On group rings with restricted minimum condition

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Submitted 28 September 2007; Accepted 18 December 2007

Abstract

In this paper we investigate the group rings $\mathbb{R}G$ satisfying the restricted minimum condition.

Keywords: restricted minimum condition, group ring

MSC: 16S34

1. Results

Let $R$ be an associative ring with unit element. $R$ is said to satisfy the left restricted minimum condition, if for each nontrivial ideal $J$ of $R$ the ring $R/J$ is left artinian. In this paper we consider the group rings with left restricted minimum condition, in the case when $\mathbb{R}G$ itself is not left artinian.

We prove the following:

Theorem 1.1. Let $G$ be a group with non-trivial center and let $R$ be a commutative ring with unit element. If the group ring $\mathbb{R}G$ satisfies the left restricted minimum condition, then $R$ is left artinian and either $G$ is finite, or $G$ is the infinite cyclic group.

For group algebras the converse assertion is also true.

Theorem 1.2. Let $G$ be a group with non-trivial center and let $R$ be a field. The group algebra $\mathbb{R}G$ satisfies the left restricted minimum condition if and only if either $G$ is finite, or $G$ is the infinite cyclic group.

By $A(\mathbb{R}G)$ we mean the augmentation ideal of $\mathbb{R}G$, that is the kernel of the ring homomorphism $\phi : \mathbb{R}G \to R$ sending each group element to 1. It is easy to see that
A(RG) is a free R-module in which the set of the elements \( g - 1 \) with \( 1 \neq g \in G \) form a basis. For a normal subgroup \( H \) of \( G \) we denote by \( I(H) \) the ideal of \( RG \) generated by all elements of the form \( h - 1 \) with \( h \in H \). As it is well-known, \( I(H) \) is the kernel of the natural epimorphism \( \overline{\phi} : RG \to R[G/H] \) induced by the group homomorphism \( \phi \) of \( G \) onto \( G/H \), furthermore

\[
RG/I(H) \cong R[G/H], \tag{1.1}
\]

and \( I(G) = A(RG) \).

The commutator subgroup and the center of the group \( G \) will be denoted by \( G' \) and \( \zeta(G) \), respectively.

### 2. Proof of Theorems

We need the following two statements.

**Proposition 2.1** (Theorem 4.12 in [2]). If \( G \) is a group whose center has finite index \( n \), then \( G' \) is finite and \( (G')^n = 1 \).

**Proposition 2.2** (Theorem 4.33 in [2]). An infinite group has each non-trivial subgroup of finite index if and only if it is infinite cyclic.

**Proof of Theorem 1.1.** It is well-known that the group ring \( RG \) is left artinian if and only if \( R \) is left artinian and \( G \) is finite. Assume that \( RG \) satisfies the left restricted minimum condition. According to (1.1) for every normal subgroup \( H \) the factor group \( G/H \) is finite and from the isomorphism \( RG/A(RG) \cong R \) it follows that \( R \) is left artinian. Furthermore, \( RG/I(\zeta(G)) \) is left artinian and therefore, by (1.1), \( G/\zeta(G) \) is finite. Then Proposition 2.1 guarantees that \( G' \) is finite. If \( G' \neq 1 \) then, by (1.1) \( G/G' \) is finite, and so \( G \) is finite. On the other hand, if \( G \) is abelian and infinite, then by (1.1) we have that every non-trivial subgroup of \( G \) has finite index. But then Proposition 2.2 states that \( G \) is the infinite cyclic group and the proof of the theorem is complete. \( \square \)

Let \( R \) be an euclidean ring with the euclidean norm \( \varphi \) such that \( \varphi(ab) \geq \varphi(a) \) for all \( a \neq 0, b \neq 0 \) \((a, b \in R\).) Then \( R \) is a principal ideal ring. Let \( I = (r) \) and \( J = (s) \) be the ideals of \( R \) generated by the element \( r \) and \( s \) respectively, and assume that \( I \supseteq J \). Then \( s = rt \) for a suitable \( t \in R \), and \( \varphi(s) = \varphi(rt) \geq \varphi(r) \). It is easy to see that \( \varphi(e) = 1 \) if and only if \( e \) is an unit in \( R \) and that \( I = J \) if and only if \( \varphi(r) = \varphi(s) \).

Let \( J = (s) \) be an arbitrary ideal of an euclidean ring \( R \) and let

\[
\overline{R} \supseteq J_1 \supseteq J_2 \supseteq \ldots \supseteq J_n \supseteq \ldots \supseteq \bigcap_{i=1}^{\infty} J_i = J_\omega \tag{2.1}
\]

a sequence of ideals, where \( \overline{R} = R/J \) and \( \omega \) the first limit ordinal. Denote by \( J_k \) the inverse image of \( J_k \) in \( R \) \((k = 1, 2, \ldots \text{ or } k = \omega) \). Then \( J_k \)'s are principal ideals.


and, in view of (2.1) we have that
\[ R \supseteq J_1 \supseteq J_2 \supseteq \ldots \supseteq J_n \supseteq \ldots \supseteq J_\omega \supseteq J = (s). \] (2.2)

Suppose that \( J_k = (s_k) \). Since \( J_k \supseteq J = (s) \), so \( \varphi(s) \geq \varphi(s_k) \) for all \( k \) (\( k = 1, 2, \ldots \) and \( k = \omega \)). But \( \varphi(s) \) and \( \varphi(s_k) \) are non-negative integers, therefore there exists a natural number \( n \) such that \( \varphi(s_n) = \varphi(s_{n+1}) = \ldots = \varphi(s) \). Thus the sequence (2.2) has finite length and consequently, the sequence (2.1) is finite, too. It follows that for each ideal \( J \) of \( R \) the ring \( R/J \) is artinian, and we have

**Lemma 2.3.** Euclidean rings satisfy the restricted minimum condition.

It was proved in [1] that the group algebra of the infinite cyclic group over a field is an euclidean ring. Hence, Theorem 1.2 is a direct consequence of Lemma 2.3 and Theorem 1.1.

**References**
