ON POLYNOMIAL VALUES OF THE PRODUCT
OF THE TERMS OF LINEAR RECURRENCE SEQUENCES

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Abstract: Let $G$ and $H$ be linear recurrence sequences and let $F(x) = dx^q + d_{p-1}x^{p-1} + \cdots + d_0$, where $d$ and $d_i$'s are rational integers, be a polynomial. In this paper we showed that for the equation $G_n H_m = F(x)$, with some restriction, there are no solutions in $n, m$ and $x$ if $q > q_0$, where $q_0$ is an effectively computable positive constant.

1. Introduction

Let $G = \{G_n\}_{n=0}^\infty$ be a linear recursive sequence of order $k \geq 2$ defined by

$$G_n = A_1 G_{n-1} + \cdots + A_k G_{n-k} \quad (n \geq k),$$

where $G_0, G_1, \ldots, G_{k-1}, A_1, A_2, \ldots, A_k$ are rational integer constants. We need another sequence, too. Let $H = \{H_n\}_{n=0}^\infty$ be another linear recurrence of order $l$ defined by

$$H_n = B_1 H_{n-1} + \cdots + B_l H_{n-l} \quad (n \geq l),$$

where the initial terms $H_0, H_1, \ldots, H_{l-1}$ and the $B_i$'s are given rational integers. We suppose that $A_k \neq 0$, $B_l \neq 0$, and that the initial values of both sequences are not all zero.

Denote the distinct zeros of the characteristic polynomial

$$g(x) = x^k - A_1 x^{k-1} - \cdots - A_k$$

by $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_s$, and similarly let $\beta = \beta_1, \beta_2, \ldots, \beta_t$ be the distinct zeros of the polynomial

$$h(x) = x^l - B_1 x^{l-1} - \cdots - B_l.$$ 

We suppose that $s > 1$, $t > 1$ and $|\alpha| = |\alpha_1| > |\alpha_2| \geq |\alpha_3| \geq \cdots \geq |\alpha_s|$ and $|\beta| = |\beta_1| > |\beta_2| \geq |\beta_3| \geq \cdots \geq |\beta_t|$. Consequently, we have $|\alpha| > 1, |\beta| > 1$.

Assume that $\alpha$ and $\beta$ have multiplicity 1 in the characteristic polynomials. As it is known the terms of the sequences $G$ and $H$ can be written in the form

$$G_n = a_n \alpha^n + r_2(n) \alpha_2^n + \cdots + r_s(n) \alpha_s^n \quad (n \geq 0),$$

(1)

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and

\[ H_n = b\beta^n + q_2(n)\beta_2^n + \cdots + q_i(n)\beta_i^n \quad (n \geq 0), \]

where \( r_i \)'s, \( q_j \)'s are polynomials and the coefficients of the polynomials, \( a \) and \( b \) are elements of the algebraic number field \( \mathbb{Q}(\alpha, \alpha_2, \ldots, \alpha_s, \beta, \beta_2, \ldots, \beta_i) \). In the following we assume that \( ab \neq 0 \) and

\[ F(x) = dx^2 + d_px^p + d_{p-1}x^{p-1} + \cdots + d_0, \]

is a polynomial with rational integer coefficients, where \( d \neq 0, q \geq 2 \) and \( q > p \).

The Diophantine equation

\[ G_n = F(x) \]

with positive integer variables \( n \) and \( x \) was investigated by several authors. It is known that if \( G \) is a nondegenerate second order linear recurrence, with some restrictions, and \( F(x) = dx^2 \) then the equation (4) have finitely many integer solutions in variables \( n \geq 0, x \) and \( q \geq 2 \).

For general linear recurrences we know a similar result (see [4]). A more general result was proved by I. Nemes and A. Pethő [3]. They proved the following theorem: let \( G_n \) be a linear recurrence sequence defined by (1) and let \( F(x) \) be a polynomial defined by (3). Suppose that \( a_2 \neq 1, |a_1| = |a_1| > |a_2| > |\alpha_i| \) for \( 3 \leq i \leq s \), \( G_n \neq a_0n \) for \( n > c_1 \) and \( p \leq q_c \). Then all integer solution \( n, |x| > 1, q \geq 2 \) of the equation (4) satisfy \( q < c_3 \), where \( c_1, c_2 \) and \( c_3 \) are effectively computable positive constants depending on the parameters of the sequence \( G \) and the polynomial \( F(x) \).

P. Kiss [2] showed that some conditions of the above result can be left out.

We prove a theorem which investigates a similar property of the product of the terms of two different linear recurrences. In the theorem and its proof \( c_4, c_5, \ldots \) will denote effectively computable positive constants which depend on the sequences, the polynomial \( F(x) \) and the constants in the following theorem.

**Theorem.** Let \( G \) and \( H \) be linear recursive sequences satisfying the above conditions. Let \( K > 1 \) and \( \delta \) \( (0 < \delta < 1) \) be real numbers. Furthermore let \( F(x) \) be a polynomial defined in (3) with the condition \( p < bq \). Assume that \( G_i \neq a\alpha_i \), \( H_j \neq b\beta_j \) if \( i, j > n_0 \) and \( \alpha \notin \mathbb{Z} \text{ or } \beta \notin \mathbb{Z} \). Then the equation

\[ G_nH_m = F(x) \]

in positive integers \( n, m, x \) for which \( m \leq n < Kn \), implies that

\[ q < q_0 \quad (n_0, G, H, K, F, \delta), \]

where \( q_0 \) is an effectively computable number (which depends on only \( n_0, G, H, K, F \) and \( \delta \)).

In the proof of the Theorem we shall use the following result due to A. Baker (see Theorem 1. in [1] with \( \delta = \frac{1}{2} \)). In this lemma the height of an algebraic number means the height of the minimal defining polynomial of the algebraic number.
Lemma. Let $\pi_1, \pi_2, \ldots, \pi_r$ be non-zero algebraic numbers of heights not exceeding $M_1, M_2, \ldots, M_r$ respectively $(M_r \geq 4)$. Further let $b_1, \ldots, b_{r-1}$ be rational integers with absolute values at most $B$ and let $b_r$ be a non-zero rational integer with absolute value at most $B'$ $(B' \geq 3)$. Then there exists a computable constant $C = C(r, M_1, \ldots, M_{r-1}, \pi_1, \ldots, \pi_r)$ such that the inequalities

$$0 \neq \sum_{i=1}^{r} b_i \log \pi_i > e^{-C(\log M_r, \log B' + \frac{1}{B'})}$$

are satisfied. (It is assumed that the logarithms have their principal values.)

Proof. Suppose that (5) holds with the conditions given in the Theorem. We may assume without loss of generality that $|\alpha| \geq |\beta|$ and that the terms of the sequences $G, H$ are positive and $x > 1$. We may assume that $n > n_0$ and $m > n_0$.

By (1), (2) and (5) we have

$$F(x) = a \alpha^n \left(1 + \frac{r_2(n)}{a} \left(\frac{\alpha_2}{\alpha}\right)^n + \cdots \right) b \beta^m \left(1 + \frac{q_2(m)}{b} \left(\frac{\beta_2}{\beta}\right)^m + \cdots \right).$$

By the assumption $|\alpha| > |\alpha_i|$ and $|\beta| > |\beta_i|$ we obtain that

$$\left(1 + \frac{r_2(n)}{a} \left(\frac{\alpha_2}{\alpha}\right)^n + \cdots \right) \to 1 \quad \text{as} \quad n \to \infty,$$

and

$$\left(1 + \frac{q_2(m)}{b} \left(\frac{\beta_2}{\beta}\right)^m + \cdots \right) \to 1 \quad \text{as} \quad m \to \infty.$$ 

Then (5) can be written in the form

$$\frac{a \alpha^n \beta^m}{dx^y} = (1 + \varepsilon_1)((1 + \varepsilon_2)(1 + \varepsilon_3))^{-1} = \left(1 + \sum_{i=0}^{p} \frac{d}{d^i} x^{i-y}\right)$$

$$\left(\left(1 + \frac{1}{a} \sum_{i=2}^{x} r_2(n) \left(\frac{\alpha_2}{\alpha}\right)^n \right) \left(1 + \frac{1}{b} \sum_{i=2}^{y} q_2(m) \left(\frac{\beta_2}{\beta}\right)^m \right)\right)^{-1}$$

where

$$|\varepsilon_1| = \left|\frac{d}{d^p} \left(\frac{1}{x}\right)^{y-p}\right| \left|1 + \frac{d_{p-1}}{d^p} \left(\frac{1}{x}\right) + \cdots\right|$$

$$|\varepsilon_2| = \left|\frac{r_2(n)}{a} \left(\frac{\alpha_2}{\alpha}\right)^n \right| \left|1 + \frac{r_2(n)}{r_2(n)} \left(\frac{\alpha_3}{\alpha_2}\right)^n + \cdots\right|$$

and
(11) \[ |e_3| = \left| \frac{q_2(m)}{b} \left( \frac{\beta_2}{\alpha} \right)^m \right| \left| 1 + \frac{q_3(m)}{q_2(m)} \left( \frac{\beta_2}{\beta} \right)^m \right| + \cdots \]

Using (8), (9), (10), (11) and \( m \leq n < Km \) we have

(12) \[ c_4 x^\frac{n}{2} < |\alpha|^\frac{n}{9} < x^{e_5}. \]

Therefore by (9), (10), (11) and (12) we have the following inequalities

(13) \[ |e_1| < \left| \frac{\alpha}{\alpha} \right| ^{\frac{x}{2^e_3}}, \quad |e_2| < \left| \frac{\alpha_2}{\alpha} \right| ^{e_5}, \quad |e_3| < \left| \frac{\beta_2}{\beta} \right| ^{e_5}. \]

We distinguish two cases. First we suppose that

\[ x^q = \frac{ab}{c} \alpha^n \beta^m, \]

moreover, without loss of generality we may assume that \( \alpha \notin \mathbb{Z} \). Let \( \alpha' \neq \alpha \) be any conjugate of \( \alpha \) and let \( \varphi \) be an automorphism of \( \mathbb{Q} \) with \( \varphi(\alpha) = \alpha' \). Then \( \varphi(\beta) = \beta' \) is a conjugate of \( \beta \) and \( |\beta'| \leq |\beta|, |\alpha'| < |\alpha| \). Moreover,

\[ \frac{ab}{c} \alpha^n \beta^m = \varphi \left( \frac{ab}{c} \right) \left( \varphi(\alpha) \right)^n \left( \varphi(\beta) \right)^m. \]

Thus

\[ \left| \left( \frac{\alpha}{\varphi(\alpha)} \right)^n \right| \leq \left| \frac{c \varphi(ab)}{ab \varphi(c)} \left( \frac{\varphi(\beta)}{\beta} \right)^m \right| \leq \left| \frac{c \varphi(ab)}{ab \varphi(c)} \right|, \]

whence \( n \) is bounded, which implies that \( q \) is bounded.

Now we can suppose that \( dx^q \neq aba^n \beta^m \). Applying the Lemma with \( M_0 = x, B' = q \) and \( B = n \), it follows that

\[ L := \log - \frac{dx^q}{a \alpha^n b \beta^m} = \left| q \log x - \log a - n \log \alpha - \log b - m \log \beta \right| > e^{-C(\log x \log q + \frac{n}{q})}. \]

On the other hand, using that \( \frac{\log x}{q} > 1 - \delta \) and (13) we can derive an upper bound for \( L \)

\[ L < 2 |e_1| + 2 |e_2| + 2 |e_3| < e^{-cn} \]

and it follows that

(14) \[ c_20 (\log x \log q + \frac{n}{q}) > c_9 n. \]
By (12) we have

\[ c_{11} \log x < \frac{n}{q} < c_{12} \log x, \]

so by (14) and (15)

\[ \log q \log x > c_{12} n > c_{13} q \log x \]

and

\[ \frac{\log q}{q} > c_{13}. \]

This can be satisfied only by finitely many positive integer \( q \) so our theorem is proved.

References


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