Distribution of terms of a logarithmic sequence*

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Abstract

The number \( L(a, b) = \frac{a - b}{\ln a - \ln b} \) for \( a \neq b \) and \( L(a, a) = a \), is said to be the logarithmic mean of the positive numbers \( a, b \). We shall say that a sequence \( (a_n)_{n=1}^{\infty} \) with positive terms is a logarithmic sequence if \( a_n = L(a_{n-1}, a_{n+1}) \). In the present paper some basic estimations of the terms of logarithmic sequences are investigated.

Keywords: logarithmic mean, power mean, logarithmic sequence.

MSC: Primary 11K31, Secondary 26E60.

1. Introduction

Let \( a, b \) be positive real numbers. The logarithmic mean of \( a, b \) is defined as follows:

\[
L(a, b) = \frac{a - b}{\ln a - \ln b} \quad \text{if} \quad a \neq b \quad \text{and} \quad L(a, a) = a
\]

(see [5]).

The logarithmic sequence is defined in paper [2] by means of logarithmic mean in the following way:

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Definition 1.1. A sequence \((a_n)_{n=1}^\infty\) of positive real numbers is called logarithmic if
\[
a_n = L(a_{n-1}, a_{n+1}) \quad \text{for each } n \geq 2.
\]

Moreover, in [2] the existence of logarithmic sequence is proved and even it is shown that if a sequence \((a_n)_{n=1}^\infty\) is logarithmic and \(a_1 < a_2 < \cdots < a_n < \cdots\). On the other hand, if \(a_1 > a_2\) then \(a_1 > a_2 > \cdots > a_n > \cdots\) (see [2], Theorem 2.1). Thus we see that the logarithmic sequence is either increasing or decreasing if \(a_1 \neq a_2\). In the case \(a_1 = a_2\) the logarithmic sequence \((a_n)_{n=1}^\infty\) is stationary and \(a_n = a_1 \quad (n = 1, 2, \ldots)\). In the present paper we will consider only the logarithmic sequences \((a_n)_{n=1}^\infty\) for which \(a_1 \neq a_2\).

The following theorem holds for logarithmic sequences.

Theorem 1.2. ([2; Th. 2.2., Th. 2.3.]) Let the sequence \((a_n)_{n=1}^\infty\) be logarithmic and \(a_1 \neq a_2\). Then the following implications hold.

(i) If \(a_1 < a_2\) then
\[
\lim_{n \to \infty} a_n = \infty.
\]

(ii) If \(a_1 > a_2\) then the series
\[
\sum_{n=1}^\infty a_n
\]
converges.

Now we introduce the power mean of degree \(\alpha \in \mathbb{R}\) of two positive numbers \(a, b\) as follows:
\[
M_\alpha(a, b) = \left(\frac{a^\alpha + b^\alpha}{2}\right)^{\frac{1}{\alpha}} \quad \text{if } \alpha \neq 0 \quad \text{and} \quad M_0(a, b) = \lim_{\alpha \to 0} M_\alpha(a, b).
\]

It is well known that \(M_0(a, b) = \sqrt{ab}\) and \(M_\alpha(a, b)\) is increasing with respect to \(\alpha\) (see [6]).

In paper [3] the following relation between \(L(a, b)\) and \(M_\alpha(a, b)\) is proved for arbitrary positive numbers \(a, b\):
\[
M_0(a, b) \leq L(a, b) \leq M_{\frac{1}{2}}(a, b), \quad (1.1)
\]
and the equality occurs if and only if \(a = b\).

As \(M_\alpha(a, b)\) is increasing with respect to \(\alpha\), from (1.1) we have
\[
M_0(a, b) \leq L(a, b) \leq M_\alpha(a, b) \quad (1.2)
\]
for all \(a, b > 0\) and \(\alpha \geq \frac{1}{3}\).

Thus, if the sequence \((a_n)_{n=1}^\infty\) is logarithmic then (1.2) implies that for all \(n \geq 2\) and \(\alpha \geq \frac{1}{3}\) the inequality
\[
\sqrt{a_{n-1}a_{n+1}} \leq a_n \leq \left(\frac{a_{n-1}^\alpha + a_{n+1}^\alpha}{2}\right)^{\frac{1}{\alpha}}
\]
Distribution of terms of a logarithmic sequence

holds. Consequently we have for all \( n \geq 2 \) and \( \alpha \geq \frac{1}{3} \)

\[
\frac{a_{n+1}}{a_n} \leq \frac{a_n}{a_{n-1}} \quad \text{and} \quad a_n^\alpha - a_{n-1}^\alpha \leq a_{n+1}^\alpha - a_n^\alpha. \tag{1.3}
\]

From (1.3) we obtain that in the case of increasing logarithmic sequence \((a_n)_{n=1}^\infty\)
for each \( n \geq 2 \) the inequalities

\[
1 < \frac{a_{n+1}}{a_n} < \frac{a_n}{a_{n-1}} \quad \text{and} \quad 0 < a_n - a_{n-1} < a_{n+1} - a_n \tag{1.4}
\]

hold.

A natural question arises. What can be said about the asymptotic behaviour
of differences \( a_{n+1} - a_n \) and fractions \( \frac{a_{n+1}}{a_n} \) if \((a_n)_{n=1}^\infty\) is an increasing logarithmic sequence? More precisely, does it hold

\[
\lim_{n \to \infty} (a_{n+1} - a_n) = \infty \quad \text{and} \quad \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 1? \tag{1.5}
\]

In the first part of the present paper, among others, we give the answer to the
previous question. We will determine the lower bounds for terms \( a_n \),
differences \( a_{n+1} - a_n \) and fractions \( \frac{a_{n+1}}{a_n} \) if \((a_n)_{n=1}^\infty\) is a logarithmic sequence.

2. Estimates for differences and quotients of consecutive terms of a logarithmic sequence

Theorem 2.1. Let \((a_n)_{n=1}^\infty\) be a logarithmic sequence. Then the following implications hold.

(i) If \((a_n)_{n=1}^\infty\) is increasing then

\[
a_n > \left( \frac{a_2^\alpha - a_1^\alpha}{2} \right)^\frac{1}{\alpha} n^{\frac{1}{\alpha}} \tag{2.1}
\]

for every \( \alpha \geq \frac{1}{3} \) and \( n \in \mathbb{N} \).

(ii) If \((a_n)_{n=1}^\infty\) is decreasing then

\[
a_n < \left( \frac{a_2^\beta - a_1^\beta}{2} \right)^\frac{1}{\beta} n^{\frac{1}{\beta}} \tag{2.2}
\]

for every \( \beta < 0 \) and \( n \in \mathbb{N} \).

Proof. (i) Let \((a_n)_{n=1}^\infty\) be an increasing logarithmic sequence. Then (1.3) implies
for \( \alpha \geq \frac{1}{3} \)

\[
a_n^\alpha - a_{n-1}^\alpha < a_{n+1}^\alpha - a_n^\alpha \quad \text{for} \quad n \geq 2.
\]
Consequently, for every \( n \geq 2 \) we have
\[
a_2^\alpha - a_1^\alpha < a_{n+1}^\alpha - a_n^\alpha, \quad \text{i.e.}
\]
\[
(a_n^\alpha + a_2^\alpha - a_1^\alpha)^{\frac{1}{\alpha}} < a_{n+1}.
\]
(2.3)

Now we will show by induction the inequality
\[
((n-1)a_2^\alpha - (n-2)a_1^\alpha)^{\frac{1}{\alpha}} \leq a_n
\]
(2.4)
for every \( n \geq 2 \). For \( n = 2 \) evidently the equality takes place in (2.4). Suppose that (2.4) holds for some \( n = k \geq 2 \). Then we obtain
\[
\begin{align*}
(ka_2^\alpha - (k-1)a_1^\alpha)^{\frac{1}{\alpha}} &= ((k-1)a_2^\alpha - (k-2)a_1^\alpha + a_2^\alpha - a_1^\alpha)^{\frac{1}{\alpha}} \\
&\leq (a_k^\alpha + a_2^\alpha - a_1^\alpha)^{\frac{1}{\alpha}}.
\end{align*}
\]
Consequently, using (2.3) we obtain
\[
(ka_2^\alpha - (k-1)a_1^\alpha)^{\frac{1}{\alpha}} \leq a_{k+1}
\]
proving (2.4) for every \( n \geq 2 \). Finally, for \( n \geq 2 \) we obtain
\[
a_n \geq ((n-1)(a_2^\alpha - a_1^\alpha) + a_1^\alpha)^{\frac{1}{\alpha}} > (n-1)^{\frac{1}{\alpha}} (a_2^\alpha - a_1^\alpha)^{\frac{1}{\alpha}} \geq n^{\frac{1}{\alpha}} \left( \frac{a_2^\alpha - a_1^\alpha}{2} \right)^{\frac{1}{\alpha}}.
\]

(ii) Let \( (a_n)_{n=1}^\infty \) be a decreasing logarithmic sequence. Then (1.2) and the fact that \( M_\alpha(a, b) \) is increasing with respect to \( \alpha \) imply the inequality
\[
\left( \frac{a_n^\beta + a_{n+1}^\beta}{2} \right)^{\frac{1}{\beta}} < a_n = L(a_{n-1}, a_{n+1})
\]
holding for every real \( \beta < 0 \). Consequently
\[
a_n^\beta - a_{n-1}^\beta < a_{n+1}^\beta - a_n^\beta
\]
holds for every \( n \geq 2 \). Especially,
\[
a_{n+1}^\beta - a_n^\beta > a_2^\beta - a_1^\beta, \quad \text{i.e.}
\]
\[
a_{n+1} < \left( a_n^\beta + a_2^\beta - a_1^\beta \right)^{\frac{1}{\beta}}
\]
(2.5)
holds for every \( n \geq 2 \). Now we will show by induction the inequality
\[
a_n \leq \left( (n-1)a_2^\beta - (n-2)a_1^\beta \right)^{\frac{1}{\beta}}
\]
(2.6)
for every $n \geq 2$. In the case $n = 2$ the equality takes place in (2.6). Suppose that (2.6) holds for some $n = k \geq 2$. The we obtain

$$
\left(ka^\beta_2 - (k - 1)a^\beta_1\right)^\frac{1}{\beta} = \left((k - 1)a^\beta_2 - (k - 2)a^\beta_1 + a^\beta_2 - a^\beta_1\right)^\frac{1}{\beta} \geq \left(a^\beta_k + a^\beta_2 - a^\beta_1\right)^\frac{1}{\beta}.
$$

Applying (2.5) we obtain

$$
\left(ka^\beta_2 - (k - 1)a^\beta_1\right)^\frac{1}{\beta} \geq a_{k+1}
$$

proving (2.6) for every integer $n \geq 2$. Finally, for every $n \geq 2$ we have

$$
a_n \leq \left((n - 1)(a^\beta_2 - a^\beta_1) + a^\beta_1\right)^\frac{1}{\beta} < \left(a^\beta_2 - a^\beta_1\right)^\frac{1}{\beta} \frac{1}{n^\beta}.
$$

\[\square\]

**Corollary 2.2.** Let $(a_n)_{n=1}^{\infty}$ be an increasing logarithmic sequence. Then for every $n \geq 2$ the inequality

$$
a_n > \left(\frac{\sqrt[3]{a_2} - \sqrt[3]{a_1}}{2}\right)^3 n^3
$$

holds.

**Proof.** Follows directly from Theorem 2.1 (i) for $\alpha = \frac{1}{3}$. \[\square\]

**Corollary 2.3.** If $(a_n)_{n=1}^{\infty}$ is an increasing logarithmic sequence then the series

$$
\sum_{n=1}^{\infty} \frac{1}{a_n}
$$

converges.

**Proof.** By Corollary 2.2 we have for every $n \geq 2$

$$
a_n > c.n^3 \text{ where } c = \left(\frac{\sqrt[3]{a_2} - \sqrt[3]{a_1}}{2}\right)^3.
$$

Evidently the series $\sum_{n=2}^{\infty} \frac{1}{a_n}$ majorises the series $\sum_{n=2}^{\infty} \frac{1}{a_n}$. Consequently the series $\sum_{n=1}^{\infty} \frac{1}{a_n}$ converges. \[\square\]
Corollary 2.4. Let \((a_n)_{n=1}^{\infty}\) be a decreasing logarithmic sequence and let \(l > 0\) be a real number. Then the inequality
\[
a_n < c_1 \frac{1}{n^{1/l}}, \quad \text{where} \quad c_1 = \left(\frac{a_2^{-l} - a_1^{-l}}{2}\right)^{-\frac{1}{l}}
\]
holds for every \(n \geq 2\).

Proof. Follows from Theorem 2.1 (ii) for \(\beta = -l, \ l > 0\).

Corollary 2.5. If \((a_n)_{n=1}^{\infty}\) is a decreasing logarithmic sequence then the series \(\sum_{n=1}^{\infty} a_n\) converges.

Theorem 2.6. Let \((a_n)_{n=1}^{\infty}\) be an increasing logarithmic sequence. Then the inequality
\[
a_{n+1} - a_n > (\sqrt{a_2} - \sqrt{a_1})^2(n + 1)
\]
holds for every \(n \geq 2\).

Proof. We will proceed by induction. From (1.3) for \(\alpha = \frac{1}{2}\) follows the inequality
\[
\sqrt{a_n} - \sqrt{a_{n-1}} < \sqrt{a_{n+1}} - \sqrt{a_n}.
\]
(2.8)

For \(n = 2\) we obtain from (2.8)
\[
\sqrt{a_3} - \sqrt{a_2} > \sqrt{a_2} - \sqrt{a_1}
\]
and
\[
a_3 - a_2 > (\sqrt{a_2} - \sqrt{a_1})(\sqrt{a_3} + \sqrt{a_2}) > 3(\sqrt{a_2} - \sqrt{a_1})(\sqrt{a_2} - \sqrt{a_1}).
\]

Suppose that (2.7) holds for some \(n = k \geq 2\). Then from (2.8) for \(n = k + 1\) we obtain
\[
\sqrt{a_{k+2}} - \sqrt{a_{k+1}} > \sqrt{a_{k+1}} - \sqrt{a_k}.
\]
Moreover
\[
a_{k+2} - a_{k+1} > (a_{k+1} - a_k)\frac{\sqrt{a_{k+2}} + \sqrt{a_{k+1}}}{\sqrt{a_{k+1}} + \sqrt{a_k}}
\]
\[
= (a_{k+1} - a_k) + (a_{k+1} - a_k)\frac{\sqrt{a_{k+2}} - \sqrt{a_k}}{\sqrt{a_{k+1}} + \sqrt{a_k}}
\]
\[
= (a_{k+1} - a_k) + (\sqrt{a_{k+1}} - \sqrt{a_k})(\sqrt{a_{k+2}} - \sqrt{a_k}) > (a_{k+1} - a_k) + (\sqrt{a_{k+1}} - \sqrt{a_k})^2.
\]
As (2.8) implies
\[
\sqrt{a_{k+1}} - \sqrt{a_k} > \sqrt{a_2} - \sqrt{a_1}
\]
we have
\[ a_{k+2} - a_{k+1} > a_{k+1} - a_k + (\sqrt{a_2} - \sqrt{a_1})^2. \]
Finally
\[ a_{k+2} - a_{k+1} > (k + 2)(\sqrt{a_2} - \sqrt{a_1})^2. \]

\[ \square \]

**Theorem 2.7.** Let \( (a_n)_{n=1}^{\infty} \) be a logarithmic sequence. Then \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} \) exists and the following implications hold.

1. If \( (a_n)_{n=1}^{\infty} \) is increasing then
\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 1.
\]

2. If \( (a_n)_{n=1}^{\infty} \) is decreasing then
\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 0.
\]

**Proof.** The sequence \( (a_n)_{n=1}^{\infty} \) is logarithmic, thus
\[
a_n = \frac{a_{n+1} - a_{n-1}}{\ln a_{n+1} - \ln a_{n-1}} \text{ for } n \geq 2.
\]
Consequently
\[
\frac{a_n}{a_{n-1}} = \frac{\frac{a_{n+1} - 1}{a_{n-1}}}{\ln \frac{a_{n+1}}{a_{n-1}}}
\]
which is equivalent with
\[
\frac{a_n}{a_{n-1}} \ln \frac{a_{n+1}}{a_n} = \frac{a_{n+1} - a_n}{a_{n-1}} - 1. \quad (2.9)
\]
The first relation in (1.3) implies that the sequence \( (\frac{a_{n+1}}{a_n})_{n=1}^{\infty} \) is decreasing and bounded from below. Consequently the limit \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} \) exists and it is finite.
Denote \( x = \lim_{n \to \infty} \frac{a_{n+1}}{a_n} \).
If the sequence \( (a_n)_{n=1}^{\infty} \) is increasing then obviously \( x \geq 1 \). Taking limit in (2.9) for \( n \to \infty \) we obtain
\[
x \ln x^2 = x^2 - 1 \text{ i.e. } 2x \ln x = x^2 - 1.
\]
The above inequality can not hold for \( x > 1 \) since for all real \( x \in (0, 1) \cup (1, \infty) \) the inequality \( 2x \ln x < x^2 - 1 \) holds. Thus \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 1 \).
If the sequence \( (a_n)_{n=1}^{\infty} \) is decreasing then obviously \( 0 \leq x < 1 \). In the case \( 0 < x < 1 \) again we obtain \( 2x \ln x = x^2 - 1 \) what is impossible. Thus we have \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 0 \) in the case of a decreasing sequence \( (a_n)_{n=1}^{\infty} \). \( \square \)
Corollary 2.8. Let \((a_n)_{n=1}^{\infty}\) be a logarithmic sequence. Then

\[
\lim_{n \to \infty} \frac{a_n}{q^n} = 0
\]

1. for every real \(q > 1\) if \((a_n)_{n=1}^{\infty}\) is increasing,
2. for every real \(q > 0\) if \((a_n)_{n=1}^{\infty}\) is decreasing.

**Proof.** 1. Consider the power series

\[
\sum_{n=1}^{\infty} a_n x^n.
\]

Then Theorem 2.7 implies that the radius of its convergence is \(R = 1\). Thus for every \(0 < x < 1\) the series \(\sum_{n=1}^{\infty} a_n x^n\) converges. Consequently

\[
\lim_{n \to \infty} a_n x^n = 0.
\]

Denoting \(q = \frac{1}{x}\) we have \(q > 1\) arbitrary and \(\frac{a_m}{q^n} \to 0\) \((n \to \infty)\) holds.

2. If \((a_n)_{n=1}^{\infty}\) is decreasing then Theorem 2.7 implies that the radius of convergence \(R\) of the considered power series is infinity. Thus for every real \(x > 0\) we have \(\lim_{n \to \infty} a_n x^n = 0\). \(\square\)

Corollary 2.9. If \((a_n)_{n=1}^{\infty}\) is an increasing logarithmic sequence then the set

\[
\left\{ \frac{a_m}{a_n} : m, n = 1, 2, \ldots \right\}
\]

is dense in \((0, \infty)\).

**Proof.** The proof follows from Theorem 2.7 and the following theorem: If for an unbounded sequence \((a_n)_{n=1}^{\infty}\) of positive real numbers

\[
\limsup_{n \to \infty} \frac{a_{n+1}}{a_n} = 1
\]

holds then the set \(\left\{ \frac{a_m}{a_n} : m, n = 1, 2, \ldots \right\}\) is dense in \((0, \infty)\) (see Theorem 1.1 of [1]). \(\square\)

3. Comparison of terms of logarithmic sequence with terms of other sequences

First we will show that the function \(L(x, b)\) is increasing in \(x > 0\) with fixed \(b > 0\). This property of the function \(L(x, b)\) will be later used in the proof of Theorem 3.3.
Theorem 3.1. Let \(a, b, c, \in \mathbb{R}^+\). Then

\[ L(c, b) \leq L(a, b) \iff c \leq a. \]

Proof. For \(0 < x \neq b\) we have

\[ L(x, b) = b \frac{x - 1}{\ln \frac{x}{b}}. \]

Thus \(L(x, b)\) is increasing with respect to \(x\) if and only if the function

\[ f(y) = b \frac{y - 1}{\ln y} \]

is increasing with respect to \(y\) \((y \neq 1)\), i.e. \(\frac{df}{dy} \geq 0\) for \(y > 0, y \neq 1\). This is equivalent to

\[ g(y) = \frac{1}{y} + \ln y - 1 \geq 0 \]

for each \(y > 0\). Since \(\frac{dg}{dy} = \frac{1}{y} - \frac{1}{y^2} = \frac{y - 1}{y^2}\) we obviously have \(\frac{dg}{dy} \leq 0\) for \(0 < y < 1\) and \(\frac{dg}{dy} \geq 0\) for \(y \geq 1\). Thus \(g(y)\) attains its minimum at \(y = 1\), i.e. \(g(y) \geq g(1) = 0\) for each \(y > 0\). \(\square\)

First we are going to compare the terms of a given logarithmic sequence with terms of another logarithmic sequence.

Theorem 3.2. Let \((a_n)_{n=1}^{\infty}\) and \((b_n)_{n=1}^{\infty}\) be such logarithmic sequences that \(a_1 = b_1\) and \(a_2 \geq b_2\). Then

\[ a_n \geq b_n \quad \text{and} \quad \frac{a_n}{a_{n-1}} \geq \frac{b_n}{b_{n-1}} \]

hold for every \(n \geq 2\).

Proof. We will proceed by induction. For \(n = 2\) the statement obviously holds. Assume that it holds for some \(n = k \geq 2\), i.e.

\[ a_k \geq b_k \quad \text{and} \quad \frac{a_k}{a_{k-1}} \geq \frac{b_k}{b_{k-1}}. \tag{3.1} \]

Let us consider the terms \(a_{k+1}, b_{k+1}\). Since both \((a_n)_{n=1}^{\infty}\) and \((b_n)_{n=1}^{\infty}\) are logarithmic sequences, we have

\[ a_k = L(a_{k-1}, a_{k+1}) \quad \text{and} \quad b_k = L(b_{k-1}, b_{k+1}) \]

Consequently

\[ \frac{a_k}{a_{k-1}} = L \left( \frac{a_{k+1}}{a_{k-1}}, 1 \right) \quad \text{and} \quad \frac{b_k}{b_{k-1}} = L \left( \frac{b_{k+1}}{b_{k-1}}, 1 \right). \tag{3.2} \]
We will use the notation 
\[ \alpha_1 = \frac{a_k}{a_{k-1}}, \alpha_2 = \frac{a_{k+1}}{a_{k-1}}, \beta_1 = \frac{b_k}{b_{k-1}} \quad \text{and} \quad \beta_2 = \frac{b_{k+1}}{b_{k-1}} \]
in the rest of the proof. Then (3.2) implies

\[
\frac{\alpha_1}{\beta_1} = \frac{L(\alpha_2, 1)}{L(\beta_2, 1)} = \frac{\alpha_2 - 1}{\beta_2 - 1} \cdot \ln \beta_2 = \frac{\alpha_2^\frac{1}{2} + 1}{\beta_2^\frac{1}{2} + 1} \cdot \frac{\alpha_2^\frac{1}{2} - 1}{\beta_2^\frac{1}{2} - 1} \cdot \ln \frac{\beta_2^\frac{1}{2}}{\alpha_2^\frac{1}{2}} = \ldots = \left( \prod_{k=1}^{n} \frac{\alpha_2^\frac{1}{k} + 1}{\beta_2^\frac{1}{k} + 1} \right) \cdot \frac{\alpha_2^\frac{1}{n} - 1}{\beta_2^\frac{1}{n} - 1} \cdot \ln \frac{\beta_2^\frac{1}{n}}{\alpha_2^\frac{1}{n}}.
\]
Taking into account that \( \frac{a+b}{b+1} \leq \frac{a}{b} \) holds in the case when \( a \geq b > 0 \), we obtain:

\[
\frac{\alpha_1}{\beta_1} \leq \left( \frac{\alpha_2}{\beta_2} \right)^\sum_{k=1}^{n} \frac{1}{k} \frac{L(\alpha_2^\frac{1}{k}, 1)}{L(\beta_2^\frac{1}{k}, 1)}.
\]
Taking limit for \( n \to \infty \) we obtain \( \frac{\alpha_1}{\beta_1} \leq \frac{\alpha_2}{\beta_2} \) as

\[
\lim_{n \to \infty} L\left( a^\frac{1}{n}, 1 \right) = 1 \quad \text{where} \quad a > 0.
\]
The inequality \( \frac{\alpha_1}{\beta_1} \leq \frac{\alpha_2}{\beta_2} \) is equivalent with the inequality \( \frac{a_{k+1}}{a_k} \geq \frac{b_{k+1}}{b_k} \). Since \( a_k \geq b_k \) using the induction assumption (3.1) we obtain \( a_{k+1} \geq b_{k+1} \) which completes the proof.

The next theorem generalizes the previous one.

**Theorem 3.3.** Let \((a_n)_{n=1}^\infty\) be a logarithmic sequence and let a sequence \((b_n)_{n=1}^\infty\) fulfils the following conditions

\[
b_1 = a_1, \quad b_2 \leq a_2 \quad \text{and} \quad b_n \geq L(b_{n-1}, b_{n+1}) \quad \text{for} \quad n \geq 2 \quad (3.3)
\]
Then for every positive integer \( n \) the inequality

\[
a_n \geq b_n
\]
holds.

**Proof.** Let \( k \geq 0 \) be a given integer. Define the sequence \((a_{k,n})_{n=1}^\infty\) as follows:

\[
a_{k,1} = b_{k+1}, \quad a_{k,2} = b_{k+2} \quad \text{and} \quad a_{k,n} = L(a_{k,n-1}, a_{k,n+1}) \quad \text{for} \quad n \geq 2. \quad (3.4)
\]
Thus the sequence \((a_{k,n})_{n=1}^\infty\) is logarithmic for every \( k \geq 0 \).
We will show that

\[
a_{k,n} \leq a_{k+n} \quad \text{and} \quad b_{k+3} \leq a_{k,3} \quad (3.5)
\]
Distribution of terms of a logarithmic sequence

holds for every integer \( k \geq 0 \) and positive integer \( n \). We will proceed by induction with respect to \( k \).

For \( k = 0 \) from (3.3), (3.4) we have

\[
a_{0,1} = b_1 = a_1, \quad a_{0,2} = b_2 \leq a_2.
\]

The assumption that both sequences \((a_n)_{n=1}^{\infty}\) and \((a_{0,n})_{n=1}^{\infty}\) are logarithmic and Theorem 3.2 imply that for every \( n \in \mathbb{N} \) the inequality

\[
a_{0,n} \leq a_n
\]

holds. On the other hand, (3.3) and (3.4) imply

\[
L(b_3, b_1) \leq b_2 = a_{0,2} = L(a_{0,3}, a_{0,1}) = L(a_{0,3}, b_1),
\]

and consequently, using Theorem 3.1, we obtain

\[
b_3 \leq a_{0,3}.
\]

Suppose that for some \( k = l \geq 0 \) inequalities (3.5) hold. In the case \( k = l + 1 \) we obtain

\[
a_{l+1,1} = b_{l+2} = a_{l,2} \quad \text{and} \quad a_{l+1,2} = b_{l+3} \leq a_{l,3}.
\]

By use of Theorem 3.2 and induction assumption we obtain

\[
a_{l+1,n} \leq a_{l,n+1} \leq a_{l+1,n}
\]

for every \( n \in \mathbb{N} \). On the other hand, (3.3) and (3.4) imply

\[
L(b_{l+4}, b_{l+2}) \leq b_{l+3} = a_{l+1,2} = L(a_{l+1,3}, a_{l+1,1}).
\]

As \( b_{l+2} = a_{l+1,1} \), Theorem 3.1 implies

\[
b_{l+4} \leq a_{l+1,3}.
\]

Thus we proved (3.5) by induction. Finally, from (3.5) we obtain

\[
b_k \leq a_{k-3,3} \leq a_k
\]

for every \( k \geq 3 \).

\[ \square \]

The proof of the following theorem is an application of the previous one.

**Theorem 3.4.** Let \((a_n)_{n=1}^{\infty}\) be such a logarithmic sequence that \( a_1 < a_2 \). Then the series \( \sum_{n=1}^{\infty} \frac{1}{a_n} \) converges and

\[
\sum_{n=1}^{\infty} \frac{1}{a_n} < \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{(\sqrt{a_2} - \sqrt{a_1})^2} \frac{\pi^2}{6}
\]

holds.
Proof. Define the sequence \((b_n)_{n=1}^{\infty}\) by:

\[ b_1 = a_1, \quad b_2 = a_2 \quad \text{and} \quad b_n = \frac{1}{2} (b_{n-1} + b_{n+1}) \quad \text{for} \quad n \geq 2. \]

As

\[ M_{\frac{1}{2}}(b_{n-1}, b_{n+1}) \geq L(b_{n-1}, b_{n+1}), \]

we have \(b_n \geq L(b_{n-1}, b_{n+1})\). Thus the sequence \((b_n)_{n=1}^{\infty}\) fulfills the assumptions of Theorem 3.2.

Consequently \(b_n \leq a_n\) for every \(n \in \mathbb{N}\). Using ([2] Th.1.1) we have

\[ b_n = \left( (n-1)\sqrt{b_2} - (n-2)\sqrt{b_1} \right)^2, \]

i.e. for every \(n > 2\)

\[ b_n = \left( (n-2)(\sqrt{b_2} - \sqrt{b_1}) + \sqrt{b_2} \right)^2 > (n-2)^2 \left( \sqrt{b_2} - \sqrt{b_1} \right)^2 = \]

\[ = (n-2)^2 (\sqrt{a_2} - \sqrt{a_1})^2 \]

holds. Finally we obtain

\[
\sum_{n=1}^{\infty} \frac{1}{a_n} \leq \sum_{n=1}^{\infty} \frac{1}{b_n} < \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{(\sqrt{a_2} - \sqrt{a_1})^2} \frac{\pi^2}{6}.
\]

\(\square\)

References


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