LINEAR RECURRENCES AND ROOTFINDING METHODS

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Abstract. Let $A, B, G_0$ and $G_1$ be fixed complex numbers, where $AB(|G_0| + |G_1|) \neq 0$. Denote by $\alpha$ and $\beta$ the roots of the equation $\lambda^2 - A\lambda + B = 0$ and suppose that $|\alpha| > |\beta|$. The sequence $\{W^{(k)}_{n, \alpha, \beta}\}_{n=0}^{\infty}$ is defined by $W^{(k)}_{n, \alpha, \beta} = (a^n \alpha^{n+k} + b^n \beta^{n+k})/(\alpha - \beta)$, where $k \geq 1$ and $d \geq 0$ are fixed integers, $a G_1 - \beta G_0 \neq 0$ and $b = G_1 - \alpha G_0$. In this paper, using new identities of the sequence $\{W^{(k)}_{n, \alpha, \beta}\}_{n=0}^{\infty}$, an other proof is presented for the Newton–Raphson and Halley transformations (accelerations) of the sequence $\{W^{(k)}_{n, \alpha, \beta}/W^{(k)}_{n-1, \alpha, \beta}\}_{n=0}^{\infty}$. It is also shown that the (transformed) sequences obtained by the secant, Newton–Raphson, Halley and Aitken transformations of the sequence $\{W^{(k)}_{n, \alpha, \beta}/W^{(k)}_{n-1, \alpha, \beta}\}_{n=0}^{\infty}$ tend to $c^k$ in order of $a(W^{(k)}_{n, \alpha, \beta}/W^{(k)}_{n-1, \alpha, \beta})$.

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1. Introduction

Let the $n^{th}$ ($n \geq 2$) term of the sequence $\{G_n\}_{n=0}^{\infty}$ be defined by the recursion

$$G_n = AG_{n-1} - BG_{n-2},$$

where $A, B, G_0$ and $G_1$ are fixed complex numbers and $AB(|G_0| + |G_1|) \neq 0$. If it is needed then the notation $G_n(A, B, G_0, G_1)$ is also used. For example, the $n^{th}$ term of the Fibonacci sequence is $F_n = G_n(1, -1, 0, 1)$. The abbreviations $U_n = G_n(A, B, 0, 1)$ and $V_n = (A, B, 2, A)$ will also be very useful for us.

Let $\alpha$ and $\beta$ be the roots of the equation $\lambda^2 - A\lambda + B = 0$ ($\alpha + \beta = A$, $\alpha \beta = B$) and suppose that $|\alpha| > |\beta|$. By the well known Binet formula we get that the explicit form of the term $G_n(A, B, G_0, G_1)$ is

$$G_n(A, B, G_0, G_1) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad (n \geq 0),$$

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where \( a = G_1 - \beta G_0 \), \( b = G_1 - \alpha G_0 \) and suppose that \( a \neq 0 \). For example, \( U_n = (\alpha^n - \beta^n) / (\alpha - \beta) \) and \( V_n = \alpha^n + \beta^n \) if \( \alpha, \beta = (A \pm \sqrt{A^2 - 4B}) / 2 \).

Z. Zhang [7] has defined the sequence \( \left\{ W_{n,d}(A, B, G_0, G_1) \right\}_{n=0}^{\infty} \) in the following manner.

\[
W_{n,d}(A, B, G_0, G_1) = (\alpha^k + \beta^k) W_{n-1,d} - \alpha^k \beta^k W_{n-2,d} \quad (n \geq 2),
\]

where \( k \geq 1 \) and \( d \geq 0 \) are fixed integers, while

\[
W_{0,d}(A, B, G_0, G_1) = \frac{\alpha^k \alpha^d - b^k \beta^d}{\alpha - \beta}, \quad W_{1,d}(A, B, G_0, G_1) = \frac{\alpha^k \alpha^{k+d} - b^k \beta^{k+d}}{\alpha - \beta}.
\]

For brevity, we write \( W_{n,d}^{(k)} \) instead of \( W_{n,d}(A, B, G_0, G_1) \).

It is obvious that \( \alpha^k \) and \( \beta^k \) are the roots of the equation

\[
\lambda^2 - (\alpha^k + \beta^k)\lambda + \alpha^k \beta^k = \lambda^2 - V_k \lambda + B^k = 0
\]

and \( |\alpha| > |\beta| \) implies \( |\alpha^k| > |\beta^k| \). Using the Binet formula for (2) we get that

\[
W_{n,d}^{(k)} = \frac{(W_{1,d}^{(k)} - \beta^k W_{0,d}^{(k)}) \alpha^{n_k} - (W_{1,d}^{(k)} - \alpha^k W_{0,d}^{(k)}) \beta^{n_k}}{\alpha^k - \beta^k},
\]

from which

\[
W_{n,d}^{(k)} = \frac{\alpha^k \alpha^{n_k+d} - b^k \beta^{n_k+d}}{\alpha - \beta}
\]

yields for \( n \geq 0 \). It can be seen that \( W_{n,d}^{(k)} \) is a generalization of \( G_n \) because e. g.

\[
G_n = G_n(A, B, G_0, G_1) = W_{n,0}^{(1)}(A, B, G_0, G_1).
\]

If \( W_{n,0}^{(k)} \neq 0 \) then let

\[
R_{n,d}^{(k)} = \frac{W_{n,d}^{(k)}}{W_{n,0}^{(k)}}.
\]

By (3), \( a \neq 0 \) and \( |\alpha| > |\beta| \), one can easily prove that

\[
\lim_{n \to \infty} R_{n,d}^{(k)} = \alpha^d,
\]
i.e., the sequence \( \{ R_{n,d}^{(k)} \}_{n=0}^{\infty} \) tends to the root \( \alpha^d \) of the polynomial

\[
f(\lambda) = \lambda^2 - (\alpha^d + \beta^d)\lambda + \alpha^d \beta^d = \lambda^2 - V_d \lambda + B^d.
\]

Recently, many authors have studied the connection between recurrences and iterative transformations. The main idea is to consider such sequence transformations \( T \) of the convergent sequence \( \{ X_n \}_{n=0}^{\infty} \) into the sequence \( \{ T_n \}_{n=0}^{\infty} \), where \( \{ T_n \}_{n=0}^{\infty} \) converges more quickly to the same limit \( X \). Thus, one can investigate the properties of these transformations or the accelerations of the convergence. We say that \( \{ T_n \}_{n=0}^{\infty} \) converges more quickly to \( X \) than \( \{ X_n \}_{n=0}^{\infty} \) if \( T_n - X = o(X_n - X) \), i.e., if \( \lim_{n \to \infty} \left((T_n - X) / (X_n - X)\right) = 0 \).

The most known four sequence transformations to accelerate the convergence of a sequence are the secant \( S(X_n, X_m) \), Newton–Raphson \( N(X_n) \), Halley \( H(X_n) \) and Aitken transformation \( A(X_n, X_m, X_l) \), namely if \( \{ X_n \}_{n=0}^{\infty} = \{ R_{n,d}^{(k)} \}_{n=0}^{\infty} \) and \( X = \alpha^d \) (i.e., the root of \( f(\lambda) = 0 \) in (5)), then

\[
S(X_n, X_m) = \frac{X_n X_m - B^d}{X_n + X_m - V_d},
\]

\[
N(X_n) = \frac{X_n^2 - B^d}{2X_n - V_d},
\]

\[
H(X_n) = \frac{X_n^3 - 3B^d X_n + V_d B^d}{3X_n^2 - 3V_d X_n + V_d^2 - B^d};
\]

\[
A(X_n, X_m, X_l) = \frac{X_n X_l - X_m^2}{X_n - 2X_m + X_l},
\]

where we assume that division by zero does not occur. (The formulae (6)-(9) can be obtained from (5) using the known forms of the transformations \( S, N, H \) and \( A \), or they can be found in [4] p. 366 and p. 369.)

Some results from the recent past: G. M. Phillips [5] proved that if \( r_n' = \frac{F_{n+1}}{F_n} \) then \( A(r_{n-1}', r_n', r_{n+1}') = r_{2n} \). J. H. McCabe and G. M. Phillips [3] generalized this for \( r_n'' = \frac{F_{n+1}}{F_n} \), and they also proved that \( S(r_n'', r_m'') = r_{n+m}'' \) and \( N(r_n'') = r_{2n}'' \). M. J. Jamieson [1] investigated the case \( r_n'' = \frac{\mu_{n+d}}{\mu_n} \) for \( d > 1 \). J. B. Muskat [4], using the notations \( r_n = \frac{\mu_{n+d}}{\mu_n} \) and \( R_n = \frac{\nu_{n+m}}{\nu_n} \) \( (d > 1) \), proved that

\[
\begin{align*}
(a) & \quad S(r_n, r_m) = r_{n+m}, & S(R_n, R_m) = r_{n+m}, \\
(b) & \quad N(r_n) = r_{2n}, & N(R_n) = r_{2n},
\end{align*}
\]
(c) \(H(r_n) = r_{3n}\), \(H(R_n) = R_{3n}\),
(d) \(A(r_{n-t}, r_n, r_{n+t}) = r_{2n}\), \(A(R_{n-t}, R_n, R_{n+t}) = r_{2n}\).

Similar results were obtained for special second order linear recurrences in [2]
by F. Mátyás, while Z. Zhang ([7],[8]) stated and partially proved that
(a) \(S \left( R_{n,d}^{(k)}, R_{n,d}^{(2k)} \right) = R_{(n+m)/2,d}^{(2k)} \), (2|n + m),
(b) \(N \left( R_{n,d}^{(k)} \right) = R_{n,d}^{(2k)} \),
(c) \(H \left( R_{n,d}^{(k)} \right) = R_{n,d}^{(3k)} \),
(d) \(A \left( R_{n-1,d}^{(k)}, R_{n,d}^{(k)}, R_{n+d,d}^{(k)} \right) = R_{n,d}^{(2k)} \).

(11)

It is easy to see that (11) implies (10) if \(k = 1, G_0 = 0, G_1 = 1\) or \(k = 1, G_0 = 2, G_1 = \alpha\). We mention that R. B. Tamer and M. Rachidi [6] investigated the so-called \(\varepsilon\)-algorithm to the ratio of the terms of linear recurrences of order \(r \geq 2\).

The purpose of this paper is to present some new properties of the sequence \(\left\{ W_{n,d}^{(k)} \right\}_{n=0}^{\infty}\) (see Lemma 1 and Lemma 2) and, using them, to give new proofs for (11)/(a) and (c), since Z. Zhang, using some other properties proven by him, presented the proof for only the cases (11)/(a) and (d) in [7] and [8]. We also show that the transformations \(S, N, H\) and \(A\) creat such sequences from \(\left\{ R_{n,d}^{(k)} \right\}_{n=0}^{\infty}\)
which tend to \(\alpha^d\) in order of \(o\left(R_{n,d}^{(k)} - \alpha^d\right)\).

2. Results

Applying the notations introduced in this paper, assume that \(k \geq 1\) and \(d \geq 0\)
are fixed integers, in (1) \(AB\left(|G_0| + |G_1|\right) \neq 0, a \neq 0\) and \(|\alpha| > |\beta|\). We always assume that division by zero does not occur. First we formulate two lemmas.

Lemma 1. Let \(n\) and \(m\) be non-negative integers with the same parity. Then
(a) \(W_{n,d}^{(k)}W_{m,d}^{(k)} - W_{n,0}^{(k)}W_{m,d}^{(k)}B^d = W_{n+2m, d}^{(k)}U^d\),
(b) \(W_{n,d}^{(k)}W_{m,0}^{(k)} + W_{m,d}^{(k)}W_{n,0}^{(k)} - W_{n,0}^{(k)}W_{m,0}^{(k)}V^d = W_{n+2m, 0}^{(k)}U^d\).

Lemma 2. Let \(n\) be a non-negative integer. Then
(a) \(W_{n,d}^{(k)}W_{n,d}^{(2k)} - W_{n,0}^{(k)}W_{n,0}^{(2k)}B^d = W_{n,d}^{(3k)}U^d\),
(b) \(W_{n,d}^{(2k)}W_{n,0}^{(k)} - W_{n,0}^{(2k)}W_{n,0}^{(k)} + W_{n,d}^{(2k)}W_{n,0}^{(k)} = W_{n,0}^{(3k)}U^d\).

Theorem 1. Let \(n\) be a non-negative integer. Then
(a) $N \left(R^{(k)}_{n,d}\right) = R^{(2k)}_{n,d}$,

(b) $H \left(R^{(k)}_{n,d}\right) = R^{(3k)}_{n,d}$.

The following theorem implies that the transformations $S$, $N$, $H$ and $A$ produce such sequences from the sequence $\left\{R^{(k)}_{n,d}\right\}_{n=0}^\infty$ which tend very quickly to $\alpha^d$.

**Theorem 2.** Let $l > k \geq 1$ be fixed integers. Then

$$R^{(l)}_{n,d} - \alpha^d = o \left(R^{(k)}_{n,d} - \alpha^d\right).$$

**Corollary.** Theorem 1 and (11) show that the transformations $S$, $N$, $A$ and $H$ transform $R^{(k)}_{n,d}$ into $R^{(2k)}_{n,d}$ and into $R^{(3k)}_{n,d}$, respectively, thus Theorem 2 implies that all of the mentioned transformations give accelerations of the convergence.

### 3. Proofs of Lemmas and Theorems

**Proof of Lemma 1.** Because of the similarity of the proofs we present only the proof of part (a). Using the explicit form (3) of $W^{(k)}_{n,d}$, we write

$$W^{(k)}_{n,d}W^{(k)}_{m,d} - W^{(k)}_{n,0}W^{(k)}_{m,0}B^d = \frac{(a^k \alpha^{nk+d} - b^k \beta^{nk+d})(a^k \alpha^{mk+d} - b^k \beta^{mk+d})}{(\alpha - \beta)^2}$$

$$- \frac{(a^k \alpha^{nk} - b^k \beta^{nk})(a^k \alpha^{mk} - b^k \beta^{mk})(\alpha^d \beta^d)}{(\alpha - \beta)^2} = \cdots = \frac{\alpha^d - \beta^d}{\alpha - \beta}$$

$$= \frac{U_d \gamma^{(2k)}_{\frac{2}{2k},d}}{\alpha - \beta}.$$

**Proof of Lemma 2.** Here we also give only the proof of part (a). By (3)

$$W^{(k)}_{n,d}W^{(2k)}_{m,d} - W^{(k)}_{n,0}W^{(2k)}_{m,0}B^d = \frac{(a^k \alpha^{nk+d} - b^k \beta^{nk+d})(a^{2k} \alpha^{2nk+d} - b^{2k} \beta^{2nk+d})}{(\alpha - \beta)^2}$$

$$- \frac{(a^k \alpha^{nk} - b^k \beta^{nk})(a^{2k} \alpha^{2nk} - b^{2k} \beta^{2nk})(\alpha^d \beta^d)}{(\alpha - \beta)^2} = \cdots = \frac{\alpha^d - \beta^d}{\alpha - \beta}.$$
\[
\frac{a^{3k} \alpha^{3n-1} + b^{3k} \beta^{3n-1}}{\alpha - \beta} = U_d W_{n,d}^{(3k)}.
\]

**Proof of Theorem 1.** (a) By (7) and (4)

\[
N \left( R_{n,d}^{(k)} \right) = \left( \frac{W_{n,d}^{(k)}}{W_{n,0}^{(k)}} \right)^2 - B_d^2 = \left( \frac{W_{n,d}^{(k)}}{W_{n,0}^{(k)}} \right)^2 - \left( \frac{W_{n,0}^{(k)}}{W_{n,0}^{(k)}} \right)^2 B_d^2.
\]

Applying Lemma 1 in the case \( n = m \), we have

\[
N \left( R_{n,d}^{(k)} \right) = \frac{U_d \cdot W_{n,d}^{(2k)}}{U_d \cdot W_{n,0}^{(2k)}} = R_{n,d}^{(2k)}.
\]

(b) By the Halley transformation (8) and (4)

\[
H \left( R_{n,d}^{(k)} \right) = \frac{\left( R_{n,d}^{(k)} \right)^3 - 3B_d^2 R_{n,d}^{(k)} + V_d B_d}{3 \left( R_{n,d}^{(k)} \right)^2 - 3V_d R_{n,d}^{(k)} + V_d^2 - B_d^2}
\]

\[
= \frac{\left( W_{n,d}^{(k)} \right)^3 - 3B_d^2 W_{n,d}^{(k)} \left( W_{n,0}^{(k)} \right)^2 + V_d B_d \left( W_{n,0}^{(k)} \right)^3}{3 \left( W_{n,d}^{(k)} \right)^2 W_{n,0}^{(k)} - 3V_d W_{n,d}^{(k)} \left( W_{n,0}^{(k)} \right)^2 + (V_d^2 - B_d) \left( W_{n,0}^{(k)} \right)^3}
\]

The numerator and the denominator of the last fraction, by Lemma 1, can be rewritten as

\[
U_d \left( W_{n,d}^{(2k)} - B_d W_{n,0}^{(2k)} W_{n,0}^{(k)} \right)
\]

and

\[
U_d \left( W_{n,d}^{(k)} W_{n,0}^{(2k)} + \left( W_{n,d}^{(k)} - V_d W_{n,0}^{(k)} \right) W_{n,0}^{(2k)} \right),
\]

respectively. From these, by Lemma 2,

\[
H \left( R_{n,d}^{(k)} \right) = \frac{U_d^2 W_{n,d}^{(3k)}}{U_d^2 W_{n,0}^{(3k)}} = R_{n,d}^{(3k)}
\]

follows.
Proof of Theorem 2. To prove the theorem we have to show that
\[ \lim_{n \to \infty} \frac{R_{n,d}^{(l)}}{R_{n,d}^{(k)}} - \alpha^d = 0. \]

Applying (4) and (3), we get that
\[ \frac{R_{n,d}^{(l)}}{R_{n,d}^{(k)}} - \alpha^d = \frac{W_{n,d}^{(l)}}{W_{n,d}^{(k)}} - \alpha^d = \frac{W_{n,0}^{(l)}}{W_{n,0}^{(k)}} = \cdots = \left( \frac{b}{a} \right)^{l-k} \left( \frac{\beta}{\alpha} \right)^{n(l-k)} \frac{1 - \left( \frac{b}{a} \right)^k \left( \frac{\beta}{\alpha} \right)^{nk}}{1 - \left( \frac{b}{a} \right)^k \left( \frac{\beta}{\alpha} \right)^{nl}}, \]
from which, by $|\alpha| > |\beta|$ and $l > k \geq 1$,
\[ \lim_{n \to \infty} \frac{R_{n,d}^{(l)}}{R_{n,d}^{(k)}} - \alpha^d = 0 \]

follows.

References


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