COMPOSITION OPERATORS BETWEEN GENERALLY WEIGHTED BLOCH SPACE AND \( Q_{\log}^q \) SPACE

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Abstract. Let \( \varphi \) be a holomorphic self-map of the open unit disk \( D \) on the complex plane and \( p, q > 0 \). In this paper, the boundedness and compactness of composition operator \( C_\varphi \) from generally weighted Bloch space \( B_{\log}^p \) to \( Q_{\log}^q \) are investigated.

1. Introduction and preliminaries

Suppose that \( D \) is the unit disc on the complex plane, \( \partial D \) its boundary and \( \varphi \) a holomorphic self-map of \( D \). We denote by \( H(D) \) the space of all holomorphic functions on \( D \), denote by \( dm(z) \) the normalized Lebesgue area measure and define the composition operator \( C_\varphi \) on \( H(D) \) by \( C_\varphi f = f \circ \varphi \).

For \( 0 < p \leq \infty \), the Hardy space \( H^p \) is the Banach space of analytic functions on \( D \) such that

\[
\|f\|_{H^p}^p = \sup_{r \in (0,1)} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty, \quad 0 < p < \infty,
\]

and

\[
\|f\|_{H^\infty} = \sup_{z \in D} |f(z)| < \infty.
\]

For more details see [15] and [16].

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We say that \( f \in H(D) \) belongs to \( \text{BMOA}_{\log} \) if \( f \in H^2 \) and has weighted bounded mean oscillation, i.e.

\[
\|f\|_{\text{BMOA}_{\log}} = \sup_{I \subseteq \partial D} \frac{\left(\log \frac{2}{|I|}\right)^2}{|I|} \int_{S(I)} |f'(z)|^2 \log \frac{1}{|z|} \, dm(z) < \infty,
\]

where

\[
S(I) = \{z \in D : 1 - |I| \leq |z| < 1, \frac{z}{|z|} \in I\}
\]
is the Carleson square of the arc \( I \) and \(|I|\) its length.

By definition it is immediate that \( \text{BMOA}_{\log} \) is exactly \( Q^1_{\log} \). In [10], the above relation helped to describe the pointwise multipliers of the Möbius invariant Banach spaces \( Q_q, q \in [0, 1] \), consisting of \( f \in H(D) \), such that

\[
\|f\|_{Q_q} = |f(0)| + \sup_{\alpha \in D} \int_D |f'(z)|^2 g^q(z, \alpha) \, dm(z) < \infty,
\]

where \( g(z, \alpha) = \log \frac{1}{|\phi_\alpha(z)|} \) is the Green’s function and \( \phi_\alpha(z) = \frac{\alpha - z}{1 - \overline{\alpha} z} \). For more details on these spaces see for example [2] and the two monographs [11] and [12].

The space of analytic functions on \( D \) such that

\[
\|f\|_{B_{\log}} = |f(0)| + \sup_{z \in D} |f'(z)|(1 - |z|^2) \log \frac{2}{1 - |z|^2} < \infty
\]
is called weighted Bloch space \( B_{\log} \).

\( B_{\log} \) and \( \text{BMOA}_{\log} \) first appeared in the study of boundedness of the Hankel operators on the Bergman space

\[
A^1 = \{f \in H(D) : \int_D |f(z)| \, dm(z) < \infty\}
\]
and the Hardy space \( H^1 \), respectively. \( \text{BMOA}_{\log} \) also appeared in the study of a Volterra type operator. For more details [1], [3], [8] and [9].

In [13], Yoneda studied the composition operators from \( B_{\log} \) to \( \text{BMOA}_{\log} \). He found one sufficient and a different necessary condition for the boundedness of the composition operators from \( B_{\log} \) to \( \text{BMOA}_{\log} \). So it is natural to ask for the approximate conditions that characterize boundedness and compactness of the composition operators \( C_\varphi : B_{\log}^p \to \text{BMOA}_{\log} \).

In [6], we introduced the space \( B_{\log}^p \). The space of analytic functions on \( D \) such that

\[
\|f\|_{B_{\log}^p} = |f(0)| + \sup_{z \in D} |f'(z)|(1 - |z|^2)^p \log \frac{2}{1 - |z|^2} < \infty
\]
is called generally weighted Bloch space \( B_{\log}^p \). When \( p = 1 \), the space \( B_{\log}^p \) is just the weighted Bloch space \( B_{\log} \).

In [5], Petros Galanopoulos considered the space \( Q_{\log}^q \), \( q > 0 \), the spaces of analytic functions on the unit disc such that

\[
\|f\|_* = \sup_{I \subseteq \partial D} \frac{(\log \frac{2}{|I|})^2}{|I|^q} \int_{S(I)} |f'(z)|^2 (\log \frac{1}{|z|})^q \, dm(z) < \infty.
\]

In this paper, we consider composition operator \( C_\varphi \) from generally weighted Bloch space \( B_{\log}^p(D) \) to \( Q_{\log}^q(D) \). We find a necessary and sufficient condition for
Taylor coefficients of a function in $B^p_\log$. Using the results for the Hadamard gap series and following a technique used before in the Bloch space in [7], we construct two functions $f, g \in B^p_\log$ such that for each $z \in D$,

$$|f'(z)| + |g'(z)| \geq \frac{C}{(1 - |z|)^p \log \frac{2}{1 - |z|}},$$

where $C$ is a positive constant. Using this fact we prove the following theorems:

**Theorem 1.1.** Let $p, q > 0$. If $\varphi$ is an analytic self-map of the unit disc, then the induced composition operator $C_\varphi : B^p_\log \to Q^q_\log$ is bounded if and only if

$$\sup_{\alpha \in D} (\log \frac{2}{1 - |\alpha|^2})^2 \int_D |\varphi'(z)|^2 \frac{(1 - |\phi_\alpha(z)|^2)^q}{(1 - |\varphi(z)|^2)^2 p (\log \frac{2}{1 - |\varphi(z)|^2})^2} \, dm(z) < \infty.$$

**Theorem 1.2.** Let $p, q > 0$. If $\varphi$ is an analytic self-map of the unit disc, then the induced composition operator $C_\varphi : B^p_\log \to Q^q_\log$ is compact if and only if $\varphi \in Q^q_\log$ and

$$\lim_{r \to 1} \sup_{\alpha \in D} (\log \frac{2}{1 - |\alpha|^2})^2 \int_{\{|\varphi(z)| > r\}} |\varphi'(z)|^2 \frac{(1 - |\phi_\alpha(z)|^2)^q}{(1 - |\varphi(z)|^2)^2 p (\log \frac{2}{1 - |\varphi(z)|^2})^2} \, dm(z) = 0.$$

By the definition of $B^p_\log$, we can easily obtain the following corollaries.

**Corollary 1.3.** Let $q > 0$. If $\varphi$ is an analytic self-map of the unit disc, then the induced composition operator $C_\varphi : B^p_\log \to Q^q_\log$ is bounded if and only if

$$\sup_{\alpha \in D} (\log \frac{2}{1 - |\alpha|^2})^2 \int_D |\varphi'(z)|^2 \frac{(1 - |\phi_\alpha(z)|^2)^q}{(1 - |\varphi(z)|^2)^2} \, dm(z) < \infty.$$

**Corollary 1.4.** Let $q > 0$. If $\varphi$ is an analytic self-map of the unit disc, then the induced composition operator $C_\varphi : B^p_\log \to Q^q_\log$ is compact if and only if $\varphi \in Q^q_\log$ and

$$\lim_{r \to 1} \sup_{\alpha \in D} (\log \frac{2}{1 - |\alpha|^2})^2 \int_{\{|\varphi(z)| > r\}} |\varphi'(z)|^2 \frac{(1 - |\phi_\alpha(z)|^2)^q}{(1 - |\varphi(z)|^2)^2} \, dm(z) = 0.$$

Throughout the remainder of this paper $C$ will denote a positive constant, the exact value of which will vary from one appearance to the next.

2. Main results

Let $f$ be a holomorphic function in $D$ with the gap series expansion

$$f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}, \quad z \in D,$$

where for a constant $\lambda > 1$, the natural numbers $n_k$ satisfy

$$\frac{n_{k+1}}{n_k} \geq \lambda, \quad k \geq 1.$$
Lemma 2.1. Let $f$ be a holomorphic function in $D$ with (a) and (b). Then for $p > 0$, $f \in B^p_{\log}$ if and only if
\[
\limsup_{k \to \infty} |a_k| \cdot n_k^{1-p} \cdot \log n_k < \infty.
\]

Proof. Let $f$ be a holomorphic function in $D$, $f(z) = \sum_{k \geq 0} a_k z^k \in B^p_{\log}$. Since $a_k = \frac{1}{2\pi i} \int_0^{2\pi} f'(re^{i\theta}) r^{1-k} e^{i(1-k)\theta} d\theta$, then
\[
|a_k| \leq \frac{1}{2\pi} \int_0^{2\pi} |f'(re^{i\theta})| r^{1-k} d\theta
\leq \frac{\|f\|_{B^p_{\log}} \cdot r^{1-k}}{k(1-r)^p \log \frac{1}{1-r}}.
\]

Let $r = 1 - \frac{1}{k}$, then
\[
|a_k| \leq \frac{\|f\|_{B^p_{\log}} (1 - \frac{1}{k})^{1-k}}{k^{1-p} \log k} = \frac{\|f\|_{B^p_{\log}} (1 + \frac{1}{k})^{-k}(1 - \frac{1}{k})}{k^{1-p} \log k},
\]

then
\[
\limsup_{k \to \infty} |a_k| \cdot k^{1-p} \cdot \log k \leq e \cdot \|f\|_{B^p_{\log}} < \infty.
\]

Conversely, Since $f(z) = \sum_{k \geq 0} a_k z^{n_k}$, then
\[
|z f'(z)| \leq \sum_{k \geq 0} |a_k| n_k |z|^{n_k} \leq C \sum_{k \geq 0} \frac{n_k^p}{\log n_k} |z|^{n_k},
\]

\[
\frac{n_{k+1}^p \log n_{k+1}}{n_k^p \log n_{k+1}} = (\frac{n_{k+1}}{n_k})^p \frac{\log n_{k+1}}{\log n_k}^{-1} = (\frac{n_{k+1}}{n_k})^p (1 + \frac{\log n_{k+1}}{\log n_k})^{-1} = \lambda^p (1 + \frac{\log \lambda}{\log n_k})^{-1}.
\]

Then for each $\varepsilon \in (0, 1)$, there exists $k_0$ such that when $k \geq k_0$ we have
\[
\frac{n_{k+1}^p \log n_{k+1}}{n_k^p \log n_{k+1}} \geq (1 - \varepsilon) \lambda^p
\]

(2.1)

thus
\[
\frac{n_k^p}{\log n_k} \leq \frac{1}{(1 - \varepsilon) \lambda^p} \cdot \frac{n_{k+1}^p}{\log n_{k+1}}.
\]

\[
\frac{|z f'(z)| \log \frac{1}{1-|z|}}{1-|z|} \leq C (\sum_{k \geq 0} \frac{n_k^p}{\log n_k} |z|^{n_k}) (\sum_{n \geq 0} |z|^n) |z| \sum_{n \geq 0} \frac{|z|^n}{n+1}
\leq C'(\sum_{n \geq n_0} (\sum_{n_0 \leq n} \frac{n_k^p}{\log n_k} |z|^{n_k})) \sum_{n \geq 0} \frac{|z|^n}{n+1}.
\]

Let $k'$ be a positive integer number such that $n_{k'} \leq n \leq n_{k'+1}$, we fix $(1 - \varepsilon) \lambda^p > 1$, $\varepsilon > 0$, then we get an index $k_0$ such that (2.1) holds.
If \( k' \geq k_0 \), then
\[
\sum_{n_k \leq n} \frac{n_k^p}{\log n_k} = \sum_{k \leq k_0} \frac{n_k^p}{\log n_k} + \sum_{k' > k_0} \frac{n_k^p}{\log n_k} \\
\leq C \frac{n_k^p}{\log n} + \frac{n_k^p}{\log n} \cdot \sum_{k' > k_0} \frac{1}{\lambda^p(1-\varepsilon)^{k'-k}} \\
\leq C \frac{n_k^p}{\log n} + \frac{n_k^p}{\log n} \cdot \frac{1}{\lambda^p(1-\varepsilon)^{k'-(k_0+1)}} (1 - [\lambda^p(1-\varepsilon)]^{k'-k_0}) \\
= C \frac{n_k^p}{\log n} + \frac{n_k^p}{\log n} \cdot \frac{\lambda^p(1-\varepsilon) - 1}{\lambda^p(1-\varepsilon) - 1} \\
\leq (C + 1) \frac{n_k^p}{\log n} + \frac{n_k^p}{\log n} \cdot \frac{1}{\lambda^p(1-\varepsilon) - 1}.
\]

Since
\[
\sum_{n=0}^{\infty} (n+1)^p |z|^n \leq \frac{C}{(1-|z|)^{1+p}}, \quad z \in D,
\]
thus
\[
\frac{|zf'(z)| \log \frac{1}{1-|z|}}{1-|z|} \leq C \left( \sum_{n \geq 3} \frac{n^p}{\log n} |z|^n \right) \left( \sum_{n \geq 0} \frac{|z|^n}{n+1} \right) \\
\leq C \sum_{n \geq 3} n^p |z|^n \\
= C |z| \sum_{n \geq 2} (n+1)^p |z|^n \\
\leq C \frac{|z|}{(1-|z|)^{1+p}}.
\]

\( \square \)

**Lemma 2.2.** There exist \( f, g \in B^{p}_{\log} \) such that
\[
|f'(z)| + |g'(z)| \geq \frac{C}{(1-|z|)^p \log \frac{2}{1-|z|}}.
\]

**Proof.** We consider the function
\[
f(z) = Kz + \sum_{j \geq 1} \frac{q^{(j+k_0)(p-1)+\frac{p}{j}}}{\log q^{j+k_0}} - z^{j+k_0}
\]
for \( q \) an appropriately large integer, \( K \) a properly small chosen positive constant and \( k_0 \) the index for which (2.1) holds for the sequence \( n_j \) such that \( n_j = q^{j+k_0} \).
So this function is a member of the \( B^{p}_{\log} \) space.
\[
1 - q^{-(k+k_0)} \leq |z| < 1 - q^{-(k+k_0+\frac{1}{2})} \quad (k \geq 1),
\]
\[ |f'(z)| = |K + \sum_{j \geq 1} q^{(j+k_0)p+\frac{p}{2}} \log q^{j+k_0} z^{q(j+k_0)-1}| \]
\[ = |K + \sum_{j=1}^{k-1} q^{(j+k_0)p+\frac{p}{2}} \log q^{j+k_0} z^{q(j+k_0)-1} \]
\[ + q^{(k+k_0)p+\frac{p}{2}} \log q^{k+k_0} z^{q(k+k_0)-1} \sum_{j=k+1}^\infty \frac{q^{(j+k_0)p+\frac{p}{2}}}{\log q^{j+k_0}} |z|^{q(j+k_0)-1} | \]
\[ \geq \frac{q^{(k+k_0)p+\frac{p}{2}}}{\log q^{k+k_0}} |z|^{q^{k+k_0}} - (K + \sum_{j=1}^{k-1} \frac{q^{(j+k_0)p+\frac{p}{2}}}{\log q^{j+k_0}} |z|^{q(j+k_0)}) \]
\[ - \sum_{j=k+1}^\infty \frac{q^{(j+k_0)p+\frac{p}{2}}}{\log q^{j+k_0}} |z|^{q(j+k_0)} \]
\[ = I_1 - I_2 - I_3. \]

Since
\[ 1 - q^{-(k+k_0)} \leq |z| < 1 - q^{-(k+k_0+\frac{1}{2})}. \]

Thus
\[ (1 - q^{-(k+k_0)})q^{k+k_0} \leq |z|q^{k+k_0} < (1 - q^{-(k+k_0+\frac{1}{2})})q^{k+k_0}. \]

Then
\[ \frac{1}{3} \leq |z|q^{k+k_0} < \left(\frac{1}{2}\right)q^{-\frac{1}{2}}. \]

\[ I_1 = q^{(k+k_0)p+\frac{p}{2}} \log q^{k+k_0} \]
\[ \geq \frac{1}{3} q^{(k+k_0)p+\frac{p}{2}} \log q^{k+k_0}. \]

\[ I_2 = K + \sum_{j=1}^{k-1} \frac{q^{(j+k_0)p+\frac{p}{2}}}{\log q^{j+k_0}} |z|^{q(j+k_0)} \]
\[ \leq K \cdot \frac{q^{(k+k_0)p+\frac{p}{2}}}{\log q^{k+k_0}} \left(1 - \frac{1}{q^{k+k_0+\frac{1}{2}}}\right) + \frac{q^{(k+k_0)p+\frac{p}{2}}}{\log q^{k+k_0}} \cdot \sum_{j=1}^{k-1} \frac{1}{((1-\varepsilon)q^p)^{k-j}} \]
\[ \leq \frac{q^{(k+k_0)p+\frac{p}{2}}}{\log q^{k+k_0}} \cdot \frac{1}{(1-\varepsilon)q^p - 1} + K \cdot \frac{q^{(k+k_0)p+\frac{p}{2}}}{\log q^{k+k_0}}. \]
where $n$.

Now with a similar argument for the function $K = j |f(q^j(z))| \geq q^{j+k+1} \sum_{j=0}^{\infty} \log q^{j+k+1} |z|^{q^j}$.

Thus

$I_3 = \sum_{j=k+1}^{\infty} \frac{q^{(j+k_0)p+\frac{p}{2}}}{\log q^{j+k_0}} |z|^{q^j}$

$= \sum_{j=0}^{\infty} \frac{q^{(j+k+1+k_0)p+\frac{p}{2}}}{\log q^{j+k+1+k_0}} |z|^{q^{j+k+1+k_0}}$

$= q^{(k+1+k_0)p+\frac{p}{2}} |z|^{q^{k+1+k_0}} \sum_{j=0}^{\infty} \frac{q^{jp}}{\log q^{j+k+1+k_0}} |z|^{q^j}$

$\leq \frac{q^{(k+1+k_0)p+\frac{p}{2}}}{\log q^{k+k_0}} |z|^{q^{k+1+k_0}} \sum_{j=0}^{\infty} (q^p |z|^q) |q^{(k+2)-q^{(k+1)})}j$

$= q^{(k+1+k_0)p+\frac{p}{2}} \frac{1 - q^p |z|^q}{1 - q^p |z|^q^{(k+2)-q^{(k+1)}}}$

$= q^{(k+k_0)p+\frac{p}{2}} \frac{q^p |z|^{q^{k+k_0}} q}{1 - q^p |z|^q^{(k+2)-q^{(k+1)}}}$

$\leq \frac{q^{(k+k_0)p+\frac{p}{2}}}{\log q^{k+k_0}} q^{\frac{1}{2}} |1 - q^p |z|^q^{(k+2)-q^{(k+1)}}$.

Thus

$|f'(z)| \geq \frac{q^{(k+k_0)p+\frac{p}{2}}}{\log q^{k+k_0}} \left( \frac{1}{3} - \frac{1}{(1 - \varepsilon)q^p - 1} - K - \frac{q^p(\frac{1}{2})q^{\frac{1}{2}}}{1 - q^p(\frac{1}{2})(q^{\frac{1}{2}}-q^{\frac{1}{2}})} \right)$.

If $K$ is so small that

$\frac{1}{3} - \frac{1}{(1 - \varepsilon)q^p - 1} - K - \frac{q^p(\frac{1}{2})q^{\frac{1}{2}}}{1 - q^p(\frac{1}{2})(q^{\frac{1}{2}}-q^{\frac{1}{2}})} > 0$,

then we have

$|f'(z)| \geq C \frac{q^{(k+k_0)p+\frac{p}{2}}}{\log q^{k+k_0}} \geq C \frac{1}{(1 - |z|)^p \log \frac{2}{1 - |z|}}$.

Now with a similar argument for the function

$g(z) = \sum_{j \geq 1} \frac{q^{(j+k_0)(p-1)+\frac{p}{2}}}{\log q^{j+k_0+\frac{1}{2}}} |z|^{q^{j+k_0+\frac{1}{2}}}$,

where $n_j = q^{j+k_0+\frac{1}{2}}$, for $q$ a large positive integer, $k = 1, 2, \ldots$, $1 - q^{-(k+k_0+\frac{1}{2})} \leq |z| < 1 - q^{-(k+k_0+1)}$,

we get

$|g'(z)| \geq \frac{C}{(1 - |z|)^p \log \frac{2}{1 - |z|}}$. 
In the case where \( f', g' \) have common zeros (\( \neq 0 \)) in \( \{ |z| < 1 - q^{-(k+\ell_0+1)} \} \), we can prove that the function \( g(e^{i\theta}z) \) for suitable \( \theta \).

In order to understand better the \( Q^q_{log} \), we need the following definition introduced in [14].

**Definition 2.3.** A positive Borel measure on \( D \) is called an \( s \)-logarithmic \( q \)-Carleson measure (\( q, s > 0 \)) if

\[
\sup_{I \subseteq \partial D} \frac{\mu(S(I))(\log \frac{2}{|I|})^s}{|I|^q} < \infty.
\]

In [14], the sufficient and necessary condition of the measure is given as follows.

**Lemma 2.4.** \( \mu \) is an \( s \)-logarithmic \( q \)-Carleson measure on \( D \) if and only if

\[
\sup_{\alpha \in D} (\log \frac{2}{1 - |\alpha|^2})^s \int_D |\phi'_\alpha(z)|^q d\mu(z) < \infty.
\]

Using techniques well known to mathematics and by Lemma 2.4 we can prove the following proposition.

**Proposition 2.5.** Let \( q > 0 \). Then the following are equivalent:

(i) \( f \in Q^q_{log} \);

(ii) \( \sup_{\alpha \in D} (\log \frac{2}{1 - |\alpha|^2})^2 \int_D |f'(z)|^2 (1 - |\phi_\alpha(z)|^2)^q dm(z) < \infty \);

(iii) \( \sup_{\alpha \in D} (\log \frac{2}{1 - |\alpha|^2})^2 \int_D |f'(z)|^2 g^q(z, \alpha) dm(z) < \infty \).

**Theorem 2.6.** Let \( p, q > 0 \). If \( \varphi \) is an analytic self-map of the unit disc, then the induced composition operator \( C_\varphi : B^p_{log} \to Q^q_{log} \) is bounded if and only if

\[
\sup_{\alpha \in D} (\log \frac{2}{1 - |\alpha|^2})^2 \int_D |\varphi'(z)|^2 \left( \frac{(1 - |\phi_\alpha(z)|^2)^q}{(1 - |\varphi(z)|^2)^2 p (\log \frac{2}{1 - |\varphi(z)|^2})^2} \right) dm(z) < \infty. \tag{2.2}
\]

**Proof.** Firstly we assume that (2.2) holds, by Proposition 2.5, then for \( f \in B^p_{log} \),

\[
\sup_{\alpha \in D} \left( \frac{2}{1 - |\alpha|^2} \right)^2 \int_D |(f \circ \varphi)'(z)|^2 (1 - |\phi_\alpha(z)|^2)^q dm(z)
\]

\[
= \sup_{\alpha \in D} \left( \frac{2}{1 - |\alpha|^2} \right)^2 \int_D |f'(\varphi(z))|^2 |\varphi'(z)|^2 (1 - |\phi_\alpha(z)|^2)^q dm(z)
\]

\[
\leq \sup_{\alpha \in D} \left( \frac{2}{1 - |\alpha|^2} \right)^2 \int_D |\varphi'(z)|^2 \left( \frac{(1 - |\phi_\alpha(z)|^2)^q}{(1 - |\varphi(z)|^2)^2 p (\log \frac{2}{1 - |\varphi(z)|^2})^2} \right) dm(z) \cdot \| f \|_{B^p_{log}}^2.
\]

By (2.2), then \( C_\varphi f \in Q^q_{log} \), thus \( C_\varphi : B^p_{log} \to Q^q_{log} \) is bounded.

Conversely, we assume that \( C_\varphi : B^p_{log} \to Q^q_{log} \) is bounded, for \( f \in B^p_{log} \), \( C_\varphi f \in Q^q_{log} \), by Lemma 2.2, there exist \( f, g \in B^p_{log} \) such that

\[
|f'(z)| + |g'(z)| \geq C \left( \frac{1 - |z|^p}{\log \frac{2}{1 - |z|^p}} \right).
\]
Then

\[ \infty > \sup_{\alpha \in D} \left( \frac{2}{1 - |\alpha|^2} \right)^2 \int_D \left[ 2\left| (f \circ \varphi)'(z) \right|^2 + (g \circ \varphi)'(z) \right]^2 (1 - |\phi_\alpha(z)|^2) \varphi \, dm(z) \]

\[ \geq \sup_{\alpha \in D} \left( \frac{2}{1 - |\alpha|^2} \right)^2 \int_D \left[ |(f \circ \varphi)'(z) + (g \circ \varphi)'(z)| \right]^2 (1 - |\phi_\alpha(z)|^2) \varphi \, dm(z) \]

\[ = \sup_{\alpha \in D} \left( \frac{2}{1 - |\alpha|^2} \right)^2 \int_D \left[ |f'(\varphi(z))| + |g'(\varphi(z))| \right]^2 |\varphi'(z)|^2 (1 - |\phi_\alpha(z)|^2) \varphi \, dm(z) \]

\[ \geq C \sup_{\alpha \in D} \left( \frac{2}{1 - |\alpha|^2} \right)^2 \int_D \left| \varphi'(z) \right|^2 \frac{(1 - |\phi_\alpha(z)|^2)^q}{\left( 1 - |\varphi(z)|^2 \right)^2 (1 - |\phi_\alpha(z)|^2)^q} \, dm(z). \]

\[ \Box \]

**Remark 2.7.** Since every element of \( Q^q_{\log} \) satisfies the following radial growth condition:

\[ |f(z) - f(0)| \leq C \log(\log \frac{1}{1 - |z|}) \|f\|_{Q^q_{\log}}, \quad C > 0, \]

then \( C_\varphi : B^p_{\log} \to Q^q_{\log} \) is compact if and only if for every sequence \( \{f_n\}_{n \in N} \subseteq Q^q_{\log} \), bounded in norm and \( f_n \to 0 \) as \( n \to \infty \), uniformly on compact subsets of the unit disk, then \( \|C_\varphi(f_n)\|_{Q^q_{\log}} \to 0 \) as \( n \to \infty \).

This is similar to [4].

We give the characterization of compactness.

**Theorem 2.8.** Let \( p, q > 0 \). If \( \varphi \) is an analytic self-map of the unit disc, then the induced composition operator \( C_\varphi : B^p_{\log} \to Q^q_{\log} \) is compact if and only if \( \varphi \in Q^q_{\log} \) and

\[ \limsup_{r \to 1} \sup_{\alpha \in D} \left( \frac{2}{1 - |\alpha|^2} \right)^2 \int_{\{\varphi(z) > r\}} |\varphi'(z)|^2 \frac{(1 - |\phi_\alpha(z)|^2)^q}{\left( 1 - |\varphi(z)|^2 \right)^2 (1 - |\phi_\alpha(z)|^2)^q} \, dm(z) \]

\[ = 0. \quad (2.3) \]

**Proof.** Firstly we assume that \( C_\varphi : B^p_{\log} \to Q^q_{\log} \) is compact, let \( f(z) = z \), then \( C_\varphi(f(z)) = \varphi(z) \in Q^q_{\log} \). Since \( \frac{z^n}{n} \|B^p_{\log} \leq C \) (in fact \( C = \frac{2p}{pq} \)) and \( z^n \to 0 \) as \( n \to \infty \), locally uniformly on the unit disc, then by the compactness of \( C_\varphi \), \( \|C_\varphi(z^n)\|_{Q^q_{\log}} \to 0 \) as \( n \to \infty \). This means that for each \( r \in (0, 1) \) and each \( \varepsilon > 0 \), there exists \( n_0 \in N \) such that

\[ r^{2(n_0 - 1)} \sup_{\alpha \in D} \left( \frac{2}{1 - |\alpha|^2} \right)^2 \int_{\{\varphi(z) > r\}} |\varphi'(z)|^2 (1 - |\phi_\alpha(z)|^2)^q \, dm(z) < \varepsilon. \]

If we choose \( r \geq 2^{-\frac{1}{(n_0 - 1)}} \), then

\[ \sup_{\alpha \in D} \left( \frac{2}{1 - |\alpha|^2} \right)^2 \int_{\{\varphi(z) > r\}} |\varphi'(z)|^2 (1 - |\phi_\alpha(z)|^2)^q \, dm(z) < 2\varepsilon. \quad (2.4) \]

Let now \( f \) with \( \|f\|_{B^p_{\log}} < 1 \). We consider the functions \( f_t(z) = f(tz) \), \( t \in (0, 1) \). By the compactness of \( C_\varphi \) we get that for each \( \varepsilon > 0 \), there exists \( t_0 \in (0, 1) \) such
that for all \( t > t_0, \)
\[
\sup_{\alpha \in D} (\log \frac{2}{1 - |\alpha|^2})^2 \int_{D} \left| (f \circ \varphi)'(z) - (f_t \circ \varphi)'(z) \right|^2 (1 - |\phi_\alpha(z)|^2)^\eta \, dm(z) < \varepsilon.
\]

Then we fix \( t, \) by (2.4)
\[
\sup_{\alpha \in D} (\log \frac{2}{1 - |\alpha|^2})^2 \int_{\{|\varphi(z)| > r\}} \left| (f \circ \varphi)'(z) \right|^2 (1 - |\phi_\alpha(z)|^2)^\eta \, dm(z)
\leq 2 \sup_{\alpha \in D} (\log \frac{2}{1 - |\alpha|^2})^2 \int_{\{|\varphi(z)| > r\}} \left| (f_t \circ \varphi)'(z) \right|^2 (1 - |\phi_\alpha(z)|^2)^\eta \, dm(z)
+ 2 \sup_{\alpha \in D} (\log \frac{2}{1 - |\alpha|^2})^2 \int_{\{|\varphi(z)| > r\}} \left| (f \circ \varphi)'(z) - (f_t \circ \varphi)'(z) \right|^2 (1 - |\phi_\alpha(z)|^2)^\eta \, dm(z)
\leq 2 \varepsilon + 2 \|f_t''\|_{H^\infty} \sup_{\alpha \in D} (\log \frac{2}{1 - |\alpha|^2})^2 \int_{\{|\varphi(z)| > r\}} \left| \varphi'(z) \right|^2 (1 - |\phi_\alpha(z)|^2)^\eta \, dm(z)
\leq 4 \varepsilon (1 + \|f_t''\|_{H^\infty}). \tag{2.5}
\]

Having in mind (2.4) and (2.5) we conclude that for each \( \|f\|_{B^p_{\log}} \leq 1 \) and \( \varepsilon > 0, \) there is \( \delta \) depending on \( f, \varepsilon, \) such that for \( r \in [\delta, 1), \)
\[
\sup_{\alpha \in D} (\log \frac{2}{1 - |\alpha|^2})^2 \int_{\{|\varphi(z)| > r\}} \left| (f \circ \varphi)'(z) \right|^2 (1 - |\phi_\alpha(z)|^2)^\eta \, dm(z) < \varepsilon. \tag{2.6}
\]

Since \( C_{\varphi} \) is compact, it maps the unit ball of \( B^p_{\log} \) to a relative compact subset of \( Q^p_{\varphi_{\log}}. \) Thus for each \( \varepsilon > 0, \) there exists a finite collection of functions \( f_1, f_2, \ldots, f_N \) in the unit ball of \( B^p_{\log}, \) such that for each \( \|f\|_{B^p_{\log}} \leq 1 \) there is a \( k \in \{1, 2, \ldots, N\} \) with
\[
\sup_{\alpha \in D} (\log \frac{2}{1 - |\alpha|^2})^2 \int_{D} \left| (f \circ \varphi)'(z) - (f_k \circ \varphi)'(z) \right|^2 (1 - |\phi_\alpha(z)|^2)^\eta \, dm(z) < \varepsilon.
\]

By (2.6), we get that for \( \delta = \max_{1 \leq k \leq N} \delta(f_k, \varepsilon) \) and \( r \in [\delta, 1), \)
\[
\sup_{\alpha \in D} (\log \frac{2}{1 - |\alpha|^2})^2 \int_{\{|\varphi(z)| > r\}} \left| (f_k \circ \varphi)'(z) \right|^2 (1 - |\phi_\alpha(z)|^2)^\eta \, dm(z) < \varepsilon.
\]
Thus we get that
\[
\sup_{\|f\|_{B^p_{\log}} \leq 1} \sup_{\alpha \in D} (\log \frac{2}{1 - |\alpha|^2})^2 \int_{\{|\varphi(z)| > r\}} \left| (f_k \circ \varphi)'(z) \right|^2 (1 - |\phi_\alpha(z)|^2)^\eta \, dm(z) < 2 \varepsilon.
\]

By Lemma 2.2, (2.3) holds.

Conversely, we assume that \( \varphi \in Q^p_{\varphi_{\log}} \) and (2.3) holds. Let \( \{f_n\}_{n \in \mathbb{N}} \) be a sequence of functions in the unit ball of \( B^p_{\log}, \) such that \( f_n \to 0 \) as \( n \to \infty, \) uniformly on the compact subsets of the unit disc.
Let \( r \in (0, 1) \), then
\[
\| f_n \circ \varphi \|^2_{Q_{\log}^q} \\
\leq 2 |f_n(\varphi(0))|^2 \\
+ 2 \sup_{\alpha \in D} (\log \frac{2}{1 - |\alpha|^2})^2 \int_{\{|\varphi(z)| \leq r\}} |(f_n \circ \varphi)'(z)|^2 (1 - |\phi_\alpha(z)|)^q dm(z) \\
+ 2 \sup_{\alpha \in D} (\log \frac{2}{1 - |\alpha|^2})^2 \int_{\{|\varphi(z)| > r\}} |(f_n \circ \varphi)'(z)|^2 (1 - |\phi_\alpha(z)|)^q dm(z) \\
= 2I_1 + 2I_2 + 2I_3.
\]
Since \( f_n \to 0 \) as \( n \to \infty \), uniformly on \( D \), then \( I_1 \to 0 \) as \( n \to \infty \) and for each \( \varepsilon > 0 \) there is \( n_0 \in \mathbb{N} \) such that for each \( n > n_0 \), 
\[
I_2 \leq \varepsilon \| \varphi \|^2_{Q_{\log}^q}.
\]
By (2.3), then for every \( n \), that means for every \( n > n_0 \) and for every \( \varepsilon > 0 \), there exists \( r_0 \) such that for every \( r > r_0 \), \( I_3 < \varepsilon \). Thus \( \| C_\varphi(f_n) \|^2_{Q_{\log}^q} \to 0 \) as \( n \to \infty \). \( \square \)

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