A MOMENT INEQUALITY
FOR THE MAXIMUM PARTIAL SUMS
WITH A GENERALIZED SUPERADDITIVE STRUCTURE

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Abstract: F. A. Móricz, R. J. Serfling and W. F. Stout (1982) proved a moment inequality with superadditive function. The theorem of this paper extends this result to multidimensional sequence.

1. Notations

In the following $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{R}$ and $d$ denotes the set of integers, positive integers, real numbers and a fixed positive integer. We define $1 = (1, 1, \ldots, 1) \in \mathbb{N}^d$ and if $k = (k_1, k_2, \ldots, k_d) \in \mathbb{Z}^d$, $l = (l_1, l_2, \ldots, l_d) \in \mathbb{Z}^d$, $k_i \leq l_i$ for each $1 \leq i \leq d$ then $k \leq l$. The $k < l$ relation is defined similarly. If there exists an index such that $k_i > l_i$ then we write $k \not< l$. Denote $|k| = \prod_{i=1}^d k_i$ and let $\{X_k : k \in \mathbb{N}^d\}$ be a $d$-multiple sequence of random variables. $S_n$ will denote the sum $\sum_{k \leq n} X_k$ if $n \in \mathbb{N}^d$, otherwise $S_n = 0$. Finally $\mathbb{E}X$ will denote the expectation of the random variable $X$.

2. Preliminary results

Let $g: \mathbb{N}^2 \to \mathbb{R}$ be a nonnegative function. If $g(i, j) + g(j + 1, k) \leq g(i, k)$ for all $1 \leq i \leq j < k$ then we say that $g$ is superadditive. F. A. Móricz, R. J. Serfling and W. F. Stout (1982) proved the next theorem: If $\{X_l : l \in \mathbb{N}\}$ sequence of random variables, $\alpha > 1$, $r \geq 1$, $g$ is a superadditive function and

$$\mathbb{E} \left| \sum_{i=1}^j X_i \right|^r \leq g^\alpha(i, j)$$

for all $1 \leq i \leq j$ integers then there exists a constant $A_{\alpha, r}$ (what depends on $\alpha$ and $r$) such that for each $n \in \mathbb{N}$

$$\mathbb{E} \left( \max_{k \leq n} \left| \sum_{i=1}^k X_i \right| \right)^r \leq A_{\alpha, r} g^\alpha(1, n).$$
F. A. Móricz (1983) generalized the definition of superadditive function for $d$-dimension as follows. Let $g : \mathbb{N}^d \times \mathbb{N}^d \to \mathbb{R}$ be a nonnegative function. If

$$
g(\hat{i}, \hat{j}) + g(\hat{i}, \hat{j}) \leq g(\hat{i}, \hat{j}),$$

(2.1)

where $\hat{i}, \hat{j} \in \mathbb{N}^d$, $i \leq j$, $1 \leq l \leq d$, $i_l \leq k_l \leq j_l$ and

$$\hat{i} = (i_1, \ldots, i_{l-1}, k_l + 1, i_{l+1}, \ldots, i_d),$$

$$\hat{j} = (j_1, \ldots, j_{l-1}, k_l, j_{l+1}, \ldots, j_d)$$

then we say that the $g$ is superadditive. F. A. Móricz (1983) proved the next theorem what is generalization of the previous theorem. If $g$ is a superadditive function, $\alpha > 1$, $r \geq 1$ and

$$
\mathbb{E} \left\{ \sum_{1 \leq i \leq j} X_{i,j} \right\}^r \leq g^\alpha(\hat{i}, \hat{j})
$$

(2.2)

for all $\hat{i}, \hat{j} \in \mathbb{N}^d$, $i \leq j$ then there exists a constant $A_{\alpha, r, d}$ (what depends on $\alpha$, $r$ and $d$) such that

$$
\mathbb{E} \left( \max_{1 \leq k \leq n} |S_k| \right)^r \leq A_{\alpha, r, d} g^\alpha(1, n)
$$

for all $n \in \mathbb{N}^d$. This paper discuss another generalization.

2. Main result

**Theorem.** Let $g : \mathbb{N}^d \times \mathbb{N}^d \to \mathbb{R}$ be a nonnegative function, $\alpha > 1$ and $r \geq 1$. Assume that for each $1 \leq i \leq j < k$

$$
g(\hat{i}, \hat{j}) + g(\hat{j} + 1, \hat{k}) \leq g(\hat{i}, \hat{k}),$$

(3.1)

and

$$
\mathbb{E}|S_k - S_{k-1}|^r \leq g^\alpha(\hat{i}, \hat{j}).
$$

(3.2)

Then

$$
\mathbb{E} \left( \max_{1 \leq k \leq n} |S_k| \right)^r \leq A_{\alpha, r} g^\alpha(1, n)
$$

(3.3)

for all $n \in \mathbb{N}^d$ where $A_{\alpha, r} = (1 - \frac{1}{2^{\alpha - 1} r})^{-r}$.

**Remark.** This theorem is generalization of result of F. A. Móricz, R. J. Serfling and W. F. Stout (1982). We remark that condition (2.1) implies (3.1) on the
other hand if $X_k$ ($k \in \mathbb{N}^d$) nonnegative random variables then (3.2) implies (2.2) moreover the constant is not depending on $d$.

**Proof of Theorem.** Assume that $1 < N = (N, N, \ldots, N) \in \mathbb{N}^d$ and $n \in \mathbb{N}^d$ where $n \leq N$ and $n \neq N$. If $|j| = 0$ then let $g(1, j) = 0$. With these notations, since

$$g(i, j) \leq g(i, j) + g(j + 1, k) \leq g(i, k) \quad \forall 1 \leq i \leq j < k,$$

there exists $m \geq 0$ integer having the property that

$$g(1, m - 1) \leq \frac{1}{2} g(1, n) \leq g(1, m),$$

where $m = n - m \cdot 1$. So if $m < n$ then

$$\frac{1}{2} g(1, n) + g(m + 1, n) \leq g(1, m) + g(m + 1, n) \leq g(1, n).$$

Consequently we have

$$g(m + 1, n) \leq \frac{1}{2} g(1, n), \quad \text{if } m < n. \quad (3.6)$$

Let us define sets

$$B = \{ k \in \mathbb{N}^d : k < m \}$$

$$C = \{ k \in \mathbb{N}^d : k \leq n, k \neq m, m \neq k \}$$

$$D = \{ k \in \mathbb{N}^d : m < k \leq n \}$$

Let $k_1 \in D$ such that $|S_{k_1}| = \max_{k \in D} |S_k|$. If $D = \emptyset$ (other words $m = n$) then let $k_1 = m$. Let $k_2 \in C$ such that $|S_{k_2}| = \max_{k \in C} |S_k|$. With these notations we have

$$\max_{k \in A} |S_k| = \max \{ \max_{k < m} |S_k|, |S_{k_1}|, |S_{k_2}| \} \leq$$

$$\max \{ \max_{k < m} |S_k|, |S_m| + |S_{k_1} - S_m|, |S_{k_2}| \} \leq$$

$$|S_{k_1}| + \max \{ \max_{k < m} |S_k|, |S_{k_1} - S_m| \} \leq$$

$$|S_{k_1}| + \left( \max_{k < m} |S_k|^r + |S_{k_1} - S_m|^r \right)^{1/r} \quad (3.7)$$

The Minkowski’s inequality states that

$$(\mathbb{E}|X + Y|^r)^{1/r} \leq (\mathbb{E}|X|^r)^{1/r} + (\mathbb{E}|Y|^r)^{1/r}$$
where \( X, Y \) random variables and \( r \geq 1 \). Therefore with \( X = |S_{k_2}| \) and \( Y = \max_{k \leq n} |S_k| - |S_{k_2}| \) substitutions the Minkowski’s inequality and (3.7) imply

\[
\mathbb{E} \left( \max_{k \leq n} |S_k|^r \right)^{1/r} \leq \left( \mathbb{E} |S_{k_2}|^r \right)^{1/r} + \left( \mathbb{E} \max_{k \leq n} |S_k| - |S_{k_2}| \right)^{1/r} \leq \left( \mathbb{E} |S_{k_2}|^r \right)^{1/r} + \left( \mathbb{E} \max_{k \leq m} |S_k|^r \right)^{1/r} + \mathbb{E} |S_{k_2} - S_m|^r \right)^{1/r}. \tag{3.8}
\]

By condition (3.2) and (3.4) we get

\[
\left( \mathbb{E} |S_{k_2}|^r \right)^{1/r} \leq g^{\alpha/r}(1, k_2) \leq g^{\alpha/r}(1, n). \tag{3.9}
\]

An elementary computation shows that \( A_{\alpha, r} \geq 1 \) so (3.2), (3.4) and (3.6) imply

\[
\mathbb{E} |S_{k_2} - S_m|^r \leq g^{\alpha}(m + 1, k_1) \leq g^{\alpha}(m + 1, n) \leq \frac{1}{2^\alpha} g^{\alpha}(1, n) \leq A_{\alpha, r} \frac{1}{2^\alpha} g^{\alpha}(1, n), \tag{3.10}
\]

if \( D \neq \emptyset \) (what means \( m < k_1 \)).

After these we prove the theorem by \( d \)-dimensional induction.

\[
\mathbb{E} \max_{k \leq 1} |S_k|^r = \mathbb{E} |S_1|^r \leq g^{\alpha}(1, 1) \leq A_{\alpha, r} g^{\alpha}(1, 1)
\]

therefore \( n = 1 \) satisfies (3.3). Now, assume that (3.3) is true if \( n < N \). Thus (3.5) implies

\[
\mathbb{E} \max_{k \leq m} |S_k|^r \leq A_{\alpha, r} g^{\alpha}(1, m - 1) \leq A_{\alpha, r} \frac{1}{2^\alpha} g^{\alpha}(1, n), \tag{3.11}
\]

Finally by (3.8), (3.9), (3.10) and (3.11) we obtain

\[
\mathbb{E} \max_{k \leq n} |S_k|^r \right)^{1/r} \leq g^{\alpha/r}(1, n) + \left( A_{\alpha, r} \frac{1}{2^\alpha} g^{\alpha}(1, n) \right)^{1/r} = A_{\alpha, r} g^{\alpha/r}(1, n)
\]

therefore (3.3) is true for each \( n \) with \( n \leq N \) and \( n \not\in N \). This completes the proof of the theorem.

References


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