Remarks on arithmetical functions

\( a_p(n) \), \( \gamma(n) \), \( \tau(n) \)

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Submitted 31 July 2005; Accepted 1 September 2006

Abstract

In this paper some properties of the arithmetical functions \( a_p(n) \), \( \gamma(n) \), \( \tau(n) \) defined by Šalát in 1994 and Mycielski in 1951, respectively are investigated from the point of view of \( I \)-convergence of sequences (\( I \)-convergence was defined by Kostyrko, Šalát and Wilczynski in 2000).

1. Introduction

We shall study some properties of the \( I \)-convergence of sequences of arithmetical functions \( f: \mathbb{N} \rightarrow \mathbb{N} \), \( a_p(n) \), \( \gamma(n) \), \( \tau(n) \). Elementary properties of the function \( a_p(n) \) were studied in [6]. We shall extend these results with properties of \( I \)-convergence of the sequence \( (a_p(n))_{n=1}^{\infty} \).

We also want to investigate the asymptotic density of the sets \( M_f = \{ n : f(n) \mid n \} \) and the \( I \)-convergence of arithmetical functions \( \gamma(n) \), \( \tau(n) \) defined by Mycielski in [4].

As usual we put for \( A \subset \mathbb{N} \): \( A(n) = |\{1, 2, \ldots n\} \cap A| \),

\[
d(A) = \lim \inf \frac{A(n)}{n}, \quad d(A) = \lim \sup \frac{A(n)}{n}
\]
the lower and upper density of \( A \). If \( d(A) = \overline{d}(A) \), then we set
\[
d(A) = d(A) = \overline{d}(A), \quad d(A) = \lim_{n \to \infty} \frac{A(n)}{n}.
\]

The system \( I \subseteq 2^\mathbb{N} \) is called an admissible ideal if \( I \) is additive \((A, B \in I \Rightarrow A \cup B \in I)\), hereditary \((A \in I, B \subseteq A \Rightarrow B \in I)\) and contains all finite sets. In this paper we are interested in ideals \( I \subseteq \mathbb{N} \) : \( d(A) = 0 \), \( I_c = \{ A \subseteq \mathbb{N} : \sum_{a \in A} a^{-1} < +\infty \} \) and \( I_c^q = \{ A \subseteq \mathbb{N} : \sum_{a \in A} a^{-q} < +\infty \} \) for \( q \in (0, 1) \).

Using this result we completely determine for which \( q \leq q' \in (0, 1) \) the following inclusions hold:
\[
I_f \subseteq I_c^q \subseteq I_c' \subseteq I_c \subseteq I_d.
\]

A given sequence \( x = (x_n)_{n=1}^\infty \) of real numbers is said to be \( I \)-convergent to \( L \in \mathbb{R} \), if for each \( \varepsilon > 0 \) we have \( A_{\varepsilon} = \{ n : |x_n - L| \geq \varepsilon \} \subseteq I \) (shortly \( I \)-lim \( x_n = L \)). The cases of \( I_f \)-convergence and \( I_d \)-convergence coincide with the usual convergence and the statistical convergence (see [3], [7]), respectively. Therefore we will write \( \lim x_n = L \) and \( \lim_{\text{stat}} x_n = L \) instead of \( I \)-lim \( x_n = L \) and \( I_d \)-lim \( x_n = L \), respectively.

In [7, Lemma 2.2] it is shown that
\[
I \subseteq I' \Rightarrow I \text{-lim } x_n = L \Rightarrow I' \text{-lim } x_n = L.
\]

Using this result we completely determine for which \( q \) the sequences \( a_p(n) \), \( (\gamma(n)) \) and \( \tau(n) \) are \( I_c^q \)-convergent.

### 2. \( I \)-convergence of \((a_p(n))_{n=1}^\infty\)

Let \( p \) be a prime number. The function \( a_p(n) \) is defined in the following way: \( a_p(1) = 0 \) and if \( n > 1 \), then \( a_p(n) \) is the unique integer \( j \geq 0 \) satisfying \( p^j | n \) but \( p^{j+1} \nmid n \), i.e., \( p^{a_p(n)} \parallel n \). At first we are going to generalize the result that the sequence \( (\log p \frac{a_p(n)}{\log n})_{n=2}^\infty \) is statistically convergent to 0 [6, Th. 4.2].

**Proposition 2.1.** Let \( g(n) > 0 \) \((n = 1, 2, \ldots)\) and \( \lim_{n \to \infty} g(n) = +\infty \). We have
\[
\lim_{n \to \infty} \text{stat}(\log p) \frac{a_p(n)}{g(n)} = 0.
\]

**Proof.** Let \( \varepsilon > 0 \). Put \( A_\varepsilon = \{ n > 1 : (\log p) \frac{a_p(n)}{g(n)} \geq \varepsilon \} \). We will show that \( d(A_\varepsilon) = 0 \). Let \( \eta > 0 \). Choose \( m \in \mathbb{N} \) such that
\[
p^{-m} < \eta.
\]

By the conditions of the proposition there exists an \( n_0 \), such that for any \( n > n_0 \) we have
\[
\frac{\varepsilon g(n)}{\log p} > m.
\]
Let $n > n_0$ and $n \in A_\varepsilon$. It follows from (2.2) and the definition of $A_\varepsilon$ that

$$
(\log p) \frac{a_p(n)}{g(n)} \geq \varepsilon,
$$

$$
a_p(n) \geq \frac{\varepsilon g(n)}{\log p} > m.
$$

Hence for the numbers $n > n_0$, $n \in A_\varepsilon$ implies $p^m | n$. This leads to the conclusion that $A_\varepsilon \subseteq \{1, 2, \ldots, n_0\} \cup \{n > n_0 : p^m | n\}$ and considering (2.1) we get $\overline{d}(A_\varepsilon) \leq p^{-m} < \eta$. Since $\eta > 0$ is an arbitrary positive number, $d(A_\varepsilon) = 0$. \hfill \square

**Remark 2.2.** It is proved [6, Th. 4.1] that the sequence $\left( (\log p) \frac{a_p(n)}{\log n} \right)_{n=2}^\infty$ is dense in interval $(0, 1)$. But $\left( (\log p) \frac{a_p(n)}{g(n)} \right)_{n=2}^\infty$ which is statistically convergent to zero if $g(n) \to +\infty$, is not always dense in $(0, 1)$: For example if we define the function $g(n) = \max\{1, \log^2 n\}$, then we have

$$
\lim_{n \to \infty} (\log p) \frac{a_p(n)}{\log^2 n} = 0
$$

and also

$$
\lim \mathrm{stat} \frac{a_p(n)}{\log^2 n} = 0,
$$

but this sequence is not dense in $(0, 1)$.

**Theorem 2.3.** The sequence $(a_p(n))_{n=1}^\infty$ is $I_c$-convergent to 0 and $I_q^c$-divergent for $q \in (0, 1)$.

**Proof.** Let $\varepsilon > 0$ and denote

$$
A_\varepsilon = \{n \in \mathbb{N} : (\log p) \frac{a_p(n)}{\log n} \geq \varepsilon\}.
$$

Let $q \in (0, 1)$. We want to show that

$$
\sum_{n \in A_\varepsilon} \frac{1}{n} < +\infty \quad \tag{2.3}
$$

and for $0 < \varepsilon < 1 - q$

$$
\sum_{n \in A_\varepsilon} \frac{1}{n^q} = +\infty. \quad \tag{2.4}
$$

For nonnegative integer $i$ denote $A^i_\varepsilon = \{n \in A_\varepsilon : n = p^i u, (u, p) = 1\}$. We have $A^i_\varepsilon \cap A^j_\varepsilon = \emptyset$ for $i \neq j$ and for any $t > 0$

$$
\sum_{n \in A_\varepsilon} \frac{1}{n^t} = \sum_{i=0}^{\infty} \sum_{n \in A^i_\varepsilon} \frac{1}{n^t}. \quad \tag{2.5}
$$
a) Consider that \( n \in A^i_{\varepsilon} \) if and only if \( n = p^iu \) where \((u, p) = 1\) and also

\[
(\log p) \frac{a_p(n)}{\log n} \geq \varepsilon.
\]

Then

\[
(\log p) \frac{i}{i \log p + \log u} \geq \varepsilon
\]

from which we obtain \( u \leq p^{i\delta} \), where \( \delta = (1 - \varepsilon)/\varepsilon \). Hence

\[
\sum_{n \in A^i_{\varepsilon}} \frac{1}{n} \leq \frac{1}{p^i} \sum_{u \leq p^{i\delta}} \frac{1}{u} \leq \frac{1}{p^i} \left( 1 + \int_1^{p^{i\delta}} \frac{dt}{t} \right) = \frac{1}{p^i} (1 + i\log p) \leq A\delta \frac{i}{p^i} \log p
\]

where \( A > 0 \) is only dependent on \( \varepsilon, p \) and not on \( i \). The series \( \sum_{i=0}^{\infty} \frac{i}{p^i} \) converges, this proves (2.3).

b) We write

\[
\sum_{n \in A^i_{\varepsilon}} \frac{1}{n^q} = \frac{1}{p^{iq}} \sum_{u \leq p^{i\delta}} \frac{1}{u^q}.
\]

Then we have

\[
\sum_{u \leq p^{i\delta}, (u, p) = 1} \frac{1}{u^q} = \sum_{u \leq p^{i\delta}} \frac{1}{u^q} - \sum_{k \leq p^{i\delta}-1} \frac{1}{(kp)^q} = \sum_{u \leq p^{i\delta}} \frac{1}{u^q} - \frac{1}{p^q} \sum_{k \leq p^{i\delta}-1} \frac{1}{k^q}
\]

\[
= \left( 1 - \frac{1}{p^q} \right) \sum_{v \leq p^{i\delta}-1} \frac{1}{v^q} + \sum_{p^{i\delta}-1 < v \leq p^{i\delta}} \frac{1}{v^q}
\]

\[
\geq \sum_{p^{i\delta}-1 < v \leq p^{i\delta}} \frac{1}{v^q} \geq (p^{i\delta} - p^{i\delta-1}) \frac{1}{p^{i\delta q}}
\]

\[
= p^{i\delta} (1 - \frac{1}{p}) \frac{1}{p^{i\delta q}} = (1 - \frac{1}{p}) p^{i\delta(1-q)}.
\]

Finally we obtain

\[
\sum_{n \in A^i_{\varepsilon}} \frac{1}{n^q} = \sum_{i=0}^{\infty} \sum_{v \in A^i_{\varepsilon}} \frac{i}{v^q} \geq (1 - \frac{1}{p}) \sum_{i=0}^{\infty} \frac{1}{p^{i[q+(q-1)i]}},
\]

The series on the right-hand side diverges if \( q + (q - 1)i < 0 \), i.e. \( \varepsilon < 1 - q \). This proves the \( I^q_{\varepsilon} \)-divergence of \( (a_p(n))_{n=1}^{\infty} \). \( \square \)
3. On the functions $\gamma(n)$ and $\tau(n)$

In [4] there were new arithmetical functions defined and investigated in connection with the representation of natural numbers of the form $n = a^b$, where $a, b$ are positive integers. Let

$$n = a_1^{b_1} = a_2^{b_2} = \cdots = a_{\gamma(n)}^{b_{\gamma(n)}}$$

be all such representations of a given natural number $n$, where $a_i, b_i \in \mathbb{N}$.

Denote by

$$\tau(n) = b_1 + \cdots + b_{\gamma(n)}, (n > 1).$$

It is clear that $\gamma(n) \geq 1$, because for any $n > 1$ there exists a representation in the form $n^1$.

We are going to study some new properties of the functions $\gamma(n)$ and $\tau(n)$.

Put $T(n) = \gamma(2) + \cdots + \gamma(n), (n \geq 2)$. It is proved in [4], that

$$T(n) = \sum_{s=1}^{[\log_2 n]} \lceil \sqrt[n]{s} \rceil - [\log_2 n] = n + \sum_{s=2}^{[\log_2 n]} \lceil \sqrt[n]{s} \rceil - [\log_2 n]. \quad (3.2)$$

**Remark 3.1.** It is easy to show that the average order of the function $\gamma(n)$ is 1, i.e.,

$$\lim_{n \to \infty} \frac{T(n)}{n} = 1.$$ 

It follows from (3.2) that

$$T(n) = n + T_1(n) - [\log_2 n],$$

where $T_1(n) = n + \sum_{s=2}^{[\log_2 n]} \lceil \sqrt[n]{s} \rceil$. Then simple estimations give

$$(\lceil \log_2 n \rceil - 1) \lceil \frac{\lceil \log_2 \sqrt[n]{s} \rceil}{\sqrt[n]{s}} \rceil \leq T_1(n) \leq (\lceil \log_2 n \rceil - 1) \sqrt[n]{s},$$

from which we get

$$\lim_{n \to \infty} \frac{T_1(n)}{n} = 0.$$

In papers [1, 2] sets of the form $M_f = \{n \in \mathbb{N} : f(n) \mid n\}, f : \mathbb{N} \to \mathbb{N}$ are investigated. For some of the known arithmetical functions the sets $M_f$ have zero asymptotic density: e.g. the functions $\omega(n)$ (the number of prime divisors of $n$), $s_g(n)$ (the digital sum of $n$ in the representation with base $g$), $\pi(n)$ (the number of primes not exceeding $n$).

**Proposition 3.2.** Put $A_k = \{n > 1 : n = p_1^{\alpha_1} \cdots p_n^{\alpha_n}, (\alpha_1, \ldots, \alpha_n) = k\} \ (k = 1, 2, \ldots)$. Then

$$d(A_1) = 1. \quad (3.3)$$
Proof. Denote by \( B = \bigcup_{k=2}^{\infty} A_k \), then \( \mathbb{N}\setminus\{1\} = A_1 \cup B \), where \( A_1 \cap B = \emptyset \). It can be easily shown that \( d(B) = 0 \), from which (3.3) follows immediately. The elements of the set \( B \) are only numbers of the form \( t^s(t > 1, s > 1) \). Denote by \( H \) the set of all numbers \( t^s(t > 1, s > 1) \). The series of reciprocal values of these numbers is equal to \( \sum_{t=2}^{\infty} \sum_{s=2}^{\infty} \frac{1}{ts} \) which is convergent to 1 (cf. [4]). Then we have \( d(H) = 0 \) and it implies that also \( d(B) = 0 \). □

Let us investigate the asymptotic density of \( M_\gamma = \{ n : \gamma(n) \mid n \} \) and \( M_\tau = \{ n : \tau(n) \mid n \} \).

Proposition 3.3. We have
(i) \( d(M_\gamma) = 1 \),
(ii) \( d(M_\tau) = 1 \).

Proof. (i) If \( n \in A_1 \), then evidently \( \gamma(n) = 1 \) and \( n \in M_\gamma \). Thus \( A_1 \subseteq M_\gamma \) and considering (3.3) we get \( d(M_\gamma) = 1 \).
(ii) Similarly. □

In [4, Th. 3, Th. 5] there are proofs of the following results:
\[
\sum_{n=2}^{\infty} \frac{\gamma(n) - 1}{n} = 1, \quad \sum_{n=2}^{\infty} \frac{\tau(n) - 1}{n} = 1 + \frac{\pi^2}{6}.
\]

In connection with these results we have investigated the convergence of series for any \( \alpha \in (0, 1) \)
\[
\sum_{n=2}^{\infty} \frac{\gamma(n) - 1}{n^\alpha}, \quad \sum_{n=2}^{\infty} \frac{\tau(n) - 1}{n^\alpha}.
\]

Theorem 3.4. The series
\[
\sum_{n=2}^{\infty} \frac{\gamma(n) - 1}{n^\alpha}
\]
diverges for \( 0 < \alpha \leq \frac{1}{2} \) and converges for \( \alpha > \frac{1}{2} \).

Proof. a) Let \( 0 < \alpha \leq \frac{1}{2} \). Put \( K = \{ k^2 : k > 1 \} \). A simple estimation gives
\[
\sum_{n=2}^{\infty} \frac{\gamma(n) - 1}{n^\alpha} \geq \sum_{n \in K} \frac{\gamma(n) - 1}{n^\alpha}.
\]
Clearly \( \gamma(n) \geq 2 \) for \( n \in K \). Therefore
\[
\sum_{n=2}^{\infty} \frac{\gamma(n) - 1}{n^\alpha} \geq \sum_{n \in K} \frac{1}{n^\alpha} = \sum_{k=2}^{\infty} \frac{1}{k^{2\alpha}} \geq \sum_{k=2}^{\infty} \frac{1}{k} = +\infty.
\] (3.4)
b) Let $\alpha > \frac{1}{2}$. We will use the formula

$$\sum_{n=2}^{\infty} \frac{\gamma(n) - 1}{n^\alpha} = \sum_{k=2}^{\infty} \sum_{s=2}^{\infty} \frac{1}{k^\alpha} = \sum_{k=2}^{\infty} \frac{1}{k^\alpha(k^\alpha - 1)}. \quad (3.5)$$

For a sufficiently large number $k$ ($k > k_0$) we have $\frac{k^\alpha}{k^\alpha - 1} < 2$. We can estimate the series on the right-hand side of (3.5) with

$$\sum_{k=2}^{\infty} \frac{1}{k^\alpha(k^\alpha - 1)} < \sum_{k=2}^{k_0} \frac{1}{k^\alpha} + 2 \sum_{k>k_0} \frac{1}{k^2\alpha}.$$

Since $2\alpha > 1$ we get

$$\sum_{n=2}^{\infty} \frac{\gamma(n) - 1}{n^\alpha} < +\infty.$$

\[ \square \]

**Corollary 3.5.** The sequence $\gamma(n)$ is

(i) $\mathcal{I}_c$-convergent to 1,

(ii) $\mathcal{I}_{q_c}$-divergent for $q \in (0, \frac{1}{2}]$ and $\mathcal{I}_c$-convergent to 1 for $q \in (\frac{1}{2}, 1)$.

**Proof.** (i) Let $\varepsilon > 0$. The set of numbers $\{n > 1 : |\gamma(n) - 1| \geq \varepsilon\}$ is a subset of $H = \{t^s, t > 1, s > 1\}$ and $\sum_{a \in H} \frac{1}{a} < +\infty$. From the definition of $I_c$-convergence (i) follows.

(ii) Let $\varepsilon > 0$ and denote $A_\varepsilon = \{n \in \mathbb{N} : |\gamma_n - 1| \geq \varepsilon\}$. When $0 < q \leq \frac{1}{2}$ then for the numbers $n \in K$, $K = \{k^2 : k > 1\}$ considering (3.4) holds

$$\sum_{n \in A_\varepsilon} \frac{1}{n^\alpha} \geq \sum_{n \in K} \frac{1}{n^\alpha} \geq +\infty.$$

Therefore $\gamma(n)$ is $\mathcal{I}_{q_c}$-divergent. When $\frac{1}{2} < q < 1$, then $A_\varepsilon \subset H$ and

$$\sum_{n=2}^{\infty} \frac{1}{n^\alpha} \leq \sum_{k=2}^{\infty} \sum_{s=2}^{\infty} \frac{1}{k^\alpha s}.$$

The convergence of the series on the right-hand side we proved previously in Theorem 3.4. Therefore $\gamma(n)$ is $\mathcal{I}_c$-convergent to 1 if $q \in (\frac{1}{2}, 1)$.

\[ \square \]

**Remark 3.6.** We have $\lim \text{stat } \gamma(n) = 1$.

**Theorem 3.7.** The series

$$\sum_{n=2}^{\infty} \frac{\tau(n) - 1}{n^\alpha}$$

diverges for $0 < \alpha \leq \frac{1}{2}$ and converges for $\alpha > \frac{1}{2}$. 


Proof. Let $0 < \alpha < 1$. We write the given series in the form
\[
\sum_{n=2}^{\infty} \frac{\tau(n) - 1}{n^\alpha} = \sum_{k=2}^{\infty} \sum_{s=2}^{\infty} \frac{s}{k^{\alpha s}},
\] (3.6)

We shall try to use a similar method to Mycielski’s proof of the convergence of $\sum_{n=2}^{\infty} \frac{\tau(n) - 1}{n^\alpha}$ to explain the equality (3.6). Since $\frac{s}{k^{\alpha s}} = \frac{1}{\alpha} \frac{d}{dt}(\frac{1}{t^{\alpha s}})_{t=k}$ and $\sum_{s=2}^{\infty} \frac{1}{t^{\alpha s}} = \frac{1}{t^{\alpha - 1}}$ the right-hand side of (3.6) is equal to
\[
\sum_{s=2}^{\infty} \frac{2k^\alpha - 1}{k^\alpha (k^\alpha - 1)^2} = \sum_{s=2}^{\infty} a_k.
\]

For the $k$-th term of $\sum a_k$ we have
\[
a_k = \frac{2 - \frac{1}{k^\alpha}}{(1 - \frac{1}{k^\alpha})^2} \cdot \frac{1}{k^{2\alpha}}.
\]

Denote by $b_k = \frac{1}{k^{2\alpha}}$ and consider that $\lim_{k \to \infty} \frac{a_k}{b_k} = 2$. Hence the series $\sum_{s=2}^{\infty} a_k$ converges (diverges) if and only if the series $\sum_{s=2}^{\infty} b_k$ converges (diverges). Since $\sum b_k$ is convergent (divergent) for any $\alpha > \frac{1}{2}$ ($0 < \alpha \leq \frac{1}{2}$) so does the series $\sum a_k$ and therefore the series $\sum \frac{\tau(n) - 1}{n^\alpha}$. \qed

Corollary 3.8. The sequence $\tau(n)$ is

(i) $\mathcal{I}_c$–convergent to 1,
(ii) $\mathcal{I}_c^q$–divergent for $q \in (0, \frac{1}{2}]$ and $\mathcal{I}_c$–convergent to 1 for $q \in (\frac{1}{2}, 1)$.

Proof. Similar to the proof of Corollary 3.5. \qed

Remark 3.9. We have $\lim \text{stat} \tau(n) = 1$.

References


