WEYL’S THEOREM FOR ALGEBRAICALLY ABSOLUTE-\((p, r)\)-PARANORMAL OPERATORS

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Abstract. An operator \(T \in B(H)\) is said to be absolute-\((p, r)\)-paranormal if \(\|T^p|T^*|r^\frac{r}{p}x\| \geq \|T^*|r^\frac{r}{r}x\|^{p+r}\) for all \(x \in H\) and for positive real number \(p > 0\) and \(r > 0\), where \(T = U|T|\) is the polar decomposition of \(T\). In this paper, we discuss some properties of absolute-\((p, r)\)-paranormal operators and show that Weyl’s theorem holds for algebraically absolute-\((p, r)\)-paranormal operators.

1. Introduction and preliminaries

Let \(H\) be an infinite dimensional complex Hilbert space and \(B(H)\) denote the algebra of all bounded linear operators acting on \(H\). Every operator \(T\) can be decomposed into \(T = U|T|\) with a partial isometry \(U\), where \(|T| = \sqrt{T^*T}\). In this paper, \(T = U|T|\) denotes the polar decomposition satisfying the kernel condition \(N(U) = N(|T|)\). Furuta, Ito and Yamazaki [10] introduced class \(A(k)\) and absolute-\(k\)-paranormal operators for \(k > 0\) as generalizations of class \(A\) and paranormal operators, respectively. An operator \(T\) belongs to class \(A(k)\) if \((T^*|T|^{2k}T)^{\frac{1}{k+1}} \geq |T|^2\) and \(T\) is said to be absolute-\(k\)-paranormal if \(\|T^kTx\| \geq \|Tx\|^{k+1}\) for every unit vector \(x\). On other hand Fujii, Izumino and Nakamoto [7] introduced \(p\)-paranormal operators for \(p > 0\) as another generalization of paranormal operators. An operator \(T\) is said to be \(p\)-paranormal if \(\|T^pU|T|^p\| \geq \|T^p\|p\) for every unit vector \(x\), where the polar decomposition of \(T\) is \(T = U|T|\).

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Fujii, Jung, S.H. Lee, M.Y. Lee and Nakamoto [8] introduced class $A(p, r)$ as a further generalization of class $A(k)$. An operator $T \in A(p, r)$ for $p > 0$ and $r > 0$ if $(\|T^*|T|^{2p}|T^*|)^{\frac{1}{2p}} \geq |T^*|^r$ and class $AI(p, r)$ is class of all invertible operators which belong to class $A(p, r)$. Yamazaki and Yanagida [18] introduced the notion of absolute-$(p, r)$-paranormal operator. It is a further generalization of the classes of both absolute-$k$-paranormal operators and $p$-paranormal operators as a parallel concept of class $A(p, r)$. An operator $T$ is said to be absolute-$(p, r)$-paranormal if $\|T^p|T^*|^rx\|^r \geq \|T^*|^rx\|^{p+r}$ for every unit vector $x$ or equivalently $\|T^p|T^*|^rx\|^{r} \geq \|T^*|^rx\|^{p+r}$ for all $x \in H$ and for positive real numbers $p > 0$ and $r > 0$.

2. ON ABSOLUTE-$(p, r)$-PARANORMAL OPERATOR

In this section, we obtain a characterization of absolute-$(p, r)$-paranormal operators using the polar decomposition $T = U|T|$ of $T$ i.e., $T = U|T|$ is absolute-$(p, r)$-paranormal operator for $p > 0$ and $r > 0$ if and only if $r|T^p|U^*|T|^{2p}U|T|^r - (p + r)\lambda^p|T|^{2r} + p\lambda^{p+r}I \geq 0$ for all real $\lambda$. Using this characterization, we also obtain some properties for absolute-$(p, r)$-paranormal operators.

**Theorem 2.1.** [9]: Let $T_1 = U_1P_1$ and $T_2 = U_2P_2$ be the polar decomposition of $T_1$ and $T_2$, respectively. Then the following are equivalent:

1. $T_1$ doubly commutes with $T_2$.
2. $U_1^*, U_1$ and $P_1$ commutes with $U_2^*, U_2$ and $P_2$.
3. $[P_1, P_2] = 0$, $[U_1, P_2] = 0$, $[P_1, U_2] = 0$, $[U_1, U_2] = 0$ and $[U_1^*, U_2] = 0$.

**Theorem 2.2.** [9]: Let $T_1 = U_1P_1$ and $T_2 = U_2P_2$ be the polar decomposition of $T_1$ and $T_2$, respectively. If $T_1$ doubly commutes with $T_2$, then $T_1T_2 = U_1U_2P_1P_2$ is also the polar decomposition of $T_1T_2$, that is, $U_1U_2$ is partial isometry with $N(U_1U_2) = N(P_1P_2)$ and $P_1P_2 = |T_1T_2|$.

In [18], Yamazaki and Yanagida gave proof in terms of operator inequalities. Here we give the proof using polar decomposition.

**Lemma 2.3.** Let an operator $T \in B(H)$ have the polar decomposition $T = U|T|$. Then $T$ is absolute-$(p, r)$-paranormal for $p > 0$, $r > 0$ if and only if

$$r|T^p|U^*|T|^{2p}U|T|^r - (p + r)\lambda^p|T|^{2r} + p\lambda^{p+r}I \geq 0$$

(2.1)

for all real $\lambda$.

**Proof.** Suppose that (2.1) holds for all real $\lambda$. Then this inequality is equivalent to

$$\|T^p|U^*|T|^{2r} - 2p\lambda^p\frac{\lambda^{p+r}}{p+r} \|T^*|^rx\|^{p+r} + p\lambda^{p+r} \geq 0$$

for all real $\lambda$ and $x \in H$. This is equivalent to

$$\|T^p|U^*|T|^{2r} \geq \|T^*|^rx\|^{2(p+r)}, x \in H$$

and thus

$$\|T^p|U^*|T|^{2r} \geq \|T|^r|x\|^{p+r}, x \in H$$

Hence $T$ is absolute-$(p, r)$-paranormal. \qed
Theorem 2.4. Let $T = U|T|$ be invertible absolute-$(p, r)$-paranormal for $p > 0$, $r > 0$. Then $T^{-1}$ is absolute-$(r, p)$-paranormal.

Proof. Suppose that $T = U|T|$ is an invertible absolute-$(p, r)$-paranormal operator. Then $U|T|^{-r} = |T^*|^{-r}U$ and $|T^*|^{-r} = U|T|^{-r}U^*$ for all $p > 0$ and $r > 0$. Since $T$ is absolute-$(p, r)$-paranormal, from Lemma 2.3, we have

$$r|T^*U|T|^{2p}U|T|^{-r} - (p + r)|\lambda^p|T|^{2r} + p|\lambda^{p+r}I \geq 0.$$ 

Since $T$ is invertible, taking inverse,

$$pI - (p + r)|\lambda^p|T^{-1}2r - r|\lambda^{p+r}U^{-1}2pU^*T^{-1}2p \geq 0$$

$$pI - (p + r)|\lambda^p|U^{-1}2rU - r|\lambda^{p+r}U^{-1}2pU^*T^{-1}2p \geq 0$$

$$U|T|^{-r}U^{-1}2p[p|T||U^*|^2pU|T|^p - (p + r)|\lambda^p|T|^{2p} + r|\lambda^{p+r}I||T|^{-p}U^*|T|^{-r}U^*$$

is positive for all real $\lambda$. Therefore by Lemma 2.3, $T^{-1}$ is absolute-$(r, p)$-paranormal.

Theorem 2.5. An operator unitarily equivalent to absolute-$(p, r)$-paranormal operator is absolute-$(p, r)$-paranormal for all $p > 0$ and $r > 0$.

Proof. Let $T_1 = W|T_1|$ be absolute-$(p, r)$-paranormal, $W$ be unitary and $T_2 = W^*T_1W$. Then $|T_2|^r = W^*|T_1|^rW$ and $|T_2|^{2p} = W^*|T_1|^{2p}W$ for every $p > 0$ and $r > 0$. Then by Theorem 2.1 and Theorem 2.2, we have $T_2 = W^*T_1W = W^*U|T_1|^W = W^*U|T_1|^W$ and $N(W^*UW) = N(W^*T_1W)$. Hence $T_2 = (W^*UW)(W^*T_1W)$ is the polar decomposition of $T_2$. Thus, we have,

$$r|T_2|^r(W^*UW)^*|T_2|^{2p}(W^*UW)|T_2|^{-r} - (p + r)|\lambda^p|T_2|^{2r} + p|\lambda^{p+r}I \geq 0$$

Since $|T_2|^r = W^*|T_1|^rW$ and $|T_2|^{2p} = W^*|T_1|^{2p}W$, we get

$$rW^*|T_1|^rU^*|T_1|^{2p}U|T_1|^rW - (p + r)|\lambda^pW^*|T_1|^{2r}W + p|\lambda^{p+r}I \geq 0$$

$$= W^*r|T_1|^rU^*|T_1|^{2p}U|T_1|^r - (p + r)|\lambda^p|T_1|^{2r} + p|\lambda^{p+r}I \geq 0$$

is true for all real $\lambda$. Since $T_1 = W|T_1|$ is the polar decomposition of $T_1$, So $T_2$ is also absolute-$(r, p)$-paranormal.

Remark 2.6. The above theorem is not true for similarly equivalent operators.

Theorem 2.7. If $T \in A(p, r)$ then $T$ is absolute-$(p, r)$-paranormal.

Proof. If $T \in A(p, r)$ for any $p > 0$ and $r > 0$, then $(|T^*|^r|T|^{2p}|T^*|^r)^{p+r} \geq |T^*|^{2r}$ for every unit vector $x \in H$ and $T = U|T|$ is the polar decomposition of $T$. Then,

$$||T^*x||^{p+r} = (|T^*|^r x, x)^{p+r}$$

$$= (U^*|T^*|^r U x, x)^{p+r}$$

$$\leq (U^*|T^*|^r U x, x)^{p+r}$$

$$\leq (U^*|T^*|^r U x, x)^{p+r}$$

$$\leq ((U^*|T^*|^r U x, x)^{p+r}$$

$$\leq (U^*|T^*|^r U x, x)^{p+r}$$

$$\leq (U^*|T^*|^r U x, x)^{p+r}$$

$$\leq (U^*|T^*|^r U x, x)^{p+r}$$

$$= ||T^*x||^{p+r}.$$
Therefore $T$ is absolute-$(p, r)$-paranormal.

3. Weyl’s theorem for algebraically absolute-$(p, r)$-paranormal operators

If $T \in B(H)$, we write $N(T)$ and $R(T)$ for null space and range of $T$, respectively. Let $\alpha(T) = \dim N(T)$, $\beta(T) = \dim N(T^*)$ and let $\sigma(T)$, $\sigma_1(T)$ and $\pi_0(T)$ denote the spectrum, approximate point spectrum and point spectrum of $T$, respectively. An operator $T \in B(H)$ is called Fredholm if it has closed range, finite dimensional null space and its range has finite co-dimension. The index of a Fredholm operator is given by $i(T) = \alpha(T) - \beta(T)$. $T$ is called Weyl if it Fredholm of index zero and Browder if it is Fredholm of finite ascent and descent. The essential spectrum $\sigma_e(T)$, the Weyl spectrum $w(T)$ and the Browder spectrum $\sigma_b(T)$ of $T$ are defined by

- $\sigma_e(T) = \{ \lambda \in C : T - \lambda$ is not Fredholm $\}$
- $w(T) = \{ \lambda \in C : T - \lambda$ is not Weyl $\}$
- $\sigma_b(T) = \{ \lambda \in C : T - \lambda$ is not Browder $\}$, respectively [11, 12].

Evidently $\sigma_e(T) \subseteq w(T) \subseteq \sigma_b(T) = \sigma_e(T) \cup \text{acc}(\sigma(T))$, where $\text{acc}(K)$ is accumulation points of $K \subseteq C$. Let $\pi_{0\lambda}(T) = \{ \lambda \in \text{iso} \sigma(T) : 0 < \alpha(T - \lambda) < \infty \}$ and $P_{0\lambda}(T) = \sigma(T) \setminus \sigma_\lambda(T)$. We say that Weyl’s theorem holds for $T$ if $\sigma(T) \setminus w(T) = \pi_{0\lambda}(T)$ and that Browder’s theorem holds for $T$ if $\sigma(T) \setminus w(T) = P_{0\lambda}(T)$. Berkani [2] says that generalized Weyl’s theorem holds for $T$ provided $\sigma(T) \setminus \sigma_{BW}(T) = E(T)$, where $E(T)$ and $\sigma_{BW}(T)$ denote the isolated point of the spectrum which are eigenvalues (no restriction on multiplicity) and the set of complex numbers $\lambda$ for which $T - \lambda I$ fails to be Weyl, respectively. An operator $T \in B(H)$ is called $B$-Fredholm if there exists $n \in N$ for which the induced operator $T_n : T^n(H) \to T^n(H)$ is Fredholm in the usual sense and $B$-Weyl if in addition $T_n$ has index zero. Note that, if the generalized Weyl’s theorem holds for $T$, then so does Weyl’s theorem. We say $T$ is algebraically absolute-$(p, r)$-paranormal if there exists a non constant complex polynomial $p$ such that $p(T)$ is absolute-$(p, r)$-paranormal.

Lemma 3.1. Let $T$ be invertible and absolute-$(p, r)$-paranormal, $\lambda \in C$ and assume that $\sigma(T) = \{ \lambda \}$ then $T = \lambda$.

Proof. Case (i): $\lambda = 0$
Since $T$ is absolute-$(p, r)$-paranormal, $T$ is normaloid by [18, Theorem 8]. Therefore $T = 0$.

Case (ii): $\lambda \neq 0$
Since $T$ is invertible and $T$ is absolute-$(p, r)$-paranormal, we have $T$ is normaloid by [18, Theorem 8]. But $T^{-1}$ is absolute-$(r, p)$-paranormal by Theorem 2.4. Therefore $T^{-1}$ is also normaloid by [18, Theorem 8]. But $\sigma(T^{-1}) = \{ \frac{1}{\lambda} \}$ then $\|T\|\|T^{-1}\| = |\lambda||\frac{1}{\lambda}| = 1$. Then by [17], $T$ is convexoid. So $w(T) = \{ \lambda \}$. Therefore $T = \lambda$.

Lemma 3.2. Let $T$ be invertible and quasi-nilpotent algebraically absolute-$(p, r)$-paranormal. Then $T$ is nilpotent.
Proof. Suppose that \( p(T) \) is absolute-(\( p, r \))-paranormal for some non constant polynomial \( p \). Since \( \sigma(p(T)) = p(\sigma(T)) \), the operator \( p(T) - p(0) \) is quasi-nilpotent. From above Lemma 3.1, we have that
\[
CT^m(T - \lambda_1)(T - \lambda_2) \cdots (T - \lambda_n) \equiv p(T) - p(0) = 0
\]
where \( m \geq 1 \). Since \( T - \lambda_i \) is invertible for every \( \lambda_i \neq 0 \) and So therefore \( T^m = 0 \).

\begin{theorem}
Let \( T \) be an invertible algebraically absolute-(\( p, r \))-paranormal operator. Then \( T \) is isoloid.
\end{theorem}

Proof. Let \( \lambda \in \text{iso}\sigma(T) \) and let \( P = \frac{1}{2\pi i} \int_{\partial D} (\mu - T)^{-1} d\mu \) be the associated Riesz idempotent, where \( D \) is a closed disk centered at \( \lambda \) which contains no other points of \( \sigma(T) \). We can then represent \( T \) as the direct sum \( T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} \) where
\[
\sigma(T_1) = \{ \lambda \} \text{ and } \sigma(T_2) = \sigma(T)/\{ \lambda \}.
\]
Since \( T \) is algebraically absolute-(\( p, r \))-paranormal, \( p(T) \) is absolute-(\( p, r \))-paranormal for some non constant polynomial \( p \). Since \( \sigma(T_1) = \{ \lambda \} \), we must have \( \sigma(p(T_1)) = p(\sigma(T_1)) = \{ p(\lambda) \} \). Therefore \( p(T_1) - p(\lambda) \) is quasi-nilpotent.

Since \( p(T_1) \) is absolute-(\( p, r \))-paranormal, it follows from Lemma 3.1, that \( p(T_1) - p(\lambda) = 0 \). Put \( q(z) = p(z) - p(\lambda) \). Then \( q(T_1) = 0 \) and hence \( T_1 \) is algebraically absolute-(\( p, r \))-paranormal. Since \( T_1 - \lambda \) is quasi-nilpotent and algebraically absolute-(\( p, r \))-paranormal, it follows from Lemma 3.2, that \( T_1 - \lambda \) is nilpotent. Therefore \( \lambda \in \pi_0(T_1) \) and hence \( \lambda \in \pi_0(T) \). This shows that \( T \) is isoloid.

\begin{lemma}
Let \( T \) be an algebraically absolute-(\( p, r \))-paranormal operator. Then \( T \) has SVEP (the single-valued extension property).
\end{lemma}

Proof. We first show that if \( T \) is absolute-(\( p, r \))-paranormal, then \( T \) has SVEP. Suppose that \( T \) is absolute-(\( p, r \))-paranormal. If \( \pi_0(T) = \phi \), then clearly \( T \) has SVEP. Suppose that \( \pi_0(T) \neq \phi \). Let \( \Delta(T) = \{ \lambda \in \pi_0(T) : N(T - \lambda) \subseteq N(T^* - \lambda) \} \). Since \( T \) is absolute-(\( p, r \))-paranormal and \( \pi_0(T) \neq \phi \), \( \Delta(T) \neq \phi \). Let \( M \) be the closed linear span of the subspaces \( N(T - \lambda) \) with \( \lambda \in \Delta(T) \). Then \( M \) reduces \( T \), and so we can write \( T \) as \( T_1 \oplus T_2 \) on \( H = M \oplus M^* \). Clearly, \( T_1 \) is normal and \( \pi_0(T_2) = \phi \). Since \( T_1 \) and \( T_2 \) have both SVEP, \( T \) has SVEP. Suppose now that \( T \) is algebraically absolute-(\( p, r \))-paranormal. Then \( p(T) \) is absolute-(\( p, r \))-paranormal for some non constant polynomial \( p \). Since \( p(T) \) has SVEP, it follows from [14, Theorem 3.3.9] that \( T \) has SVEP.

Let \( H(\sigma(T)) \) be the set of all analytic functions in an open neighborhood of \( \sigma(T) \).

\begin{theorem}
Let \( T \) be an algebraically absolute-(\( p, r \))-paranormal operator. Then Weyl’s theorem holds for \( T \).
\end{theorem}

Proof. Suppose that \( \lambda \in \sigma(T) \setminus \text{w}(T) \). Then \( T - \lambda \) is Weyl and not invertible. We claim that \( \lambda \in \partial \sigma(T) \). Assume that \( \lambda \) is an interior point of \( \sigma(T) \). Then there exists a neighborhood \( U \) of \( \lambda \), such that \( \dim N(T - \mu) > 0 \) for all \( \mu \in U \). It follows from [6, Theorem 10] that \( T \) doesnot have SVEP. On the other hand, Since \( p(T) \) is absolute-(\( p, r \))-paranormal for some non constant polynomial \( p \), it follows from
Lemma 3.4 that $T$ has SVEP. It is a contradiction. Therefore, $\lambda \in \partial \sigma(T) \setminus \omega(T)$ and it follows from the punctured neighborhood theorem that $\lambda \in \pi_{00}(T)$. Conversely, suppose that $\lambda \in \pi_{00}(T)$. Using the Riesz idempotent $E = \frac{1}{2\pi i} \int (\mu - T)^{-1} d\mu$ for $\lambda$, we can represent $T$ as the direct sum $T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$, where $\sigma(T_1) = \{\lambda\}$ and $\sigma(T_2) = \sigma(T) \setminus \{\lambda\}$. Now we consider two cases:
Case(i) : $\lambda = 0$: Then $T_1$ is algebraically absolute-($p, r$)-paranormal and quasi-nilpotent. It follows from Lemma 3.2 that $T_1$ is nilpotent. We claim that $\dim R(E) < \infty$. For, if $N(T_1)$ is infinite dimensional, then $0 \notin \pi_{00}(T)$. It is a contradiction. Therefore $T_1$ is an operator on the finite dimensional space $R(E)$. So it follows that $T_1$ is Weyl. But since $T_2$ is invertible, we can conclude that $T$ is Weyl. Therefore $0 \in \sigma(T) \setminus \omega(T)$.

Case(ii) : $\lambda \neq 0$: Then by the proof of Theorem 3.3, $T_1 - \lambda$ is nilpotent. Since $\lambda \in \pi_{00}(T)$, $T_1 - \lambda$ is an operator on the finite dimensional space $R(E)$. So $T_1 - \lambda$ is Weyl. Since $T_2 - \lambda$ is invertible, $T - \lambda$ is Weyl.

By Case (i) and Case (ii), Weyl’s theorem holds for $T$. This completes the proof. \hfill $\square$

**Theorem 3.6.** Let $T$ be an algebraically absolute-($p, r$)-paranormal operator. Then Weyl’s theorem holds for $f(T)$ for every $f \in H(\sigma(T))$.

**Proof.** Let $f \in H(\sigma(T))$. Since it generally holds $\omega(f(T)) \subseteq f(\omega(T))$, it suffices to show that $f(\omega(T)) \subseteq \omega(f(T))$. Suppose $\lambda \notin \omega(f(T))$, then $f(T) - \lambda$ is Weyl and

$$f(T) - \lambda = c(T - \alpha_1)(T - \alpha_2)(T - \alpha_3)\cdots(T - \alpha_n)g(T) \quad (3.1)$$

where $c, \alpha_1, \alpha_2, \alpha_3, \cdots \alpha_n \in C$ and $g(T)$ is invertible. Since the operators in the right side of (3.1) commute, every $T - \alpha_i$ is Fredholm. Since $T$ is algebraically absolute-($p, r$)-paranormal, $T$ has SVEP by Lemma 3.4. It follows from [1, Theorem 2.6] that $\text{ind}(T - \alpha_i) \leq 0$ for each $i = 1, 2, 3, \cdots n$. Therefore $\lambda \notin f(\omega(T))$ and hence $f(\omega(T)) = \omega(f(T))$.

Now by [16], that if $T$ is isoloid, then

$$f(\sigma(T) \setminus \pi_{00}(T)) = \sigma(f(T)) \setminus \pi_{00}(f(T))$$

for every $f \in H(\sigma(T))$.

Since $T$ is isoloid by Theorem 3.3 and Weyl’s theorem holds for $T$ by Theorem 3.5,

$$\sigma(f(T)) \setminus \pi_{00}(f(T)) = f(\sigma(T) \setminus \pi_{00}(T)) = f(\omega(T)) = \omega(f(T))$$

which implies that Weyl’s theorem holds for $f(T)$. This completes the proof. \hfill $\square$

**Theorem 3.7.** Let $T$ be an algebraically absolute-($p, r$)-paranormal operator. Then generalized Weyl’s theorem holds for $T$.

**Proof.** Assume that $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$. Then $T - \lambda I$ is $B$-Weyl and not invertible. We claim that $\lambda \in \partial \sigma(T)$. Assume to the contrary that $\lambda$ is an interior point of $\sigma(T)$. Then there exists a neighborhood $U$ of $\lambda$ such that $\dim(T - \mu) > 0$ for all $\mu \in U$. It follows from [6, Theorem 10], that $T$ does not have SVEP. On the other hand, since $p(T)$ is absolute-($p, r$)-paranormal for non constant polynomial $p$, it follows from Lemma 3.4 that $p(T)$ has SVEP. Hence by [14, Theorem 3.3.9], $T$ is
SVEP, a contradiction. Therefore $\lambda \in \partial \sigma(T)$. Conversely, assume that $\lambda \in E(T)$, then $\lambda$ is isolated in $\sigma(T)$. From [13, Theorem 7.1], we have $X = M \oplus N$, where $M$, $N$ are closed subspaces of $X$, $U = (T - \lambda I)|_N$ is an invertible operator and $V = (A - \lambda I)|_N$ is a quasi-nilpotent operator. Since $T$ is algebraically absolute-$(p, r)$-paranormal, $V$ is also algebraically absolute-$(p, r)$-paranormal, from Lemma 3.2, $V$ is nilpotent. Therefore $T - \lambda I$ is Drazin invertible [5, Proposition 19] and [15, Corollary 2.2]. By [3, Lemma 4.1], $T - \lambda I$ is a $B$-Fredholm operator of index 0. \(\square\)

Let $\sigma_{BF}(T) = \{\lambda \in C : T - \lambda I$ is not a $B$-Fredholm operator\} be the $B$-Fredholm spectrum of $T$ and $\rho_{BF}(T) = C \setminus \sigma_{BF}(T)$, the resolvent set of $T$.

**Definition 3.8.** Let $T \in B(H)$, we say that $T$ is of stable index if for each $\lambda, \mu \in \rho_{BF}(T)$, $\text{ind}(T - \lambda I)$, $\text{ind}(T - \mu I)$ have the same sign index.

**Lemma 3.9.** Let $T \in B(H)$ be absolute-$(p, r)$-paranormal, then $T$ is of stable index.

**Proof.** If $T$ is absolute-$(p, r)$-paranormal, then $\|T^p(T^*)^r x\| \geq \|T^r x\|^{p+r}$ for all $x \in H$. So $N(T) \subset N(T^*) = R(T)$. Since $N(T^2)/N(T) \approx N(T) \cap R(T)$, implies that $N(T^2) = N(T)$. Moreover, if $T$ is also $B$-Fredholm, then there exists an integer $n$, such that $R(T^n)$ is closed and such that $T_n : R(T^n) \to R(T^n)$ is a Fredholm operator. We have,

$$\text{ind}(T) = \text{ind}(T_n)$$
$$= \dim N(T) \cap R(T^n) - \dim R(T^n)/R(T^{n+1})$$
$$= -\dim R(T^n)/R(T^{n+1}).$$

Hence it follows that $\text{ind}(T) \leq 0$.

Further, if $\lambda \in \rho_{BF}(T)$, then $T - \lambda I$ is a $B$-Fredholm operator and $T - \lambda I$ is also absolute-$(p, r)$-paranormal. By the same way as above, we have $\text{ind}(T - \lambda I) \leq 0$. Therefore $T$ is of stable index. \(\square\)

**Theorem 3.10.** Let $T$ be an invertible algebraically absolute-$(p, r)$-paranormal operator. Then generalized Weyl’s theorem holds for $f(T)$ for every function $f$ analytic on a neighborhood of $\sigma(T)$.

**Proof.** Assume that $T$ be an algebraically absolute-$(p, r)$-paranormal operator. We prove that $f(\sigma_{BW}(T)) = \sigma_{BW}(f(T))$ for every function $f$ analytic on a neighborhood of $\sigma(T)$. Let $f$ be an analytic function on a neighborhood of $\sigma(T)$. Since $\sigma_{BW}(f(T)) \subseteq f(\sigma_{BW}(T))$ with no restriction on $T$, it is sufficient to prove that $f(\sigma_{BW}(T)) \subseteq \sigma_{BW}(f(T))$. Assume that $\lambda \notin \sigma_{BW}(f(T))$. Then $f(T) - \lambda$ is $B$-Weyl and $f(T) - \lambda = C(T - \alpha_1 I)(T - \alpha_2 I) \cdots (T - \alpha_n I)g(t)$

where $c, \alpha_1, \alpha_2, \cdots , \alpha_n \in C$ and $g(T)$ is invertible. Since $f(T) - \lambda I$ is a $B$-Fredholm operator from [2, Theorem 3.4], it follows that for each $i$, $1 \leq i \leq n$, $T - \alpha_i I$ is a $B$-Fredholm operator. Moreover, since $\text{ind}(f(T) - \lambda I) = 0$ and $T$ is of stable sign index by Lemma 3.9, from [3, Theorem 3.2], we have for each
\[i, 1 \leq i \leq n, \text{ind}(T - \alpha_i I) = 0. \] So for each \(i, 1 \leq i \leq n, \alpha_i \notin \sigma_{BW}(T). \) If \(\lambda \in f(\sigma_{BW}(T)), \) there exists \(\alpha \in \sigma_{BW}(T)\) such that \(\lambda = f(\alpha). \) Hence \(0 = f(\alpha) - \lambda = (\alpha - \alpha_1)(\alpha - \alpha_2) \cdots (\alpha - \alpha_n)g(\alpha). \) This implies that \(\alpha \in \{\alpha_1, \alpha_2, \ldots, \alpha_n\}. \) Hence, there exists \(i, 1 \leq i \leq n, \) such that \(\alpha_i \in \sigma_{BW}(T), \) contradiction. Hence \(\lambda \notin f(\sigma_{BW}(T)). \) It is known \([4, \text{Lemma 2.9}]\) that if \(T\) is isoloid then \(f(\sigma(T) \backslash E(T)) = \sigma(f(T) \backslash E(f(T)))\) for every analytic function on a neighborhood of \(\sigma(T). \) Since \(T\) is isoloid, by Theorem 3.3, and generalized Weyl’s theorem holds for \(T\) by Theorem 3.5, \[\sigma(f(T) \backslash E(f(T))) = f(\sigma(T) \backslash E(T)) \]

\[= f(\sigma_{BW}(T)) = \sigma_{BW}(f(T))\] by \([4, \text{Theorem 2.10}]. \]

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References

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