ON ADDITIVE FUNCTIONS SATISFYING CONGRUENCE PROPERTIES

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Dedicated to the memory of Professor Péter Kiss

Abstract. In this paper, we consider those integer-valued additive functions $f_1$ and $f_2$ for which the congruence $f_1(mn+b) \equiv f_2(cn)+d \pmod{n}$ is satisfied for all positive integers $n$ and for some fixed integers $a \geq 1$, $b \geq 1$, $c \geq 1$ and $d$. Our result improve some earlier results of K. Kovács, I. Joó, I. Joó & B. M. Phong and F. V. Chung concerning the above congruence.

1. Introduction

The problem concerning the characterization of some arithmetical functions by congruence properties initiated by Subbarao [10] was studied later by several authors. M. V. Subbarao proved that if an integer-valued multiplicative function $g(n)$ satisfies the congruence

$$g(n + m) \equiv g(m) \pmod{n}$$

for all positive integers $n$ and $m$, then there is a non-negative integer $\alpha$ such that $g(n) = n^\alpha$

holds for all positive integers $n$. Recently some authors generalized and improved this result in a variety of ways. A. Iványi [3] obtained that the same result holds when $m$ is a fixed positive integer and $g$ is an integer-valued completely multiplicative function. For further results and generalizations of this problem we refer to the works of B. M. Phong [7]–[8], B. M. Phong & J. Fehér [9], I. Joó [4] and I. Joó & B. M. Phong [5]. For example, it follows from [8] that if an integer-valued multiplicative function $g(n)$ satisfies the congruence

$$g(An + B) \equiv C \pmod{n}$$

for all positive integers $n$ and for some fixed integers $A \geq 1$, $B \geq 1$ and $C \neq 0$ with $(A, B) = 1$, then there are a non-negative integer $\alpha$ and a real-valued Dirichlet character $\chi_A \pmod{A}$ such that

$$g(n) = \chi_A(n)n^\alpha$$

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holds for all positive integers $n$ which are prime to $A$.

In the following let $\mathcal{A}$ and $\mathcal{A}^*$ denote the set of all integer-valued additive and completely additive functions, respectively. Let $\mathbb{N}$ denote the set of all positive integers. A similar problem concerning the characterisation of a zero-function as an integer-valued additive function satisfying a congruence property have been studied by K. Kovács [6], P. V. Chung [1]–[2], I. Joó [4] and I. Joó & B. M. Phong [5]. It was proved by K. Kovács [6] that if $f \in \mathcal{A}^*$ satisfies the congruence

$$f(Au + B) \equiv C \pmod{n}$$

for some integers $A \geq 1$, $B \geq 1$, $C$ and for all $n \in \mathbb{N}$, then

$$f(n) = 0$$

holds for all $n \in \mathbb{N}$ which are prime to $A$. This result was extended in [1], [2], [4] and [5] for integer-valued additive functions $f$. It follows from the results of [2] and [4] that for integers $A \geq 1$, $B \geq 1$, $C$ and functions $f_1 \in \mathcal{A}$, $f_2 \in \mathcal{A}^*$ the congruence

$$f_1(Au + B) \equiv f_2(n) + C \pmod{n} \quad (\forall n \in \mathbb{N})$$

implies that $f_2(n) = 0$ for all $n \in \mathbb{N}$ and $f_1(n) = 0$ for all $n \in \mathbb{N}$ which are prime to $A$.

Our purpose in this paper is to improve the above results by showing the following

**Theorem 1.** Assume that $a \geq 1$, $b \geq 1$, $c \geq 1$ and $d$ are fixed integers and the functions $f_1$, $f_2$ are additive. Then the congruence

(1) \hspace{1cm} f_1(an + b) \equiv f_2(cn) + d \pmod{n}

is satisfied for all $n \in \mathbb{N}$ if and only if the equation

(2) \hspace{1cm} f_1(an + b) = f_2(cn) + d

holds for all $n \in \mathbb{N}$.

**Theorem 2.** Assume that $a \geq 1$, $b \geq 1$, $c \geq 1$ and $d$ are fixed integers. Let $a_1 = \frac{a}{(a, b)}$, $b_1 = \frac{b}{(a, b)}$ and

$$\mu := \begin{cases} 1 & \text{if } 2 \mid a_1 b_1 \\ 2 & \text{if } 2 \nmid a_1 b_1. \end{cases}$$

If the additive functions $f_1$ and $f_2$ satisfy the equation (2) for all $n \in \mathbb{N}$, then

$$f_1(n) = 0 \quad \text{for all } n \in \mathbb{N}, \quad (n, \mu a_1 b_1) = 1$$
and
\[ f_2(n) = 0 \quad \text{for all} \quad n \in \mathbb{N}, \quad (n, \mu c_1) = 1. \]

2. Lemmas

**Lemma 1.** Assume that \( f^* \in A^* \) satisfies the congruence
\[ f^*(An + B) \equiv f^*(n) + D \pmod{n} \]
for some fixed integers \( A \geq 1, B \geq 1 \) and \( D \). Then \( f^*(n) = 0 \) holds for all \( n \in \mathbb{N} \).

**Proof.** Lemma 1 follows from Theorem 2 of [4].

**Lemma 2.** Assume that \( f \in A \) satisfies the congruence
\[ f(An + B) \equiv D \pmod{n} \]
for some fixed integers \( A \geq 1, B \geq 1 \) and \( D \). Then \( f(n) = 0 \) holds for all \( n \in \mathbb{N} \) which are prime to \( A \).

**Proof.** This is the result of [1].

**Lemma 3.** Assume that \( f_1, f \in A \) satisfy the congruence
\[ f_1(An + 1) \equiv f(Cn) + D \pmod{n} \]
holds for all \( n \in \mathbb{N} \) with some integers \( A \geq 1, C \geq 1 \) and \( D \). Then
\[ f(n) = f\left(\lceil n, 6C^2 \rceil\right) \quad \text{for all} \quad n \in \mathbb{N} \]
and \( f_1(m) = 0 \) holds for all \( m \in \mathbb{N} \), which are prime to \( 6AC \). Here \( (x, y) \) denotes the greatest common divisor of the integers \( x \) and \( y \).

**Proof.** In the following we shall denote by \( n^* \) the product of all distinct prime divisors of positive integer \( n \).

For each positive integer \( M \) let \( P = P(M) \) be a positive integer for which
\[ (M^2 - 1)^* | ACP. \]

It is obvious from (4) that
\[ (ACM(M + 1)Pn + 1, ACM(M + 1)Pn + 1) = 1, \]
\[ (C^2(M + 1)^2 Pn, ACM Pn + 1) = 1 \]
and

\[(AC(M + 1)Pn + 1)(AC(M + 1)Pn + 1) = AC(M + 1)^2 Pn[ACMPn + 1] + 1\]

hold for all \(n \in \mathbb{N}\). Using these relations and appealing to the additive nature of the functions \(f_1\) and \(f\), we can deduce from (3) that

\[(5) \quad f(ACMPn + 1) \equiv -f(C^2(M + 1)^2 Pn) + f(C^2(M + 1)Pn) + f(C^2(M + 1)Pn) + D \pmod{n}\]

is satisfied for all \(n, M \in \mathbb{N}\), where \(P = P(M)\) satisfies the condition (4).

Let \(M = 2, P(2) = 3\) and \(M = 3, P(3) = 2\). In these cases (4) is true and so it follows from (5) that

\[(6) \quad f(6ACn + 1) \equiv -f(27C^2n) + f(18C^2n) + f(9C^2n) + D \pmod{n}\]

and

\[(7) \quad f(6ACn + 1) \equiv -f(32C^2n) + f(24C^2n) + f(8C^2n) + D \pmod{n}\]

are satisfied for all \(n \in \mathbb{N}\). Let \(N\) and \(n\) be positive integers with the condition

\[(8) \quad (N(N + 1), 6ACn + 1) = 1.\]

By using the relation

\[(6ACn + 1)(6^2 A^2C^2 Nn^2 + 1) = 6ACn[6ACNn(6ACn + 1) + 1] + 1\]

and that

\[(6ACn + 1, 6^2 A^2C^2 Nn^2 + 1) = (6ACn + 1, N + 1) = 1,\]

\[(6ACNn, 6ACn + 1) = (6ACn + 1, N) = 1,\]

it follows from (6) and (7) that

\[(9) \quad -f(162AC^3Nn^2) + f(108AC^3Nn^2) + f(54AC^3Nn^2) \equiv -f(27C^2Nn) + f(18C^2Nn) + f(9C^2n) + f(9C^2n) + D \pmod{n}\]

and

\[(10) \quad -f(192AC^3Nn^2) + f(144AC^3Nn^2) + f(48AC^3Nn^2) \equiv -f(32C^2Nn) + f(24C^2Nn) + f(8C^2n) + f(8C^2n) + D \pmod{n}\]

hold for all \(n, N \in \mathbb{N}\) satisfying (8).
Let $Q$ be a fixed positive integer. First we apply (9) when $N = 1$, $n = Qm$, $(m, Q) = 1$ and $m \to \infty$. It is obvious that (8) holds, and so by (9) we have

\begin{equation}
(11) \quad f(Q^2) = 2f(Q) \quad \text{for} \quad Q \in \mathbb{N}, (Q, 6AC) = 1.
\end{equation}

Now let $N = Q$ and $n = Q^k(6CQm + 1)$ with $k, m \in \mathbb{N}$. It is obvious that (8) holds for infinity many integers $m$, because $(36AC^2Q^{k+1}, 6ACQ^k + 1) = 1$. These with (9) show that

\begin{equation}
(12) \quad f(Q^{2k+1}) = f(Q^k) + f(Q^{k+1}) \quad \text{for all} \quad Q \in \mathbb{N}, (Q, 6AC) = 1.
\end{equation}

From (11) and (12) we obtain that

\begin{equation}
(13) \quad f(Q^k) = kf(Q) \quad \text{for all} \quad Q \in \mathbb{N}, (Q, 6AC) = 1.
\end{equation}

Thus, by using the additivity of $f$ it follows from (8) and (13) that (9) and (10) hold for all $N, n \in \mathbb{N}$, and they with $n = Qm$, $(m, 6ACNQ) = 1, m \to \infty$ imply that

\begin{align*}
&-f(162AC^3NQ^2) + f(108AC^3NQ^2) + f(54AC^3NQ^2) = -f(27C^2NQ) \\
&+f(18C^2NQ) + f(9C^2NQ) - f(27C^2Q) + f(18C^2Q) + f(9C^2Q)D
\end{align*}

and

\begin{align*}
&-f(192AC^3NQ^2) + f(144AC^3NQ^2) + f(48AC^3NQ^2) = -f(32C^2NQ) \\
&+f(24C^2NQ) + f(8C^2NQ) - f(32C^2Q) + f(24C^2Q) + f(8C^2Q) + D
\end{align*}

hold for all $N, Q \in \mathbb{N}$. Consequently

\begin{equation}
(14) \quad f(27C^2NQ) = f(18C^2NQ) + f(9C^2NQ) - f(27C^2Q) + f(18C^2Q) + f(9C^2Q)
\end{equation}

\begin{equation}
(15) \quad f(32C^2NQ) = f(24C^2NQ) + f(8C^2NQ) - f(32C^2Q) + f(24C^2Q) + f(8C^2Q)
\end{equation}

are satisfied for all $N, Q \in \mathbb{N}$.

For each prime $p$ let $e = e(p)$ be a non-negative integer for which $p^e \| C^2$. 

First we consider the case when \((p, 6) = 1\). By applying (14) with \(Q = p, N = p^l (l \geq 0)\), we have

\[
f \left( p^{l+1}p^{e(p)+2} \right) - f \left( p^{l+1}p^{e(p)+1} \right) = f \left( p^{e(p)+1} \right) - f \left( p^{e(p)} \right) \quad \text{for all} \quad l \geq 0,
\]

which shows that for all integers \(\beta \geq e(p)\)

\[
f \left( p^{\beta+1} \right) - f \left( p^{\beta} \right) = f \left( p^{e(p)+1} \right) - f \left( p^{e(p)} \right).
\]

Now we consider the case \(p = 2\). Applying (14) with \(Q = 2, n = 2^l, (l \geq 0)\) one can check as above that

\[
f \left( 2^{2l+1} \right) - f \left( 2^l \right) = f \left( 2^{(2l+2)} \right) - f \left( 2^{(2l+1)} \right).
\]

Finally, we consider the case \(p = 3\). Applying (15) with \(Q = 3, N = 3^l, l \geq 0\) we also get

\[
f \left( 3^{3l+1} \right) - f \left( 3^l \right) = f \left( 3^{(3l+2)} \right) - f \left( 3^{(3l+1)} \right).
\]

Now we write

\[
f(n) = f^*(n) + F(n),
\]

where \(f^*\) is a completely additive function defined as follows:

\[
f^*(p) := \begin{cases} f \left( p^{e(p)+1} \right) - f \left( p^{e(p)} \right) & \text{for } (p, 6) = 1 \\ f \left( p^{e(p)+2} \right) - f \left( p^{e(p)+1} \right) & \text{for } p = 2 \text{ or } p = 3 \end{cases}.
\]

Then, from (16)-(19) it follows that

\[
F \left( p^k \right) = F \left[ (p^k, 6C^2) \right] \quad \text{for } (k = 0, 1, \ldots).
\]

Thus, we have proved that

\[
F(n) = F \left[ (n, 6C^2) \right]
\]

is satisfied for all \(n \in 6N\).

We shall prove that \(f^*(n) = 0\) for all \(n \in 6N\) and \(f_1(m) = 0\) for all \(m \in 6N\) which are prime to \(6AC\).

We note that, by considering \(n = 2m\) and taking into account (6), we have

\[
f(12ACm + 1) \equiv -f(54C^2m) + f(36C^2m) + f(18C^2m) + D \pmod{m}
\]
Since \( f = f^* + F \), from the last relation and (20) we get
\[
f^*(12ACm + 1) \equiv f^*(m) + [f^*(12C^2) + F(6C^2) + D] \pmod{m},
\]
which with Lemma 1 shows that \( f^*(n) = 0 \) for all \( n \in \mathbb{N} \). This shows that \( f \equiv F \), i.e.
\[
f(n) = f([n, 6C^2])
\]
holds for all \( n \in \mathbb{N} \). Now, by applying (3) with \( n = 6Cm \) and using the last relation and Lemma 2, we have that \( f_1(n) = 0 \) holds for all \( n \in \mathbb{N} \) which are prime to \( 6AC \).

The proof of Lemma 3 is completed.

3. Proof of Theorem 1

It is obvious that (1) follows from (2). We shall prove that if (1) is true, then (2) holds.

Assume that the functions \( f_1 \) and \( f_2 \in A \) satisfy the congruence (1) for some integers \( a \geq 1, b \geq 1, c \geq 1 \) and \( d \). It is obvious that (1) implies the fulfilment of
\[
f_1(abn + 1) \equiv f_2(b^2cn + d - f_1(b)) \pmod{n}
\]
for all \( n \in \mathbb{N} \). By Lemma 3,
\[
f_2(n) = f_2([n, 6b^4c^2]) \quad \text{for all} \quad n \in \mathbb{N}
\]
and
\[
f_1(n) = 0
\]
for all \( n \in \mathbb{N} \) which are prime to \( 6abc \).

We shall prove that
\[
f_1(an + b) = f_2(cn + d)
\]
is true for all \( n \in \mathbb{N} \).

Let \( K \) be a positive integer. By (21) and (22), we have
\[
f_1(6ab^4ct + 1) = 0,
\]
\[
f_2(6b^4c^2(aK + b)t + cK) = f_2(cK)
\]
hold for all positive integers $t$, consequently
\[
\begin{align*}
    f_1(aK + b) - f_2(cK) - d &= f_1(aK + b) + f_1(6ab^4t + 1) - f_2(cK) - d \\
    &= f_1[6b^4c(aK + b)t + K] + b] - f_2[6b^4c^2(aK + b) + 1] - d
\end{align*}
\]
holds for every positive integer $t$. Thus, by applying (1) with $n = 6b^4c(aK + b)t + K$, the last relation proves that (23) holds for $n = K$.

This completes the proof of Theorem 1.

4. Proof of Theorem 2

As we have shown in the proof of Theorem 1, if the functions $f_1, f_2 \in A$ satisfy (2), then (21) and (22) imply
\[
(24) \quad f_1(m) = 0 \quad \text{for all} \quad m \in \mathbb{N}, \ (m, 6abc) = 1
\]
and
\[
(25) \quad f_2(n) = 0 \quad \text{for all} \quad n \in \mathbb{N}, \ (m, 6bc) = 1.
\]

Let $D = (a, b)$, $a_1 = \frac{a}{d}$, $b_1 = \frac{b}{d}$. It is clear that for each positive integer $M$, $(M, a_1) = 1$ there are $m_0, n_0 \in \mathbb{N}$ such that
\[
(26) \quad Mm_0 = a_1n_0 + b_1, \ (m_0, a_1) = 1 \quad \text{and} \quad (M, n_0) = (M, b_1).
\]

Let
\[
(27) \quad u(M) := \begin{cases} 
1, \quad \text{if} \quad 2 \mid a_1(M, b_1) \ (M, b_1), \\
2, \quad \text{if} \quad 2 \notmid a_1(M, b_1) \ (M, b_1).
\end{cases}
\]

By applying the Chinese Remainder Theorem and using (26)–(27), we can choose a positive integer $t_1$ such that $m_1 = a_1t_1 + m_0$, $n_1 = Mt_1 + n_0$ satisfy the following conditions:

\[
Mm_1 = a_1n_1 + b_1\ ,
\]

\[
\frac{n_1}{u(M)(M, b_1)} \text{ is an integer,}
\]

and
\[
(m_1, 6abc) = \left(\frac{n_1}{u(M)(M, b_1)}, 6bc\right) = 1.
\]

Hence, we infer from (2) and (24)-(25) that
\[
f_1(DM) = f_1(DMm_1) = f_1(an_1 + b) = f_2(cn_1) + d = f_2 \left[\frac{a_1n_1}{u(M)(M, b_1)}\right] + d,
\]
consequently

(28) \[ f_1[DM] = f_2[\mu v(M)(M, b_1)] + d \]

hold for all \( M \in \mathbb{N}, \ (M, a_1) = 1 \). This implies that

(29) \[ f_1(n) = 0 \quad \text{for all} \quad n \in \mathbb{N}, \ (n, \mu ab_1) = 1, \]

where \( \mu \in \{1, 2\} \) such that \( 2|\mu a_1 b_1 \).

Now we prove that

(30) \[ f_2(n) = 0 \quad \text{for all} \quad n \in \mathbb{N}, \ (n, \mu cb_1) = 1. \]

For each positive integer \( n \), let \( M(n) := a_1 n + b_1 \) and \( U(n) := v(a_1 n + b_1) \).

Since \( (M(n), b_1) = (n, b_1) \) and

\[ \frac{M(n)}{b_1} - \frac{b_1}{(M(n), b_1)} = a_1 \frac{b_1}{(n, b_1)} \left[ \frac{n}{(n, b_1)} + 1 \right] \pmod{2}, \]

we have

\[ U(n) := \begin{cases} 1, & \text{if } 2 \mid a_1 \frac{b_1}{(n, b_1)} \left[ \frac{n}{(n, b_1)} + 1 \right], \\ 2, & \text{if } 2 \not\mid a_1 \frac{b_1}{(n, b_1)} \left[ \frac{n}{(n, b_1)} + 1 \right]. \end{cases} \]

Hence, (2) and (28) show that

\[ f_2(\mu v n) = f_1(an + b) - d = f_1[DM(n)] - d = f_2[\mu v U(n), b_1] \]

is satisfied for all \( n \in \mathbb{N} \), which implies (29). Thus, (29) is proved.

By (29) and (30), the proof of Theorem 2 is completed.

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