REMARKS ON UNIFORM DENSITY OF SETS OF INTEGERS

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Dedicated to the memory of Professor Péter Kiss

Abstract. The concept of the uniform density is introduced in papers [1], [2]. Some properties of this concept are studied in this paper. It is proved here that the uniform density has the Darboux property.

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Introduction

Let \( A \subseteq \mathbb{Z} = \{1, 2, 3, \ldots \} \) and \( m, n \in \mathbb{N}, m < n \). Denote by \( A(m, n) \) the cardinality of the set \( A \cap \mathbb{N} \cap \mathbb{Z} \). The numbers

\[
\underline{d}(A) = \lim_{n \to \infty} \frac{A(1, n)}{n}, \quad \overline{d}(A) = \lim_{n \to \infty} \frac{A(1, n)}{n}
\]

are called the lower and the upper asymptotic density of the set \( A \). If there exists

\[
d(A) = \lim_{n \to \infty} \frac{A(1, n)}{n}
\]

then it is called the asymptotic density of \( A \).

According to [1], [2] we set

\[
\alpha_s = \min_{t \geq 0} A(t + 1, t + s), \quad \alpha^s = \max_{t \geq 0} A(t + 1, t + s).
\]

Then there exist

\[
\underline{u}(A) = \lim_{s \to \infty} \frac{\alpha_s}{s}, \quad \overline{u}(A) = \lim_{s \to \infty} \frac{\alpha^s}{s}
\]

and they are called the lower and the upper uniform density of \( A \), respectively.
It is obvious that for every $A \subseteq N$

$$u(A) \leq d(A) \leq \overline{d}(A) \leq \underline{u}(A).$$

Hence if $u(A)$ exists then $d(A)$ exists as well and $u(A) = d(A)$. The converse is not true. For example put

$$A = \bigcup_{k=1}^{\infty} \{10^k + 1, 10^k + 2, \ldots, 10^k + k\}.$$ 

Then $d(A) = 0$, but $\underline{u}(A) = 0$, $\overline{u}(A) = 1$.

Note that the numbers $\alpha_s$ and $\alpha^s$ can be replaced by the numbers $\beta_s$ and $\beta^s$, respectively, where

$$\beta_s = \lim_{t \to \infty} A(t + 1, t + s), \quad \beta^s = \lim_{t \to \infty} A(t + 1, t + s)$$

(cf. [1], [2]).

In this paper we introduce some elementary remarks, observations on the concept of the uniform density and prove that this density has the Darboux property.

1. **Uniform density $u(A)$ and $\lim_{s \to \infty} \frac{A(t + 1, t + s)}{s}$ (uniformly with respect to $t \geq 0$)**

We introduce the following observation.

**Theorem 1.1.** If there exists

$$\lim_{s \to \infty} \frac{A(t + 1, t + s)}{s} = L$$

uniformly with respect to $t \geq 0$, then there exists $u(A)$ and $u(A) = L$.

**Proof.** Let $\varepsilon > 0$. By the assumption there exists an $s_0 = s_0(\varepsilon) \in N$ such that for each $s > s_0$ and each $t \geq 0$ we have

$$(L - \varepsilon)s < A(t + 1, t + s) < (L + \varepsilon)s.$$ 

By the definition of the numbers $\beta_s, \beta^s$ we get from this for $s > s_0$

$$L - \varepsilon \leq \frac{\beta_s}{s} \leq \frac{\beta^s}{s} \leq L + \varepsilon.$$
If \( s \to \infty \) we get
\[
L - \varepsilon \leq u(A) \leq \bar{u}(A) \leq L + \varepsilon.
\]
Since \( \varepsilon > 0 \) is an arbitrary positive number, we get \( u(A) = L \).

The foregoing theorem can be conversed.

**Theorem 1.2.** If there exists \( u(A) \) then
\[
\lim_{s \to \infty} \frac{A(t + 1, t + s)}{s} = u(A)
\]
uniformly with respect to \( t \geq 0 \).

**Proof.** Put \( u(A) = L \). Since
\[
L = \lim_{p \to \infty} \frac{\alpha_p}{p} = \lim_{p \to \infty} \frac{\alpha^p}{p}
\]
for every \( \varepsilon > 0 \), there exists a \( p_0 \) such that for each \( p > p_0 \) we have
\[
(L - \varepsilon)p < \alpha_p \leq \alpha^p < (L + \varepsilon)p.
\]
So we get
\[
(L - \varepsilon)p < \min_{t \geq 0} A(t + 1, t + p) \leq \max_{t \geq 0} A(t + 1, t + p) < (L + \varepsilon)p.
\]
By the definition of \( A(t + 1, t + p) \) we get from this
\[
\left| \frac{A(t + 1, t + p)}{p} - L \right| \leq \varepsilon
\]
for each \( p > p_0 \) and each \( t \geq 0 \). Hence
\[
\lim_{p \to \infty} \frac{A(t + 1, t + p)}{p} = L \quad (= u(A))
\]
uniformly with respect to \( t \geq 0 \).

2. Uniform density and almost convergence

The concept of almost convergence was introduced in [5] (see also [10], p. 60).
A sequence \( (x_n)_{n=1}^{\infty} \) of real numbers almost converges to \( L \) if
\[
\lim_{p \to \infty} \frac{x_{n+1} + x_{n+2} + \cdots + x_{n+p}}{p} = L
\]
uniformly with respect to \( n \geq 0 \). If \((x_n)_{1}^{\infty}\) almost converges to \( L \), we write

\[
F - \lim x_n = L. 
\]

One can conjecture that there is a relationship between the uniform density of a set \( A \subseteq N \) and the characteristic function \( \chi_{A} \) of this set \((\chi_{A}(n) = 1 \text{ if } n \in A, \chi_{A}(n) = 0 \text{ if } n \in N \setminus A)\).

**Theorem 2.1.** Let \( A \subseteq N \). Then \( u(A) = v \) if and only if \( F - \lim \chi_{A}(n) = v \).

**Proof.** Let \( t \geq 0, s \in N \). By the definition of the sequence \((\chi_{A}(n))_{1}^{\infty}\) we see that

\[
\frac{A(t+1, t+s)}{s} = \chi_{A}(t+1) + \chi_{A}(t+2) + \cdots + \chi_{A}(t+s) - t.
\]

The assertion follows from this equality by Theorem 1.1 and 1.2.

3. Another way for defining the uniform density of sets

If \( A = \{a_1 < a_2 < \cdots < a_n < \cdots\} \subseteq N \) is an infinite set then it is well-known that

\[
d(A) = \lim_{n \to \infty} \frac{n}{a_n}, \quad \overline{d}(A) = \lim_{n \to \infty} \frac{n}{a_n}
\]

and

\[
d(A) = \lim_{n \to \infty} \frac{n}{a_n}
\]

(if \( d(A) \) exists) (cf. [8], p. 247). A similar result can be stated also for the uniform density.

**Theorem 3.1.** Let \( A = \{a_1 < a_2 < \cdots < a_n < \cdots\} \subseteq N \) be an infinite set. Then \( u(A) = L \) if and only if

\[
\lim_{p \to \infty} \frac{p}{a_{k+p} - a_{k+1}} = L
\]

uniformly with respect to \( k \geq 0 \).

**Proof.** 1. Let \( u(A) = L \). Consider that for \( p \geq 2 \)

\[
\frac{p}{a_{k+p} - a_{k+1}} = \frac{A(a_{k+1}, a_{k+p})}{a_{k+p} - a_{k+1}}.
\]

By Theorem 1.2 (see (1)) the right-hand side converges by \( p \to \infty \) (uniformly with respect to \( k \geq 0 \)) to \( u(A) = L \). Hence (2) holds.

2. Suppose that (2) holds (uniformly with respect to \( k \geq 0 \)). By Theorem 1.1 it suffices to prove that

\[
\lim_{p \to \infty} \frac{A(t+1, t+p)}{p} = L
\]
uniformly with respect to \( t \geq 0 \).

We shall show it. Suppose in the first place that \( t \geq a_1 \). Then there exist

\[
a_k < t + 1 \leq a_{k+1} < \cdots < a_{k+s} \leq t + p < a_{k+s+1}.
\]

Then \( A(t + 1, t + p) \) equals to \( s \) and so

\[
\frac{A(t + 1, t + p)}{p} = \frac{s}{p}.
\]

Further on the basis of choice of the numbers \( k, s \) we get

\[
a_{k+s} - a_{k+1} \leq p - 1 < a_{k+s+1} - a_k.
\]

Therefore

\[
\frac{s}{a_{k+s+1} - a_k + 1} < \frac{A(t + 1, t + p)}{p} < \frac{s}{a_{k+s} - a_k + 1}.
\]

But \( -a_k + 1 \leq -a_{k-1} \), so that

\[
\frac{s}{a_{k+s+1} - a_k + 1} \geq \frac{s}{a_{k+s+1} - a_{k-1}} = \frac{s + 3}{a_{k+s+1} - a_{k-1}} \frac{s}{s + 3} = \frac{s + 3}{a_{k+s+1} - a_{k-1}} \left( 1 - \frac{3}{s + 3} \right).
\]

So we get wholly

\[
(3) \quad \frac{s + 3}{a_{k+s+1} - a_{k-1}} \left( 1 - \frac{3}{s + 3} \right) < \frac{A(t + 1, t + p)}{p} < \frac{s}{a_{k+s} - a_k + 1}.
\]

Let \( \gamma > 0 \). Then by assumption (see (2)) there exists a \( v_0 \) such that for each \( v > v_0 \) we have

\[
(4) \quad -\gamma < \frac{v}{a_{k+v} - a_{k+1}} - L < \gamma
\]

for all \( k \geq 0 \).

Using (4) we get from (3)

\[
(5) \quad \frac{s + 3}{a_{k+s+1} - a_{k-1}} - L - \frac{3}{a_{k+s+1} - a_{k-1}} \leq \frac{A(t + 1, t + p)}{p} - L \leq \frac{s}{a_{k+s} - a_k + 1} - L.
\]

Let \( s > v_0 \). Then by (4) the right-hand side of (5) is less than \( \gamma \). On the left-hand side we get

\[
\frac{s + 3}{a_{k+s+1} - a_{k-1}} - L > -\gamma.
\]
Further
\[ \frac{-3}{a_{k+1} - a_{k-1}} \geq \frac{-3}{s + 2}, \]

since
\[ a_{k+1} - a_{k-1} = (a_k - a_{k-1})+(a_{k+1} - a_k) + \cdots + (a_{k+1} - a_{k+s}) \]

and each summand on the right-hand side is \( \geq 1 \).

Hence for every \( t \geq a_1 \) we get from (5) \( (s > v_0) \)

\[ -\gamma - \frac{3}{s + 2} < \frac{A(t+1, t+p)}{p} - L < \gamma \]  

From this
\[ \lim_{p \to \infty} \frac{A(t+1, t+p)}{p} = L \]

uniformly with respect to \( t \geq a_1 \).

It remains the case if \( 0 \leq t < a_1 \). Since there is only a finite number of such \( t \)'s, it suffices to show that for each fixed \( t \), \( 0 \leq t < a_1 \), we have

\[ \lim_{p \to \infty} \frac{A(t+1, t+p)}{p} = L. \]  

If \( t \) is fixed, \( 0 \leq t < a_1 \) and \( p \) is sufficiently large we can determine a \( k \) such that \( a_k \leq t + p < a_{k+1} \). Then
\[ 0 \leq t < a_1 < a_2 < \cdots < a_k \leq t + p < a_{k+1} \]

and
\[ A(t+1, t+p) = A(t+1, a_1) + A(a_2, a_k). \]

From this
\[ p < a_{k+1}, \quad p > a_k - a_1 \]

and so from (8), (8') we obtain

\[ \frac{A(t+1, a_1)}{p} + \frac{A(a_2, a_{k+1}) - 1}{a_{k+1}} \leq \frac{A(t+1, t+p)}{p} \]

\[ \leq \frac{A(t+1, a_1)}{p} + \frac{k - 1}{a_k - a_1}. \]
Obviously we have $A(t+1, a_1) \leq a_1$ and so
\[
\frac{A(t+1, a_1)}{p} = o(1) \quad (p \to \infty).
\]

We arrange the left-hand side of (9). We get
\[
\frac{A(a_2, a_{k+1}) - 1}{a_{k+1}} = -\frac{1}{a_{k+1}} + \frac{k}{a_{k+1} - a_2} \frac{a_{k+1} - a_2}{a_{k+1}} = o(1) + \frac{k}{a_{k+1} - a_2}
\]
(if $p \to \infty$ then $k \to \infty$, as well).

Wholly we have
\[
\frac{k}{a_{k+1} - a_2} + o(1) \leq \frac{A(t+1, t+p)}{p} \leq \frac{k - 1}{a_k - a_1} + o(1).
\]

If $p \to \infty$, then $k \to \infty$ and by assumption (cf (2)) the terms
\[
\frac{k - 1}{a_k - a_1} - L, \quad \frac{k}{a_{k+1} - a_2} - L
\]
converge to zero. But then (9) yields
\[
\lim_{p \to \infty} \frac{A(t+1, t+p)}{p} = L
\]
uniformly with respect to $t \geq 0$. So $u(A) = L$.

The following theorem is a simple consequence of Theorem 3.1

**Theorem 3.2.** Let $A = \{a_1 < a_2 < \cdots \} \subseteq \mathbb{N}$ be a lacunary set, i.e.
\[
(10) \quad \lim_{n \to \infty} (a_{n+1} - a_n) = +\infty.
\]

Then $u(A) = 0$.

**Proof.** Let $\varepsilon > 0$. Choose $M \in \mathbb{N}$ such that $M^{-1} < \varepsilon$. By the assumption there exists an $n_0$ such that for each $n > n_0$ we get $a_{n+1} - a_n > M$.

Let $k > n_0$, $s \in \mathbb{N}$, $s > 1$. Then
\[
a_{k+s} - a_{k+1} = (a_{k+2} - a_{k+1}) + (a_{k+3} - a_{k+2}) + \cdots + (a_{k+s} - a_{k+s-1}) > (s-1)M
\]
and so
\[
\frac{s}{a_{k+s} - a_{k+1}} < \frac{1}{(s-1)M} < 2\varepsilon.
\]
Hence for each \( k > n_0 \) and \( s \geq 2 \) we have
\[
\frac{s}{a_{k+s} - a_{k+1}} < 2\varepsilon.
\]

If \( 0 \leq k \leq n_0, \) \( k \) is fixed, then
\[
\lim_{s \to \infty} \frac{s}{a_{k+s} - a_{k+1}} = 0,
\]
(11) since, for sufficiently large \( s \)

\[
a_{k+s} - a_{k+1} = [(a_{k+2} - a_{k+1}) + \cdots + (a_{n_0+1} - a_{n_0})] \\
+ [(a_{n_0+2} - a_{n_0+1}) + \cdots + (a_{k+s} - a_{k+s-1})] > M(k + s - n_0 - 1) \\
\geq M(s - (n_0 + 1)).
\]

There exists only a finite number of \( k \)'s with \( 0 \leq k \leq n_0, \) so we see that (11) holds uniformly with respect to \( k, \) \( 0 \leq k \leq n_0. \) So we get wholly
\[
\lim_{s \to \infty} \frac{s}{a_{k+s} - a_{k+1}} = 0
\]
uniformly with respect to \( k \geq 0. \) So according to Theorem 3.1, \( u(A) = 0. \)

**Remark.** The assumption (10) in Theorem 3.2 cannot be replaced by the weaker assumption
\[
(10') \quad \lim_{n \to \infty} (a_{n+1} - a_n) = +\infty.
\]

This can be shown by the following example:

\[
A = \bigcup_{k=1}^{\infty} \{k! + 1, k! + 2, \ldots, k! + k\} = \{a_1 < a_2 < \cdots < a_n < \cdots\}.
\]

Here we have \( g(A) = 0, \) \( \bar{u}(A) = 1 \) and (10') is satisfied.

**Example 3.1** Let \( \alpha \in R, \) \( \alpha > 1. \) Put \( a_k = [\alpha k], \) \( (k = 1, 2, \ldots) \), where \( [x] \) denotes the integer part of \( x. \) We show that the uniform density of the set \( A \) is \( \frac{1}{\alpha}. \) This follows from Theorem 3.1, since
\[
\lim_{p \to \infty} \frac{p}{a_{k+p} - a_{k+1}} = \frac{1}{\alpha}
\]
uniformly with respect to \( k \geq 0 \). This uniform convergence can be shown by a simple calculation which gives the estimates (\( p \geq 2 \))
\[
\frac{p}{(p-1)\alpha + 1} \leq \frac{p}{a_{k+p} - a_{k+1}} \leq \frac{p}{(p-1)\alpha - 1}.
\]

4. Darboux property of the uniform density

For every \( A \subseteq N \) having the uniform density the number \( u(A) \) belongs to \([0, 1]\). The natural question arises whether also conversely for every \( t \in [0, 1] \) there is a set \( A \subseteq N \) such that \( u(A) = t \). The answer to this question is positive.

Theorem 4.1.
If \( t \in [0, 1] \) then there is a set \( A \subseteq N \) with \( u(A) = t \).

Proof. We can already suppose that \( 0 < t < 1 \). Construct the set
\[
A = \left\{ \left\lceil \frac{1}{t} \right\rceil, \left\lceil \frac{2}{t} \right\rceil, \ldots, \left\lceil \frac{k}{t} \right\rceil, \ldots \right\} = \{ a_1 < a_2 < \cdots \}.
\]
Put \( a_k = \left\lceil \frac{k}{t} \right\rceil (k = 1, 2, \ldots) \) and set in Example 3.1 \( \alpha = \frac{1}{t} > 1 \). So we get
\[
\lim_{p \to \infty} \frac{p}{a_{k+p} - a_{k+1}} = \frac{1}{\alpha} = t
\]
uniformly with respect to \( k \geq 0 \). The assertion follows by Theorem 3.1.

Let \( v \) be a non-negative set function defined on a class \( S \subseteq 2^N \). The function \( v \) is said to have the Darboux property provided that if \( v(A) > 0 \) for \( A \in S \) and \( 0 < t < v(A) \), then there is a set \( B \subseteq A, B \in S \) such that \( v(B) = t \) (cf. [6], [7], [9]).

Theorem 4.2. The uniform density has the Darboux property.

Proof. Let \( u(A) = \delta > 0 \),
\[
A = \{ a_1 < a_2 < \cdots < a_k < \cdots \}
\]
and \( 0 < t < \delta \). Construct the set
\[
B = \{ b_1 < b_2 < \cdots < b_k < \cdots \}
\]
in such a way that we set
\[
b_k = a_{\left\lceil \frac{k}{t} \right\rceil} \quad (k = 1, 2, \ldots).
\]
Put \( n_k = \lceil k^\alpha \rceil \) \((k = 1, 2, \ldots)\). Then \( n_1 < n_2 < \cdots < n_k < \cdots \),

\[ B = \{ a_{n_1} < a_{n_2} < \cdots < a_{n_k} < \cdots \}, \quad B \subseteq A. \]

We prove that \( u(B) = t \).

By Theorem 3.1 it suffices to show that

\[ \lim_{p \to \infty} \frac{p}{b_{m+p} - b_{m+1}} = t \]

uniformly with respect to \( m \geq 0 \).

We have \((p > 1)\)

\[ \frac{p}{b_{m+p} - b_{m+1}} = \frac{p}{a_{n_{m+p}} - a_{n_{m+1}}}. \]

By a simple arrangement we get

\[ \frac{p}{b_{m+p} - b_{m+1}} = \frac{p}{a_{n_{m+p}} - a_{n_{m+1}} + 1} \]

A simple estimation gives

\((p - 1) \frac{\delta}{t} - 1 < n_{m+p} - n_{m+1} < (p - 1) \frac{\delta}{t} + 1.\)

Using this in (13) we get

\[ \lim_{p \to \infty} \frac{p}{n_{m+p} - n_{m+1} + 1} = \frac{t}{\delta} \]

uniformly with respect to \( m \geq 0 \).

Further by assumption

\[ \lim_{p \to \infty} \frac{p}{a_{s+p} - a_{s+1}} = \delta \]

uniformly with respect to \( s \geq 0 \) (Theorem 3.1).

So we get

\[ \lim_{p \to \infty} \frac{n_{m+p} - n_{m+1} + 1}{a_{n_{m+p}} - a_{n_{m+1}}} = \delta \]

uniformly with respect to \( m \geq 0 \) since the sequence

\[ \left( \frac{n_{m+p} - n_{m+1} + 1}{a_{n_{m+p}} - a_{n_{m+1}}} \right)^\infty_{p=2} \]
is a subsequence of the sequence
\[
\left( p \frac{a_{x+p} - a_{x+1}}{a_{x+p} - a_{x+1}} \right)_{p=1}^{\infty}.
\]

By (13), (14), (15) we get (12) uniformly with respect to \( m \geq 0 \).

References


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