THE CONDITION FOR GENERALIZING INVERTIBLE SUBSPACES IN CLIFFORD ALGEBRAS

Nguyen Canh Luong (Hanoi, Vietnam)

Abstract. Let $\mathcal{A}$ be a universal Clifford algebra induced by $m$-dimensional real linear space with basis $\{e_1, e_2, \ldots, e_m\}$. The necessary and sufficient condition for the subspaces of form $L_i = \text{lin}\{e_0, e_1, \ldots, e_m, e_{m+1}, \ldots, e_{m+i}\}$ to be invertible is $m \equiv 2 \pmod{4}$, $i \neq 1$ and $e_{m+i} = e_{12} \ldots m$ (see [2]). In this paper we improve this assertion for the subspaces of the form $L_i = \text{lin}\{e_0, e_{A_1}, \ldots, e_{A_m}, e_A, e_{m+1}, \ldots, e_{m+i}\}$, where $A_i \subseteq \{1,2,\ldots,m\}$ ($i=1,2,\ldots,m+i$).

1. Introduction

Let $V_m$ be an $m$-dimensional ($m \geq 1$) real linear space with basis $\{e_1, e_2, \ldots, e_m\}$. Consider the $2^m$-dimensional real space $\mathcal{A}$ with basis

$$E = \{e_0, e_{[1]}, \ldots, e_{[m]}, e_{[1,2]}, \ldots, e_{[m-1,m]}, \ldots, e_{[1,2,\ldots,m]}\},$$

where $e_{[i]} := e_i$ ($i = 1, 2, \ldots, m$).

In the following, for each $K = \{k_1, k_2, \ldots, k_t\} \subseteq \{1, 2, \ldots, m\}$ we write $e_K = e_{k_1, k_2, \ldots, k_t}$ with $e_0 = e_0$, and so

$$E = \{e_0, e_{1}, \ldots, e_{m}, e_{12}, \ldots, e_{m-1m}, \ldots, e_{12\ldots m}\}.$$

The product of two elements $e_A, e_B \in E$ is given by

$$(1)\quad e_A e_B = (-1)^{|A \cap B|} (-1)^{p(A,B)} e_{A \Delta B}; \quad A, B \subseteq \{1, 2, \ldots, m\},$$

where

$$\begin{cases}
  p(A, B) = \sum_{j \in B^c} p(A, j), \\
  p(A, j) = \mathbb{1}_{\{i \in A : i > j\}}, \\
  A \Delta B = (A \backslash B) \cup (B \backslash A)
\end{cases}$$

and $\sharp A$ denotes the number of elements of $A$. 

Each element \( a = \sum_A a_A e_A \in \mathcal{A} \) is called a Clifford number. The product of two Clifford numbers \( a = \sum_A a_A e_A; \quad b = \sum_B b_B e_B \) is defined by the formula

\[
ab = \sum_A \sum_B a_A b_B e_A e_B.
\]

It is easy to check that in this way \( \mathcal{A} \) is turned into a linear associative noncommutative algebra over \( \mathbb{R} \). It is called the Clifford algebra over \( V_m \).

It follows at once from the multiplication rule (1) that \( e_\# \) is identity element, which is denoted by \( e_0 \) and in particular

\[
e_i e_j + e_j e_i = 0 \text{ for } i \neq j; \quad e_i^2 = -1 \quad (i, j = 1, 2, \ldots, m)
\]

and

\[
e_{k_1, k_2, \ldots, k_t} = e_{k_1} e_{k_2} \cdots e_{k_t}; \quad 1 \leq k_1 < k_2 < \ldots < k_t \leq m.
\]

The involution for basic vectors is given by

\[
\bar{e}_{k_1, k_2, \ldots, k_t} = (-1)^{\frac{t(t+1)}{2}} e_{k_1, k_2, \ldots, k_t}.
\]

For any \( a = \sum_A a_A e_A \in \mathcal{A} \), we write \( \bar{a} = \sum_A a_A \bar{e}_A \). For any Clifford number \( a = \sum_A a_A e_A \), we write \( |a| = (\sum_A a_A^2)^{\frac{1}{2}} \).

2. Result and Proof

We use the following definitions.

(i) An element \( a \in \mathcal{A} \) is said to be invertible if there exists an element \( a^{-1} \) such that \( aa^{-1} = a^{-1}a = e_0 \); \( a^{-1} \) is said to be the inverse of \( a \).

(ii) A subspace \( X \subset \mathcal{A} \) is said to be invertible if every non-zero element in \( X \) is invertible in \( \mathcal{A} \).

(iii) \( L(u_1, u_2, \ldots, u_n) = \text{lin} \{ u_1, u_2, \ldots, u_n \}, u_i \in \mathcal{A} \quad (i = 1, 2, \ldots, n) \).

It is well-known (see [1]) that for any special Clifford number of the form

\[
a = \sum_{i=0}^m a_i e_i \neq 0 \text{ we have } a^{-1} = \frac{\bar{a}}{|a|^2}.
\]

So \( L(e_0, e_1, \ldots, e_m) \) is invertible, and if \( m \equiv 2 \mod 4 \) (see [2]), then every \( a = \sum_{i=0}^{m+1} a_i e_i \neq 0 \), where \( e_{m+1} = e_{12} \cdots e_m \) is invertible

and \( a^{-1} = \frac{\bar{a}}{|a|^2} \). So \( L(e_0, e_1, \ldots, e_m, e_{m+1}) \) is invertible.
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We shall need the following lemmas.

**Lemma 1.** (see Lemma 1 [3]) If \( L(e_{A_1}, e_{A_2}, \ldots, e_{A_k}) \), where \( e_{A_i} \in E, e_{A_i} \neq e_{A_j} \) for \( i \neq j, \ i, j \in \{1, 2, \ldots, k\} \), is invertible if and only if \( L(e_{A_1}, e_{A_2}e_{A_3}, \ldots, e_{A_k}e_{A_k}) \) is invertible.

By Lemma 1 we shall study subspaces of \( \mathcal{A} \) in the form \( L(e_0, e_{A_1}, \ldots, e_{A_l}) \).

**Lemma 2.** (see Lemma 3 [3]) \( L(e_0, e_{A_1}, \ldots, e_{A_l}) \), \( e_{A_i} \in E, e_{A_i} \neq e_{A_j} \) for \( i \neq j \), is invertible if and only if \( e_{A_i}e_{A_j} + e_{A_j}e_{A_i} = 0 \) for \( i \neq j, \ i, j \in \{0, 1, 2, \ldots, m\} \), where \( e_{A_0} = e_0 \).

**Lemma 3.** (see Theorem 3 [3]) If \( L(e_0, e_{A_1}, e_{A_2}, \ldots, e_{A_l}) \), \( e_{A_i} \in E, e_{A_i} \neq e_{A_j} \) for \( i \neq j, \ i, j \in \{1, 2, \ldots, l\} \) is invertible, then

(i) \( l \leq m + 1 \).

(ii) If \( l = m + 1 \), then

either \( e_{A_1} = e_{A_1}e_{A_2} \ldots e_{A_m} \) or \( e_{A_1} = -e_{A_1}e_{A_2} \ldots e_{A_m} \).

The purpose of this paper is to prove the following.

**Theorem.** \( L(e_0, e_{A_1}, \ldots, e_{A_m}, e_{A_{m+1}}, \ldots, e_{A_{n+m}}) \) is invertible if and only if the following conditions simultaneously hold:

1. \( e_{A_i}e_{A_j} + e_{A_j}e_{A_i} = 0 \) for \( i \neq j, \ i, j \in \{0, 1, 2, \ldots, m\} \), where \( e_{A_0} = e_0 \).
2. \( m \equiv 2 \pmod{4} \),
3. \( s = 1 \),
4. Either \( e_{A_{m+1}} = e_{A_1}e_{A_2} \ldots e_{A_m} \) or \( e_{A_{m+1}} = -e_{A_1}e_{A_2} \ldots e_{A_m} \).

**Proof.** First we prove the sufficiency. From \( e_{A_i}e_{A_j} + e_{A_j}e_{A_i} = 0 \) for \( i \neq j, \ i, j \in \{0, 1, 2, \ldots, m\} \) we have

\[
e_{A_i}e_{A_i} + e_{A_j}e_{A_i} = 0 \quad \text{and} \quad e_{A_i} + e_{A_j}e_{A_i} = 0 \quad \text{for} \quad i \neq j, \ i, j \in \{1, \ldots, m\}.
\]

We shall prove that \( e_{A_k}e_{A_{m+1}} + e_{A_{m+1}}e_{A_k} = 0 \) for \( k \in \{0, 1, \ldots, m\} \). For \( k = 0 \), by \( ab = ba \) and by \( m \equiv 2 \pmod{4} \), we get that

\[
e_0e_{A_{m+1}} + e_{A_{m+1}}e_0 = e_{A_1}e_{A_2} \ldots e_{A_m} + e_{A_1}e_{A_2} \ldots e_{A_m}
\]

\[
= e_{A_m}e_{A_{m-1}} \ldots e_{A_1} + e_{A_1}e_{A_2} \ldots e_{A_m}
\]

\[
= (-1)^m e_{A_m}e_{A_{m-1}} \ldots e_{A_1} + e_{A_1}e_{A_2} \ldots e_{A_m}
\]
\[ (-1)^m (-1)^{\frac{m(m-1)}{2}} e_{A_1} e_{A_2} \cdots e_{A_m} + e_{A_1} e_{A_2} \cdots e_{A_m} = 0. \]

For \( k \in \{1, 2, \ldots, m\} \) we have

\[
\begin{align*}
\epsilon_{A_k} e_{A_{m+1}} + e_{A_{m+1}} \bar{e}_{A_k} &= \epsilon_{A_k} \bar{e}_{A_m} \cdots \bar{e}_{A_k} \cdots \epsilon_{A_1} + e_{A_1} \cdots e_{A_k} \cdots e_{A_m} \bar{e}_{A_k} \\
&= (-1)^{m-k} e_{A_m} \cdots e_{A_k} e_{A_1} + (-1)^{m-k} e_{A_1} \cdots e_{A_k} e_{A_m} \\
&= (-1)^{m-k} \left( (-1)^{m-1} e_{A_m} \cdots e_{A_{k+1}} e_{A_{k-1}} \cdots e_{A_1} + e_{A_1} \cdots e_{A_{k-1}} e_{A_{k+1}} \cdots e_{A_m} \right) \\
&= (-1)^{m-k} \left[ -(-1)^{\frac{(m-1)(m-2)}{2}} e_{A_1} \cdots e_{A_{k+1}} e_{A_{k+1}} \cdots e_{A_m} + e_{A_1} \cdots e_{A_{k-1}} e_{A_{k+1}} \cdots e_{A_m} \right] = 0.
\end{align*}
\]

Take \( 0 \neq a = a_0 e_0 + \sum_{i=1}^{m+1} a_i e_{A_i} \in L(\epsilon_0, e_{A_1}, \ldots, e_{A_m}, e_{A_{m+1}}). \)

Let \( a^{-1} = \frac{1}{|a|^2} \left( a_0 e_0 + \sum_{i=1}^{m+1} a_i e_{A_i} \right). \) Then

\[
\begin{align*}
\begin{array}{c}
a \cdot a^{-1} = \frac{1}{|a|^2} \left( a_0 e_0 + \sum_{i=1}^{m+1} a_i e_{A_i} \right) \left( a_0 e_0 + \sum_{i=1}^{m+1} a_i \bar{e}_{A_i} \right) \\
= \frac{1}{|a|^2} \left[ a_0^2 e_0 + a_0 \left( \sum_{i=1}^{m+1} a_i e_{A_i} + \sum_{j=1}^{m+1} a_j \bar{e}_{A_i} \right) + \sum_{i=1}^{m+1} a_i^2 e_{A_i} \bar{e}_{A_i} \\
+ \sum_{i<j} a_i a_j (e_{A_i} \bar{e}_{A_j} + e_{A_j} \bar{e}_{A_i}) \right] = \frac{1}{|a|^2} \left( \sum_{i=0}^{m+1} a_i^2 \right) e_0 = e_0.
\end{array}
\end{align*}
\]

Similarly, one can check the equality \( a^{-1} \cdot a = e_0. \)

Now we prove the necessity. By Lemma 2 we have \( e_{A_i} \bar{e}_{A_i} + e_{A_j} \bar{e}_{A_i} = 0 \) for \( i \neq j; \ i, j \in \{0, 1, \ldots, m\} \) and by Lemma 3 we get that \( s = 1 \) and

either \( e_{A_{m+1}} = e_{A_1} e_{A_2} \cdots e_{A_m} \) or \( e_{A_{m+1}} = -e_{A_1} e_{A_2} \cdots e_{A_m}. \)

We shall prove that \( m \equiv 2 \pmod{4}. \) From \( e_{A_i} \bar{e}_{A_i} + e_{A_j} \bar{e}_{A_i} = 0 \) for \( i \neq j; \ i, j \in \{0, 1, \ldots, m\} \) one gets
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c_{A_i} + \overline{c}_{A_i} = 0, \ i \in \{1, 2, \ldots, m\}. Hence either \#A_i = 4p_i + 1 or \#A_i = 4p_i + 2 (p_i \in \mathbb{N}), i \in \{1, 2, \ldots, m\}. So \ c_{A_i} c_{A_i} = -c_0 \ (i = 1, 2, \ldots, m).

Let \ m \equiv 0 \ (\text{mod } 4) \ or \ m \equiv 3 \ (\text{mod } 4). Choosing \ a = c_0 + c_{A_{m+1}} \ and \ b = c_0 - c_{A_{m+1}} \ we \ find

\begin{align*}
ab &= c_0 + c_{A_{m+1}} - c_{A_{m+1}} - c_{A_{m+1}} c_{A_{m+1}} = c_0 - c_{A_1} \cdots c_{A_m} c_{A_1} \cdots c_{A_m} \\
&= c_0 - \left( (-1)^m (-1)^{\frac{m(m-1)}{2}} c_0 \right) = c_0 - (-1)^{\frac{m(m+1)}{2}} c_0 = c_0 - c_0 = 0.
\end{align*}

Hence the non-zero numbers \ a \ and \ b \ are \ not \ invertible.

Let \ m \equiv 1 \ (\text{mod } 4). Choosing \ a = c_{A_1} + c_{A_{m+1}} \ and \ b = c_{A_1} - c_{A_{m+1}} \ we \ get

\begin{align*}
ab &= (c_{A_1} + c_{A_{m+1}}) (c_{A_1} - c_{A_{m+1}}) \\
&= c_{A_1} c_{A_1} - c_{A_1} c_{A_{m+1}} + c_{A_{m+1}} c_{A_1} - c_{A_{m+1}} c_{A_{m+1}} \\
&= -c_0 - c_{A_1} c_{A_2} \cdots c_{A_m} + c_{A_1} c_{A_2} \cdots c_{A_m} c_{A_1} - (-1)^{\frac{m(m+1)}{2}} c_0 \\
&= c_{A_2} \cdots c_{A_m} (-1)^{m-1} c_{A_1} c_{A_2} \cdots c_{A_m} = c_{A_2} \cdots c_{A_m} - c_{A_2} \cdots c_{A_m} = 0.
\end{align*}

Hence \ a \ and \ b \ are not invertible. So \ m \equiv 2 \ (\text{mod } 4). The theorem is proved.

References


Nguyen Canh Luong
Department of Mathematics
University of Technology of Hanoi
Hanoi, F105-Nha C14, 1 Dai Co Viet
Vietnam
e-mail: nchuong@hn.vnn.vn