. Introduction

Consider the following class of the polynomials:

\[(1) \quad W_n(x, c) = \left( \frac{x + \sqrt{x^2 + c}}{2} \right)^n + \left( \frac{x - \sqrt{x^2 + c}}{2} \right)^n \]

with respect to \(c\), where \(n \geq 1\) is the degree of the polynomial \(W_n(x, c)\). It is known (see [2], p. 94) that the Dickson polynomial \(D_n(x, a)\) of degree \(n \geq 1\) and integer parameter \(a\) can be represent in the form:

\[(D) \quad D_n(x, a) = \left( \frac{x + \sqrt{x^2 - 4a}}{2} \right)^n + \left( \frac{x - \sqrt{x^2 - 4a}}{2} \right)^n .\]

We note that the Dickson polynomial belongs to class (1) if we take \(c = -4a\). Taking \(c = -1\) in (1) we obtain the Chebyshev polynomial of the second kind. For \(c = 1\) we get the Pell polynomial and for \(c = 4\) the Fibonacci polynomial.

We prove the following:

Theorem. The general solution of the differential equation

\[(*) \quad (x^2 + c) y'' + xy' - n^2 y = 0; \quad x^2 + c > 0\]

is of the form

\[(** \quad y = C_1 \left( \frac{x + \sqrt{x^2 + c}}{2} \right)^n + C_2 \left( \frac{x - \sqrt{x^2 + c}}{2} \right)^n ,\]

where \(C_1, C_2\) are arbitrary constants.
We remark that the general solution (***) is strictly connected with the polynomials \( W_\alpha(x, c) \) defined by (1).

### 2. Basic Lemmas

**Lemma 1.** (see [1], Thm. 2.) Let the real-valued functions \( s_0, t_0 u, v \in C^2(J) \), where \( J \subset \mathbb{R} \) and \( u \neq 0, v \neq 0 \). Then the functions

\[
y_1 = s_0 u^\lambda, \quad y_2 = t_0 v^\lambda,
\]

where \( \lambda \) is non-zero real constant, are the particular solutions of the differential equation

\[
D_0 y'' + D_1 y' + D_2 y = 0,
\]

where

\[
D_0 = \det \begin{pmatrix} s_0 & s_1 \\ t_0 & t_1 \end{pmatrix}, \quad D_1 = \det \begin{pmatrix} s_2 & s_0 \\ t_2 & t_0 \end{pmatrix}, \quad D_2 = \det \begin{pmatrix} s_1 & s_2 \\ t_1 & t_2 \end{pmatrix}
\]

and

\[
s_1 = s_0' + \lambda s_0 \frac{u'}{u}, \quad t_1 = t_0' + \lambda t_0 \frac{v'}{v}
\]

\[
s_2 = s_1' + \lambda s_1 \frac{u'}{u}, \quad t_2 = t_1' + \lambda t_1 \frac{v'}{v}.
\]

**Lemma 2.** Let \( \lambda, s_0, t_0 \) be non-zero real constants and let non-zero real functions \( u, v \in C^2(J), J \subset \mathbb{R} \) be linearly independent over the real number field \( \mathbb{R} \). Then the general solution of the differential equation:

\[(***) \quad \det \begin{pmatrix} 1 & \frac{u'}{u} & \frac{v'}{v} \\ 1 & \frac{v'}{v} & \frac{u'}{u} \\ 1 & 1 & 1 \end{pmatrix} g'' + \lambda \det \begin{pmatrix} \frac{u'}{u} & 1 \\ \frac{v'}{v} & 1 \\ g & h \end{pmatrix} g' y = 0,
\]

where

\[
g = \frac{u''}{u} - (1 - \lambda) \left( \frac{u'}{u} \right)^2, \quad h = \frac{v''}{v} - (1 - \lambda) \left( \frac{v'}{v} \right)^2
\]

is of the form

\[
y = C_1 s_0 u^\lambda + C_2 t_0 v^\lambda,
\]
where $C_1, C_2$ are arbitrary constants.

**Proof.** By the assumptions of Lemma 1 and Lemma 2 it follows that

$$s_1 = \lambda s_0 \frac{u'}{u}, \quad t_1 = \lambda t_0 \frac{v'}{v}. \tag{9}$$

From (9) and (6) we obtain

$$s_2 = s_1' + \lambda s_1 \frac{u'}{u} = \lambda s_0 \left( \frac{u''}{u} - (1 - \lambda) \left( \frac{u'}{u} \right)^2 \right)\tag{10}$$

and

$$t_2 = t_1' + \lambda t_1 \frac{v'}{v} = \lambda t_0 \left( \frac{v''}{v} - (1 - \lambda) \left( \frac{v'}{v} \right)^2 \right). \tag{11}$$

Let us denote by $g = \frac{u''}{u} - (1 - \lambda) \left( \frac{u'}{u} \right)^2$ and by $h = \frac{v''}{v} - (1 - \lambda) \left( \frac{v'}{v} \right)^2$. Then the formulae (10) and (11) have the form:

$$s_2 = \lambda s_0 g, \quad t_2 = \lambda t_0 h. \tag{12}$$

By (12), (9) and Lemma 1 it follows that the differential equation (3) reduce to $(***)$). On the other hand from Lemma 1 it follows that the functions $y_1 = s_0 u^\lambda$ and $y_2 = t_0 v^\lambda$ are the particular solutions of $(***)$. Now we observe that the functions $u, v$ are linearly independent over $\mathbb{R}$ if and only if the functions $u^\lambda$ and $v^\lambda$ are linearly independent over $\mathbb{R}$. Indeed, denote by $W(u^\lambda, v^\lambda)$ the Wronskian of the functions $u^\lambda$ and $v^\lambda$ and let

$$D_0 = \det \begin{pmatrix} 1 & \frac{u'}{u} \\ 1 & \frac{v'}{v} \end{pmatrix}. \tag{13}$$

Then we have

$$D_0 = (uv)^{-1} \det \begin{pmatrix} u & v \\ u' & v' \end{pmatrix},$$

and

$$W(u^\lambda, v^\lambda) = \det \begin{pmatrix} u^\lambda & v^\lambda \\ (u^\lambda)' & (v^\lambda)' \end{pmatrix} = \lambda (uv)^\lambda \det \begin{pmatrix} 1 & 1 \\ \frac{u'}{u} & \frac{v'}{v} \end{pmatrix}. \tag{14}$$
Since \( \det \begin{pmatrix} 1 & 1 \\ \frac{u}{u'} & \frac{v}{v'} \end{pmatrix} = \det \begin{pmatrix} 1 & 1 \\ \frac{u}{u'} & \frac{v}{v'} \end{pmatrix} \), from the definition of \( D_0 \), (13) and (14) we get

\[
W \left( u^\lambda, v^\lambda \right) = \lambda(uc)^\lambda D_0 = \lambda(uc)^\lambda^{-1} \det \begin{pmatrix} u & v \\ u' & v' \end{pmatrix}.
\]

(15)

From (15) easily follows that the functions \( u^\lambda, v^\lambda \) are linearly independent over \( \mathbb{R} \) if and only if the functions \( u, v \) have the same property. Using the assumption of Lemma 2 about the functions \( u, v \) we obtain that the functions \( u^\lambda, v^\lambda \) and also \( y_1 = s_0 u^\lambda, y_2 = t_0 v^\lambda \) are linearly independent over \( \mathbb{R} \). Since the functions \( y_1, y_2 \) are the particular solutions of \((\ast \ast \ast)\), the function \( y = C_1 y_1 + C_2 y_2 = C_1 s_0 u^\lambda + C_2 t_0 v^\lambda \) is a general solution of \((\ast \ast \ast)\). The proof of Lemma 2 is complete.

3. Proof of the Theorem

Let \( \lambda = n \) be a natural number and let \( s_0 = t_0 = 1 \). Moreover, let \( u = a(x) + b(x) \sqrt{k} \) and \( v = a(x) - b(x) \sqrt{k} \), where \( k \) is fixed non-zero constant. If the functions \( u, v \) are linearly independent over \( \mathbb{R} \) then by Lemma 2 it follows that the general solution of the differential equation

\[
\det \begin{pmatrix} 1 & \frac{n}{u'} \\ \frac{u}{u'} & \frac{v}{v'} \end{pmatrix} y'' + \det \begin{pmatrix} g & 1 \\ h & 1 \end{pmatrix} y' + n \det \begin{pmatrix} \frac{n}{u'} & g \\ \frac{v}{v'} & h \end{pmatrix} y = 0
\]

is of the form

\[
y = C_1 \left( a(x) + b(x) \sqrt{k} \right)^n + C_2 \left( a(x) - b(x) \sqrt{k} \right)^n,
\]

where \( g = \frac{u^{n'}}{u} - (1 - n) \left( \frac{n'}{u} \right)^2 \) and \( h = \frac{v^{n'}}{v} - (1 - n) \left( \frac{n'}{v} \right)^2 \) and \( C_1, C_2 \) are arbitrary constants. Now, we put \( a(x) = \frac{x}{2}, \quad b(x) = \frac{\sqrt{x^2 + c}}{2}, \quad k = 1, \) where \( x^2 + c > 0 \). Then we have

\[
u = \frac{x + \sqrt{x^2 + c}}{2}, \quad v = \frac{x - \sqrt{x^2 + c}}{2}.
\]

From (18) we obtain

\[
u' = \frac{1}{2} \left( \frac{x + \sqrt{x^2 + c}}{\sqrt{x^2 + c}} \right), \quad v' = -\frac{1}{2} \left( \frac{x - \sqrt{x^2 + c}}{\sqrt{x^2 + c}} \right).
\]
By (18) and (19) easily follows that the functions $u, v$ are linearly independent over $\mathbb{R}$, because the Wronskian $W(u, v) \neq 0$. On the other hand from (19) we obtain

\begin{align}
(20) \quad u'' &= \frac{1}{2} \frac{c}{(x^2 + c) \sqrt{x^2 + c}}, \quad v'' = -\frac{1}{2} \frac{c}{(x^2 + c) \sqrt{x^2 + c}}.
\end{align}

From (19) and (18) we get

\begin{align}
(21) \quad \frac{u'}{u} &= \frac{1}{\sqrt{x^2 + c}}, \quad \frac{v'}{v} = -\frac{1}{\sqrt{x^2 + c}},
\end{align}

hence by (21) it follows that

\begin{align}
(22) \quad \left(\frac{u'}{u}\right)^2 = \left(\frac{v'}{v}\right)^2 = \frac{1}{x^2 + c}.
\end{align}

Similarly from (20) and (18) we obtain

\begin{align}
\frac{u''}{u} &= \frac{c}{(x^2 + c) (x + \sqrt{x^2 + c}) \sqrt{x^2 + c}},
\end{align}

(23)

\begin{align}
\frac{v''}{v} &= -\frac{c}{(x^2 + c) (x - \sqrt{x^2 + c}) \sqrt{x^2 + c}}.
\end{align}

From (21) we calculate that

\begin{align}
(24) \quad D_0 &= \det \begin{pmatrix} 1 & \frac{u'}{u} \\ 1 & \frac{v'}{v} \end{pmatrix} = \frac{v'}{v} - \frac{u'}{u} = -\frac{2}{\sqrt{x^2 + c}}.
\end{align}

In similar way from (22) and (23) we get

\begin{align}
(25) \quad D_1 &= \det \begin{pmatrix} g & 1 \\ h & 1 \end{pmatrix} = g - h = -\frac{2x}{(x^2 + c) \sqrt{x^2 + c}}.
\end{align}

On the other hand by (21) and (23) it follows that

\begin{align}
(26) \quad D_2 &= \det \begin{pmatrix} \frac{u'}{u} & g \\ \frac{v'}{v} & h \end{pmatrix} = h \frac{u'}{u} - g \frac{v'}{v} = \frac{2u}{(x^2 + c) \sqrt{x^2 + c}}.
\end{align}
Now, we see that from (24), (25) and (26) the differential equation (16) has the following form:

\[(x^2 + c) y'' + xy' - n^2 y = 0,\]

so denote that (27) is the same equation as in our Theorem. Thus, by Lemma 2 it follows that the general solution of (27) is given by the formula

\[y = C_1 \left( \frac{x + \sqrt{x^2 + c}}{2} \right)^n + C_2 \left( \frac{x - \sqrt{x^2 + c}}{2} \right)^n\]

and the proof of the Theorem is complete.

Remark. Consider the following functional matrix;

\[M(x) = \frac{1}{2} \begin{pmatrix} x & \sqrt{x^2 + c} \\ \sqrt{x^2 + c} & x \end{pmatrix} .\]

Then we can calculate that the functions \( u = \frac{x + \sqrt{x^2 + c}}{2} \) and \( v = \frac{x - \sqrt{x^2 + c}}{2} \) are the characteristic roots of this matrix. Hence, we observe that the general solution of the differential equation (16) is linear combination of the powers such roots.

References


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