SECOND ORDER LINEAR RECURRENCES
AND PELL’S EQUATIONS OF HIGHER DEGREE

Ferenc Mátyás (Eger, Hungary)

Dedicated to the memory of Professor Péter Kiss

Abstract. In this note solutions are given to an infinite family of Pell’s equations of degree \( n \geq 2 \) based on second order linear recursive sequences of integers.

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1. Introduction

Let \( A \) and \( B \) be non-zero integers. The second order linear recursive sequences \( R = \{ R_n \}_{n=0}^{\infty} \) and \( V = \{ V_n \}_{n=0}^{\infty} \) are defined by the recursions

\[
R_n = AR_{n-1} + BR_{n-2} \quad \text{and} \quad V_n = AV_{n-1} + BV_{n-2},
\]

for \( n \geq 2 \), while \( R_0 = 0, R_1 = 1, V_0 = 2 \) and \( V_1 = A \). If \( A = B = 1 \) then \( R_n = F_n \) and \( V_n = L_n \), where \( F_n \) and \( L_n \) denote the \( n^{th} \) Fibonacci and Lucas numbers, respectively.

The polynomial \( g(x) = x^2 - Ax - B \) is said to be the characteristic polynomial of the sequences \( R \) and \( V \), the complex numbers \( \alpha \) and \( \beta \) are the roots of \( g(x) = 0 \). In this note we suppose that \( \alpha^2 + 4B \neq 0 \), i.e. \( \alpha \neq \beta \). Then, by the well-known Binet formulae, for \( n \geq 0 \)

\[
R_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n.
\]

The classical Pell’s equation \( x^2 - dy^2 = \pm 1 \) \((d \in \mathbb{Z})\) can be rewritten as

\[
\det \begin{pmatrix} x & dy \\ y & x \end{pmatrix} = \pm 1.
\]

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To generalize this Lin Dazheng [1] investigated the quasi-cyclic matrix

$$
\begin{pmatrix}
x_1 & dx_n & dx_{n-1} & \cdots & dx_2 \\
x_2 & x_1 & dx_n & \cdots & dx_3 \\
x_3 & x_2 & x_1 & \cdots & dx_4 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
x_n & x_{n-1} & x_{n-2} & \cdots & x_1
\end{pmatrix}
$$

(3) \quad C_n = C_n(d; x_1, x_2, \ldots, x_n)

i.e. every entry of the upper triangular part (not including the main diagonal) of the cyclic matrix of entries $x_1, x_2, \ldots, x_n$ is multiplied by $d$. The equation

$$
\det (C_n) = \pm 1
$$

is called Pell’s equation of degree $n \geq 2$. For example, if $n = 3$ then (4) has the form

$$
x_1^2 + dx_2^2 + d^2 x_3^2 - 3dx_1 x_2 x_3 = \pm 1.
$$

Lin Dazheng [1] proved that $\det (C_n (L_n; F_{2n-1}, F_{2n-2}, \ldots, F_n)) = 1$, i.e. if $d = L_n$ then $(x_1, x_2, \ldots, x_n) = (F_{2n-1}, F_{2n-2}, \ldots, F_n)$ is a solution of (4). The aim of this paper is to extend and generalize this result for more general sequences defined by (1) with $A^2 + 4B \neq 0$. In the proofs of our theorems we’ll apply the methods and algorithms developed and presented in [1] by Lin Dazheng.

2. Results

Using (1) with $A^2 + 4B \neq 0$ and (3), we can state our results.

**Theorem 1.** For $n \geq 2$

$$
\det (C_n (V_n; R_{2n-1}, R_{2n-2}, \ldots, R_n)) = B^{n(n-1)},
$$

i.e. $(x_1, x_2, \ldots, x_n) = (R_{2n-1}, R_{2n-2}, \ldots, R_n)$ is a solution of the generalized Pell’s equation of degree $n$

$$
\det (C_n (V_n; x_1, x_2, \ldots, x_n)) = B^{n(n-1)}.
$$

**Corollary 1.** For $n \geq 2$

$$
\prod_{k=0}^{n-1} \left( \sum_{j=1}^{n} R_{2n-j} \left( \sqrt[n]{V_n} \right)^{j-1} e^{k(j-1)} \right) = B^{n(n-1)},
$$

where $\sqrt[n]{V_n}$ denotes a fixed $n$th complex root of $V_n$ and $\varepsilon = e^{2\pi i/n}$. 

It is known from [3] that the inverse of a quasi-cyclic matrix is quasi-cyclic. In
our case we can prove the following result, too.

**Theorem 2.** For \( n \geq 3 \) the matrix \( C_n (V_n; R_{2n-1}, R_{2n-2}, \ldots, R_n) \) is invertible and
its inverse matrix \( C_n^{-1} \) is as follows:

\[
C_n^{-1} (V_n; R_{2n-1}, R_{2n-2}, \ldots, R_n) = (-1)^{n-1} B^{-n} (BL_n + AE_n - E_n^2),
\]

where \( L_n \) and \( E_n \) denotes the identity matrix of order \( n \) and the \( n \) by \( n \) matrix

\[
E_n = \begin{pmatrix}
0 & 0 & \cdots & 0 & V_n \\
1 & 0 & \cdots & 0 & 0 \\
& 1 & 0 & \cdots & 0 \\
& & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & 1 & 0
\end{pmatrix},
\]

respectively.

**Remark.** Naturally, if \( |B| \neq 1 \) then the entries of the matrix

\[
C_n^{-1} (V_n; R_{2n-1}, R_{2n-2}, \ldots, R_n)
\]

are not integers.

**Corollary 2.**

\[
(x_1, x_2, \ldots, x_n) = \begin{cases}
(1, A, -1, 0, \ldots, 0), & \text{if } n \geq 3 \text{ odd and } B = 1, \\
(1, -A, 1, 0, \ldots, 0), & \text{if } n \geq 3 \text{ odd and } B = -1, \\
(-1, -A, 1, 0, \ldots, 0), & \text{if } n \geq 4 \text{ even and } B = 1, \\
(1, -A, 1, 0, \ldots, 0), & \text{if } n \geq 4 \text{ even and } B = -1
\end{cases}
\]

is another solution of the generalized Pell's equation

\[
\det (C_n (V_n; x_1, x_2, \ldots, x_n)) = 1.
\]

3. Proofs

To prove our theorems we need the following

**Lemma.** Let the sequences \( R \) and \( V \) be defined by (1) and we suppose that \( \alpha \neq \beta \)
in (2). Then

\[
R_{n+1} R_{n-1} - R_n^2 = (-1)^n B^{n-1} \ (n \geq 1),
\]

(7/1)
**(7/2)** \( R_n V_n = R_{2n} \) \((n \geq 0)\),

**(7/3)** \( V_n R_{n+1} = R_{2n+1} + (-B)^n \) \((n \geq 0)\),

**(7/4)** \( E_n^n = V_n L_n \) and \( E_n^{n+1} = V_n E_n \) \((n \geq 3)\),

where \( E_n \) is defined by \((5)\).

**Proof.** The first three properties of the Lemma are known or, using \((2)\), they can be proven easily. For the proof of \((7/4)\) consider the multiplication of matrices. For example:

\[
E_n^2 = E_n \cdot E_n = \begin{pmatrix}
0 & 0 & \ldots & 0 & V_n & 0 \\
0 & 0 & \ldots & 0 & 0 & V_n \\
1 & 0 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & 0 & 0
\end{pmatrix}
\]

\[
E_n^3 = E_n^2 \cdot E_n = \begin{pmatrix}
0 & 0 & \ldots & 0 & V_n & 0 & 0 \\
0 & 0 & \ldots & 0 & 0 & V_n & 0 \\
0 & 0 & \ldots & 0 & 0 & 0 & V_n \\
1 & 0 & \ldots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & 0 & 0 & 0
\end{pmatrix}
\]

\[
E_n^n = \begin{pmatrix}
V_n & 0 & \ldots & 0 & 0 \\
0 & V_n & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & V_n & 0 \\
0 & 0 & \ldots & 0 & V_n
\end{pmatrix} = V_n L_n
\]

and so \( E_n^{n+1} = E_n^n \cdot E_n = (V_n L_n) E_n = V_n E_n \).

**Proof of Theorem 1.** For \( n = 2 \) we get that

\[
\det \left( C_2(V_2; R_3, R_2) \right) = \begin{vmatrix}
A^2 + B & A^3 + 2AB \\
A & A^2 + B
\end{vmatrix} = B^2.
\]
If $n > 2$, let us consider the $n$ by $n$ matrices

$$
T_n = \begin{pmatrix}
1 & -A & -B & 0 & \ldots & 0 & 0 \\
0 & 1 & -A & -B & \ldots & 0 & 0 \\
0 & 0 & 1 & -A & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & -A & -B \\
0 & 0 & 0 & 0 & \ldots & 1 & -A \\
0 & 0 & 0 & 0 & \ldots & 0 & 1 \\
\end{pmatrix}
$$

and

$$
C_n = C_n(V_n, R_{2n-1}, R_{2n-2}, \ldots, R_n) = \begin{pmatrix}
R_{2n-1} & V_n R_n & \ldots & V_n R_{2n-2} \\
R_{2n-2} & R_{2n-1} & \ldots & V_n R_{2n-3} \\
\vdots & \vdots & \ddots & \vdots \\
R_n & R_{n+1} & \ldots & R_{2n-1} \\
\end{pmatrix}.
$$

Then, by (1), (2) and (7/1)-(7/3), one can verify that

$$
C_n T_n = \begin{pmatrix}
R_{2n-1} & B R_{2n-2} & (-B)^n & 0 & \ldots & 0 \\
R_{2n-2} & B R_{2n-3} & 0 & (-B)^n & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
R_{n+2} & B R_{n+1} & 0 & 0 & \ldots & (-B)^n \\
R_{n+1} & B R_n & 0 & 0 & \ldots & 0 \\
R_n & B R_{n-1} & 0 & 0 & \ldots & 0 \\
\end{pmatrix}.
$$

Developing the $\det(C_n T_n)$ we get that

$$
\det(C_n T_n) = (-1)^{2n+2} \det \left( \begin{array}{cc}
R_{n+1} & B R_n \\
R_n & B R_{n-1}
\end{array} \right) \det((-B)^n I_{n-1})
= B(R_{n+1} R_{n-1} - R_n^2) (-B)^n = B(-1)^n B^{n-1} (-B)^{n(n-2)}
= (-1)^{n(n-1)} B^{n(n-1)} = B^{n(n-1)}.
$$

But, since $\det(T_n) = 1$, $\det(C_n T_n) = \det(C_n) \cdot \det(T_n) = \det(C_n)$, therefore $\det(C_n) = B^{n(n-1)}$, i.e. Theorem 1 is true.

**Proof of Corollary 1.** In [2] it is proven that if $C_n$ is as in (3) then

$$
\det(C_n(d, x_1, x_2, \ldots, x_n)) = \prod_{k=0}^{n-1} \left( \sum_{j=1}^{n} x_j \left( \sqrt{d} \right)^{j-1} \varepsilon^{k(j-1)} \right),
$$

where $\varepsilon = e^{2\pi i / n}$. Substituting in (8)

$$
d = V_n \quad \text{and} \quad (x_1, x_2, \ldots, x_n) = (R_{2n-1}, R_{2n-2}, \ldots, R_n),
$$
by Theorem 1, the statement of Corollary 1 immediately yields.

**Proof of Theorem 2.** Theorem 1 implies that $C_{n}^{-1}(V_n; R_{2n-1}, R_{2n-2}, \ldots, R_n)$ exists. It is easily verifiable that

$$C_n(V_n; R_{2n-1}, R_{2n-2}, \ldots, R_n) = R_{2n-1}I_n + R_{2n-2}E_n + \cdots + R_nE_n^{n-1},$$

therefore we have to show that

$$(9) \quad (R_{2n-1}I_n + R_{2n-2}E_n + \cdots + R_nE_n^{n-1})(-1)^{n-1}B^{-n}(B\mathbf{I}_n + A\mathbf{E}_n - E_n^2) = I_n.$$

By (1), the left hand side of (9) can be written as

$$(10) \quad (-1)^{n-1}B^{-n}(R_{2n-1}B\mathbf{I}_n + R_{2n-2}B\mathbf{E}_n + R_{2n-1}A\mathbf{E}_n + R_nA\mathbf{E}_n^n - R_{n+1}E_n^n + \mathbf{O}_n + \cdots + \mathbf{O}_n),$$

where $\mathbf{O}_n$ is the zero-matrix of order $n$.

Thus, applying (1), (7/1) - (7/4) and (2), the form (10) is equal to

$$(-1)^{n-1}B^{-n}(R_{2n-1}B\mathbf{I}_n + (BR_{2n-2} + AR_{2n-1})\mathbf{E}_n + R_nA\mathbf{E}_n^n - R_{n+1}E_n^n + \mathbf{O}_n + \mathbf{O}_n)$$

$$= (-1)^{n-1}B^{-n}(R_{2n-1}B\mathbf{I}_n + (R_{2n-2} + AR_{2n-1})\mathbf{E}_n + R_nA\mathbf{E}_n^n - R_{n+1}E_n^n)$$

$$= (-1)^{n-1}B^{-n+1}(R_{2n-1}B\mathbf{I}_n + \mathbf{O}_n - V_nB\mathbf{I}_n - R_{n+1}I_n$$

$$= (-1)^{n-1}B^{-n+1}(-B)^{n-1}I_n = (-1)^{2n-2}B^nI_n = I_n,$$

which completes the proof of Theorem 2.

**Proof of Corollary 2.** By Theorem 2

$$\det(C_n(V_n; R_{2n-1}, R_{2n-2}, \ldots, R_n)) \cdot \det(C_n^{-1}(V_n; R_{2n-1}, R_{2n-2}, \ldots, R_n)) = 1$$

thus, if $|B| = 1$ then, by Theorem 1,

$$\det(C_n^{-1}(V_n; R_{2n-1}, R_{2n-2}, \ldots, R_n)) = 1.$$

E.g. let $n \geq 3$ be an odd integer and $B = 1$. Then, by Theorem 2,

$$C_n^{-1}(V_n; R_{2n-1}, R_{2n-2}, \ldots, R_n) = I_n + A\mathbf{E}_n - E_n^2$$

$$= \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & -V_n & AV_n \\
A & 1 & 0 & \cdots & 0 & 0 & -V_n \\
-1 & A & 1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -1 & A & 1
\end{pmatrix}.$$
i.e. \((x_1, x_2, \ldots, x_n) = (1, A, -1, 0, \ldots, 0)\) is a solution of (6).

The proof is similar when \(n \geq 3\) odd and \(B = -1\), or \(n \geq 4\) even and \(|B| = 1\).

References


Ferenc Mátyás
Department of Mathematics
Károly Eösterházy College
H-3301, Eger, P. O. Box 43,
Hungary
E-mail: matyas@ektf.hu