WEIGHTED CLASSES OF QUATERNION-VALUED FUNCTIONS

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Abstract. In this paper, we define the classes $F(p,q,s)$ of quaternion-valued functions, then we characterize quaternion Bloch functions by quaternion $F(p,q,s)$ functions in the unit ball of $\mathbb{R}^3$. Further, some important basic properties of these functions are also considered.

1. Introduction and preliminaries

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the complex unit disk. Let $0 < p < \infty$. An analytic function $f$ in $\mathbb{D}$ belongs to the Hardy space $H^p$ (see [11, 18]), if

$$\sup_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta}|^p d\theta < \infty;$$

$f$ is in $H^\infty$, if

$$\sup_{z \in \mathbb{D}} |f(z)| < \infty.$$

It is well known that $f \in H^2$ if and only if

$$\int_{\mathbb{D}} |f(z)|^2(1 - |z|^2) \, dA(z) < \infty,$$

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where $dA(z)$ is the Euclidean area element $dx\,dy$. For $0 < p < \infty$, an analytic function $f$ in $\mathbb{D}$ belongs to the Bergman space $L^p_\alpha$ (see [12]), if
\[ \int_{\mathbb{D}} |f(z)|^2 dA(z) < \infty. \]
The well known $\alpha$-Bloch space (see [29]) is defined by:
\[ \mathcal{B}^\alpha = \{ f : f \text{ analytic in } \mathbb{D} \text{ and } \mathcal{B}^\alpha(f) = \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f'(z)| < \infty \}, \]
where $0 < \alpha < \infty$. The space $\mathcal{B}^1$ is called the Bloch space $\mathcal{B}$. The little $\alpha$-Bloch space $\mathcal{B}_0^\alpha$ is a subspace of $\mathcal{B}^\alpha$ consisting of all $f \in \mathcal{B}^\alpha$ such that
\[ \lim_{|z| \to 1} (1 - |z|^2)^\alpha |f'(z)| = 0. \]
The Dirichlet space is given by:
\[ \mathcal{D} = \{ f : f \text{ analytic in } \mathbb{D} \text{ and } \int_{\mathbb{D}} |f'(z)|^2 dA(z) < \infty \}. \]

Let $0 < p < \infty$. Then the Besov-type spaces
\[ B^p = \{ f : f \text{ analytic in } \mathbb{D} \text{ and } \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^{p-2} (1 - |\varphi_a(z)|^2)^2 dA(z) < \infty \} \]
are introduced and studied intensively (see [24]). Here, $\varphi_a$ always stands for the Möbius transformation $\varphi_a(z) = \frac{a - z}{1 - az}$. From [24] it is known that the $B^p$ spaces can be used to describe the Bloch space $\mathcal{B}$ equivalently by the integral norms of $B^p$. Composing the Möbius transform $\varphi_a(z)$, which maps the unit disk $\mathbb{D}$ onto itself, and the fundamental solution of the two-dimensional real Laplacian on $\mathbb{D}$, we obtain the Green’s function $g(z, a) = \ln \left| \frac{1 - \frac{z}{a}}{1 - \frac{a}{z}} \right|$ with logarithmic singularity at $a \in \mathbb{D}$. Then the spaces
\[ Q_p = \{ f : f \text{ analytic in } \mathbb{D} \text{ and } \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^q g^p(z, a) dA(z) < \infty \} \]
are defined in [6]. The idea of these $Q_p$-spaces is to find a scale of spaces with $\mathcal{D}$ and $\mathcal{B}$, respectively, "at the both end points" of the scale. In [28] Zhao gave the following definition:

**Definition 1.1.** Let $f$ be an analytic function in $\mathbb{D}$ and let $0 < p < \infty$, $-2 < q < \infty$ and $0 < s < \infty$. If
\[ \|f\|_F(p, q, s) = \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^q g^s(z, a) dA(z) < \infty, \]
then $f \in F(p, q, s)$. Moreover, if
\[ \lim_{|a| \to 1} \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^q g^s(z, a) dA(z) = 0, \]
then $f \in F_0(p, q, s)$. 


The spaces $F(p, q, s)$ were intensively studied by Zhao in [28] and Rättyä in [21]. It is known from ([28], Theorem 2.10) that, for $p \geq 1$, the spaces $F(p, q, s)$ are Banach spaces under the norm

$$
\|f\| = \|f\|_{F(p, q, s)} + |f(0)|.
$$

Moreover, it is known that in (Definition 1.1) the Green’s function $g(z, a)$ can be replaced by the weight function $1 - |\varphi_a(z)|^2$ and that for $q + s \leq -1$ the spaces $F(p, q, s)$ and $F_0(p, q, s)$ both reduce to the space of constant functions (see [28], theorem 2.4 and proposition 2.12 ). It is sometimes convenient to replace the parameter $q$ by $p - 2$ and consider the spaces $F(p, p - 2, s)$ and $F_0(p, p - 2, s)$ instead of the spaces $F(p, q, s)$ and $F_0(p, q, s)$ (see [21]).

If $q = p - 2$ and $s = 0$, we denote $F(p, p - 2, 0) = F_0(p, p - 2, 0) = B^p$.

**Remark 1.2.** The interest of the spaces $F(p, q, s)$ come from that these spaces cover a lot of known spaces. Zhao in [28] collected the following immediate relations of $F(p, q, s)$ and $F_0(p, q, s)$:

1. $F(p, q, s) = \mathcal{B}^{q+2}_{p^2}$ and $F_0(p, q, s) = \mathcal{B}^{q+2}_0$, for $s > 1$.
2. $F(2, 0, s) = Q_s$, $F_0(2, 0, s) = Q_{s,0}$.
3. $F(2, 1, 0) = H^2$.
4. $F(p, p, 0) = L^p_0$, for $1 \leq p < \infty$.
5. $F(p, p - 2, 0) = B^p$, for $1 < p < \infty$.

For more studies on the spaces $F(p, q, s)$ in the unit disk or in the unit ball of $\mathbb{C}^n$, we refer to [4, 17, 19, 20, 21, 26, 27, 28, 30].

2. Quaternion function spaces

Let $\mathbb{H}$ be the skew field of quaternions. This means we can write each element $z \in \mathbb{H}$ in the form

$$
z = z_0 + z_1 i + z_2 j + z_3 k, \quad z_0, z_1, z_2, z_3 \in \mathbb{R},
$$

where $1, i, j, k$ are the basis elements of $\mathbb{H}$. For these elements we have the multiplication rules

$$
i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad k j = -jk = i, \quad ki = -ik = j.
$$

The conjugate element $\bar{z}$ is given by $\bar{z} = z_0 - z_1 i - z_2 j - z_3 k$ and we have the property

$$
z \bar{z} = \bar{z} z = \|z\|^2 = z_0^2 + z_1^2 + z_2^2 + z_3^2.
$$

Moreover, we can identify each vector $\vec{x} = (x_0, x_1, x_2) \in \mathbb{R}^3$ with a quaternion $x$ of the form

$$
x = x_0 + x_1 i + x_2 j.
$$

In what follows we will work in $\mathbb{B} \subset \mathbb{R}^3$, the unit ball in the real three-dimensional space. $\mathbb{B}$ is a bounded, simply connected domain with a $C^\infty$-boundary $S_1(0)$. Moreover, we will consider functions $f$ defined on $\mathbb{B}$ with values in $\mathbb{H}$. Let $\Omega$ be a domain in $\mathbb{R}^3$, then we will consider $\mathbb{H}$-valued functions defined in $\Omega$ (depending on $x = (x_0, x_1, x_2)$):

$$
f : \Omega \rightarrow \mathbb{H}.$$
The notation \( C^p(\Omega; \mathbb{H}), p \in \mathbb{N} \cup \{ 0 \} \), has the usual component-wise meaning. On \( C^1(\Omega; \mathbb{H}) \) we define a generalized Cauchy-Riemann operator \( D \) by
\[
Df = \frac{\partial f}{\partial x_0} + i \frac{\partial f}{\partial x_1} + j \frac{\partial f}{\partial x_2}
\]
and it’s conjugate operator by
\[
\overline{D}f = \frac{\partial f}{\partial x_0} - i \frac{\partial f}{\partial x_1} - j \frac{\partial f}{\partial x_2}.
\]
The solutions of \( Df = 0, \ x \in \Omega \), are called (left) hyperholomorphic (or monogenic) functions and generalize the class of holomorphic functions from the one-dimensional complex function theory. For more details about quaternionic analysis and general Clifford analysis, we refer to [9], [14], [16] and [25] and others.

For \( |a| < 1 \), we will denote by
\[
\varphi_a(x) = (a - x)(1 - \bar{a}x)^{-1}
\]
the Möbius transform, which maps the unit ball onto itself. Furthermore, let
\[
g(x, a) = \frac{1}{4\pi} \left( \frac{1}{|\varphi_a(x)|} - 1 \right)
\]
be the modified fundamental solution of the Laplacian in \( \mathbb{R}^3 \) composed with the Möbius transform \( \varphi_a(x) \). Especially, we denote for all \( p \geq 0 \)
\[
g^p(x, a) = \frac{1}{4^p \pi^p} \left( \frac{1}{|\varphi_a(x)|} - 1 \right)^p.
\]
Let \( f : \mathbb{B} \mapsto \mathbb{H} \) be a hyperholomorphic function. Then from [13], we have the seminorms
\[
\bullet \ \mathcal{B}(f) = \sup_{x \in \mathbb{B}} (1 - |x|^2)^{3/2} |\overline{D}f(x)|,
\]
\[
\bullet \ \mathcal{Q}_p(f) = \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} |\overline{D}f(x)|^2 g^p(x, a) d\mathbb{B}_x,
\]
which lead to the following definitions:

**Definition 2.1.** (see [13]) The spatial (or three-dimensional) Bloch space \( \mathcal{B} \) is the right \( \mathbb{H} \)-module of all hyperholomorphic functions \( f : \mathbb{B} \mapsto \mathbb{H} \) with \( \mathcal{B}(f) < \infty \).

**Definition 2.2.** (see [13]) The right \( \mathbb{H} \)-module of all quaternion-valued functions \( f \) defined on the unit ball, which are hyperholomorphic and satisfy \( \mathcal{Q}_p(f) < \infty \), is called \( \mathcal{Q}_p \)-space.

**Remark 2.3.** Because of the special structure of \( g(x, a) \) the seminorms \( \mathcal{Q}_p(f) \) make sense for \( p < 3 \) only. Consequently, we will consider in this paper \( \mathcal{Q}_p \)-spaces for \( p < 3 \) only.

With the generalized Cauchy-Riemann operator \( D \), its adjoint \( \overline{D} \), the hypercomplex Möbius transformation \( \varphi_a(x) = (a - x)(1 - \bar{a}x)^{-1} \), and a modified fundamental solution \( g \) of the real Laplacian Gürlebeck et al. [13] considered generalized
$Q_p$-spaces defined by

$$Q_p = \{ f \in \ker D : \sup_{a \in B} \int_B |\overline{D}f(x)|^2 (g(\varphi_a(x)))^p \, d\mathbb{B}_x < \infty \}.$$ 

where $\mathbb{B}$ stands for the unit ball in $\mathbb{R}^3$.

**Definition 2.4.** The right $\mathbb{H}$-module of all quaternion-valued functions $f$ defined on the unit ball, which are hyperholomorphic and satisfy the condition

$$\int_B |\overline{D}f(x)|^2 d\mathbb{B}_x < \infty,$$

is called spatial (or three-dimensional) Dirichlet space $D$.

The quaternion $\alpha$-Bloch space (see [2]) is defined by:

$$B^\alpha = \{ f : f \in KerD \text{ and } B^\alpha(f) = \sup_{x \in B} (1 - |x|^2)^{\frac{3}{2}\alpha} |\overline{D}f(x)| < \infty \},$$

where $0 < \alpha < \infty$. The space $B^1$ is called the Bloch space $B$. The little quaternion $\alpha$-Bloch space $B^\alpha_0$ is a subspace of $B^\alpha$ consisting of all $f \in B^\alpha$ such that

$$\lim_{|x| \to 1} (1 - |x|^2)^{\frac{3}{2}\alpha} |\overline{D}f(x)| = 0.$$ 

Now, we give the following definition:

**Definition 2.5.** Let $f$ be quaternion-valued function in $\mathbb{B}$. For $0 < p < \infty$, $-2 < q < \infty$ and $0 < s < \infty$. If

$$\|f\|_{F(p,q,s)} = \sup_{a \in B} \int_B |\overline{D}f(x)|^p (1 - |x|^2)^{\frac{3q}{2}} \left(1 - \varphi_a(x)^2\right)^s \, d\mathbb{B}_x < \infty,$$

then $f \in F(p,q,s)$. Moreover, if

$$\lim_{|a| \to 1} \int_B |\overline{D}f(x)|^p (1 - |x|^2)^{\frac{3q}{2}} \left(1 - \varphi_a(x)^2\right)^s \, d\mathbb{B}_x = 0,$$

then $f \in F_0(p,q,s)$.

**Remark 2.6.** Obviously, these spaces are not Banach spaces. Nevertheless, if we consider a small neighborhood of the origin $N_\epsilon$, with an arbitrary but fixed $\epsilon > 0$, then we can add the $L_1$-norm of the function $f$ over $N_\epsilon$ to the seminorms, so $F(p,q,s)$ spaces will become Banach spaces. Also, $F(p,q,s)$ spaces are not linear spaces.

**Remark 2.7.** It should be remarked that if we put $q = 0$ and $p = 2$, then $F(2,0,s) = Q_s$. Also, if $p = 2$ and $s = q = 0$, then $F(2,0,0) = D$, the quaternion Dirichlet space.

The main aim of this paper is to study these $F(p,q,s)$ spaces and their relations to the above mentioned quaternionic Bloch space. It will be shown that these exponents $p$ and $q$ generate a new scale of spaces, equivalent to the Bloch space for all $p$ and $q$. The concept may be generalized in the context of Clifford analysis to arbitrary real dimensions. We will restrict us for simplicity to $\mathbb{R}^3$ and quaternion-valued functions as (the lowest non-commutative case) a model case.
For more studies on quaternion function spaces, we refer to [1, 2, 3, 5, 7, 8, 10, 13, 15, 22] and others.

Let $U(a, R) = \{ x : |\varphi_a(x)| < R \}$ be the pseudo-hyperbolic ball with radius $R$, where $0 < R < 1$. Analogously to the complex case (see [24]), for a point $a \in \mathbb{B}$ and $0 < R < 1$, we can get that $U(a, R)$ with pseudo-hyperbolic center $a$ and pseudo hyperbolic radius $R$ is a Euclidean disc: its Euclidean center and Euclidean radius are $(1 - R^2)a$ and $(1 - |a|^2)R$, respectively.

We will need the following lemma in the sequel:

Lemma 2.8. [22] Let $f : \mathbb{B} \rightarrow \mathbb{H}$ be a hyperholomorphic function. Suppose that $0 < R < 1$ and $1 < q < \infty$. Then for every $a \in \mathbb{B}$, we have

$$|Df(a)|^q \leq \frac{3(4)^{2+q}}{\pi R^3 (1 - R^2)^{2q} (1 - |a|^2)^3} \int_{U(a, R)} |Df(x)|^q d\mathbb{B}_x.$$ 

3. $F(p, q, s)$-spaces in Clifford Analysis

In this section, relations between $F(p, q, s)$ and Bloch spaces, which have been attracted considerable attention are given in quaternion sense. Our results are extensions of the results due to Zhao (see [28]) in quaternion sense. We consider some essential properties of $F(p, q, s)$ spaces of quaternion-valued functions as basic scale properties.

Proposition 3.1. Let $f$ be a hyperholomorphic function in $\mathbb{B}$ and $f \in \mathcal{B}^{\frac{3(q+2)}{2p}}$. Then for $0 < p < \infty$, $-2 < q < \infty$ and $2 < s < \infty$, we have that

$$\int_{\mathbb{B}} |Df(x)|^p (1 - |x|^2)^{\frac{q}{2}} (1 - |\varphi_a(x)|^2)^s d\mathbb{B}_x \leq \lambda(\mathcal{B}(f))^{\frac{q}{2}(q+2)}.$$ 

Proof. For $\alpha > 0$, we have

$$(1 - |x|^2)^{\frac{q}{2}} |Df(x)| \leq \mathcal{B}^\alpha(f).$$

Then, for $\alpha = \frac{3(q+2)}{2p}$, we deduce that

$$\int_{\mathbb{B}} |Df(x)|^p (1 - |x|^2)^{\frac{q}{2}} (1 - |\varphi_a(x)|^2)^s d\mathbb{B}_x \leq \lambda(\mathcal{B}(f))^{\frac{q}{2}(q+2)}$$

$$\leq \lambda(\mathcal{B}(f))^{\frac{q}{2}(q+2)} (1 - |x|^2)^{\frac{3}{2p}} (1 - |\varphi_a(x)|^2)^s d\mathbb{B}_x.$$

Here, we used that the Jacobian determinant is $\frac{(1 - |a|^2)^2}{|1 - \bar{a}x|^2}$. Now, using the equality

$$(1 - |\varphi_a(x)|^2) = \frac{(1 - |a|^2)(1 - |x|^2)}{|1 - \bar{a}x|^2}.$$
we obtain that,

\[
\int_{ \mathbb{B} } |Df(x)|^p (1 - |x|^2)^{\frac{3}{2}q} (1 - \varphi_a(x)|^2)^s d\mathbb{B}_x
\]

\[
\leq \lambda (B(f))^{\frac{3}{2}(q+2)} \int_0^1 (1 - r^2)^{s-3} r^2 dr,
\]

where \( \lambda \) is a positive constant. The integral

\[
\int_0^1 (1 - r^2)^{s-3} r^2 dr < \infty
\]

for \( 2 < s < \infty \). This completes the proof.

**Corollary 3.2.** From proposition 3.1, for \( 0 < p < \infty \), \(-2 < q < \infty \) and \( 2 < s < \infty \), then we have that

\[
B^{\frac{3(q+2)}{2p}} \subset F(p, q, s).
\]

**Proposition 3.3.** Let \( f \) be a hyperholomorphic function in the unit ball \( \mathbb{B} \). Then for \( 1 < p < \infty \), \(-2 < q < \infty \) and \( 0 < s < \infty \), we have

\[
(1 - |a|^2)^{\frac{3}{2}(q+2)} |Df(a)|^p \leq \frac{48(2)^{2p}}{\pi R^3(1 - R^2)^{s+2p}} \int_{ \mathbb{B} } |Df(x)|^p (1 - |x|^2)^{\frac{3}{2}q} (1 - \varphi_a(x)|^2)^s d\mathbb{B}_x,
\]

where \( 0 < R < 1 \).

**Proof.** For a fixed \( R \in (0, 1) \), let

\[
E(a, R) = \{ x \in \mathbb{B} : |x - a| < R|1 - a| \}
\]

Then, we have

\[
\int_{ \mathbb{B} } |Df(x)|^p (1 - |x|^2)^{\frac{3}{2}p} (1 - \varphi_a(x)|^2)^s d\mathbb{B}_x
\]

\[
\geq \int_{U(a, R)} |Df(x)|^p (1 - |x|^2)^{\frac{3}{2}q} (1 - \varphi_a(x)|^2)^s d\mathbb{B}_x
\]

\[
\geq (1 - R^2)^s \int_{U(a, R)} |Df(x)|^p (1 - |x|^2)^{\frac{3}{2}q} d\mathbb{B}_x
\]

\[
\geq (1 - R^2)^s \int_{E(a, R)} |Df(x)|^p (1 - |x|^2)^{\frac{3}{2}q} d\mathbb{B}_x
\]

\[
\geq (1 - R^2)^s (1 - |a|^2)^{\frac{3}{2}q} \int_{E(a, R)} |Df(x)|^p d\mathbb{B}_x.
\]

Then, applying Lemma 2.8, we obtain

\[
\int_{ \mathbb{B} } |Df(x)|^p (1 - |x|^2)^{\frac{3}{2}q} (1 - \varphi_a(x)|^2)^s d\mathbb{B}_x
\]

\[
\geq \frac{48(2)^{2p}}{\pi R^3(1 - R^2)^{s+2p}} (1 - |a|^2)^{\frac{3}{2}q+3} |Df(a)|^p,
\]

which implies that,

\[
(1 - |a|^2)^{\frac{3}{2}(q+2)} |Df(a)|^p \leq \frac{48(2)^{2p}}{\pi R^3(1 - R^2)^{s+2p}} \int_{ \mathbb{B} } |Df(x)|^p (1 - |x|^2)^{\frac{3}{2}q} (1 - \varphi_a(x)|^2)^s d\mathbb{B}_x,
\]
This completes the proof.

**Corollary 3.4.** From proposition 3.2, we get for $1 < p < \infty$, $-2 < q < \infty$ and $0 < s < \infty$ that

$$F(p, q, s) \subset \mathcal{B}^{\frac{3(q+2)}{2p}}.$$

The following result gives a characterization of the quaternion Bloch space by quaternion $F(p, q, s)$ spaces.

**Theorem 3.5.** Let $f$ be hyperholomorphic in the unit ball $\mathbb{B}$. Then for $1 < p < \infty$, $-2 < q < \infty$ and $2 < s < \infty$, we have that

$$F(p, q, s) = \mathcal{B}^{\frac{3(q+2)}{2p}}.$$

**Proof.** The proof follows from Corollaries 3.2 and 3.4.

The importance of the above theorem is to give us a characterization for the hyperholomorphic Bloch space by the help of integral norms of $F(p, q, s)$ spaces of hyperholomorphic functions.

Also, with the same arguments used to prove the previous theorem, we can prove the following theorem for characterization of little hyperholomorphic Bloch space.

**Theorem 3.6.** Let $f$ be hyperholomorphic in the unit ball $\mathbb{B}$. Then, for $1 < p < \infty$, $-2 < q < \infty$ and $2 < s < \infty$, we have that

$$F_0(p, q, s) = \mathcal{B}_0^{\frac{3(q+2)}{2p}}.$$

4. **Weights in quaternion $F(p, q, s)$-spaces**

In this section, we give a characterization for the quaternion $F(p, q, s)$ spaces in terms of some different weighted functions in the unit ball of $\mathbb{R}^3$.

**Theorem 4.1.** Let $f$ be a hyperholomorphic function in $\mathbb{B}$. Then, for $1 < q < 4$ and $1 \leq p \leq 2 + \frac{q}{4}$, we have that

$$f \in F(p, q, s) \iff \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} |Df(x)|^p (1 - |x|^2)^\frac{3q}{2} (g(x, a))^s d\mathbb{B}_x < \infty.$$

**Proof.** First, we consider the equivalence

$$\int_{\mathbb{B}} |Df(x)|^p (1 - |x|^2)^\frac{3q}{2} (1 - |\varphi_a(x)|^2)^s d\mathbb{B}_x \simeq \int_{\mathbb{B}} |Df(x)|^p (1 - |x|^2)^\frac{3q}{2} (g(x, a))^s d\mathbb{B}_x,$$

with $g(x, a) = \frac{1}{4\pi} \left( \frac{1}{|\varphi_a(x)|} - 1 \right)$ and $\varphi_a(x) = (a - x)(1 - \bar{a}x)^{-1}$ the Möbius-transform, which maps the unit ball onto itself. After a change of variables $w = \varphi_a(x)$ (the Jacobian determinant $\left( \frac{1 - |a|^2}{1 - \bar{a}w} \right)^3$ has no singularities) we get

$$\int_{\mathbb{B}} |D_{\varphi_a(x)} f(x)|^p (1 - |\varphi_a(x)|^2)^\frac{3q}{2} (1 - |w|^2)^s \left( \frac{1 - |a|^2}{1 - \bar{a}w} \right)^3 d\mathbb{B}_w \simeq \int_{\mathbb{B}} |D_{\varphi_a(x)} f(x)|^p (1 - |\varphi_a(x)|^2)^\frac{3q}{2} g^s(w, 0) \left( \frac{1 - |a|^2}{1 - \bar{a}w} \right)^3 d\mathbb{B}_w,$$
where $D_x$ means the Cauchy-Riemann-operator with respect to $x$.

The problem here is, that $\overline{D}_xf(x)$ is hyperholomorphic, but after the change of variables $\overline{D}_xf(\varphi_a(w))$ is not hyperholomorphic. But we know from [23] that $\frac{1-\bar{w}a}{|1-\bar{w}a|^3} \overline{D}_xf(\varphi_a(w))$ is again hyperholomorphic. We also refer to [25] who studied this problem for the four-dimensional case already in 1979. Therefore, we get

$$\int_{\mathbb{B}} |\psi(w,a)|^p (1-|w|^2)^{\frac{3}{2}q+s} \frac{(1-|a|^2)^{\frac{3}{2}q+3}}{|1-\bar{a}w|^{2(\frac{3}{2}q+p+3)}} \, d\mathbb{B}_w,$$

with $\psi(w,a) = \frac{1-\bar{w}a}{|1-\bar{w}a|^3} \overline{D}_xf(\varphi_a(w))$. This means we have to find constants $C_1(s)$ and $C_2(s)$ with

$$\frac{1}{(4\pi)^s} C_1(s) \int_{\mathbb{B}} |\psi(w,a)|^p \left( \frac{1}{|w|} - 1 \right)^s \frac{(1-|w|^2)^{\frac{3}{2}q}(1-|a|^2)^{\frac{3}{2}q+3}}{|1-\bar{a}w|^{2(\frac{3}{2}q+p+3)}} \, d\mathbb{B}_w,$$

$$\leq \int_{\mathbb{B}} |\psi(w,a)|^p (1-|w|^2)^{\frac{3}{2}q+s} \frac{(1-|a|^2)^{\frac{3}{2}q+3}}{|1-\bar{a}w|^{2(\frac{3}{2}q+p+3)}} \, d\mathbb{B}_w,$$

$$\leq \frac{1}{(4\pi)^s} C_2(s) \int_{\mathbb{B}} |\psi(w,a)|^p \left( \frac{1}{|w|} - 1 \right)^s \frac{(1-|w|^2)^{\frac{3}{2}q}(1-|a|^2)^{\frac{3}{2}q+3}}{|1-\bar{a}w|^{2(\frac{3}{2}q+p+3)}} \, d\mathbb{B}_w.$$

**Part 1**

Let $C_2(s) = 2^s (4\pi)^s$. Then, using the inequalities

$$1 - |a| \leq |1 - \bar{a}w| \leq 1 + |a| \quad \text{and} \quad 1 - |w| \leq |1 - \bar{a}w| \leq 1 + |w|,$$

we obtain that

$$I_1 = \int_{\mathbb{B}} |\psi(w,a)|^p (1-|w|^2)^{\frac{3}{2}q+s} \frac{(1-|a|^2)^{\frac{3}{2}q+3}}{|1-\bar{a}w|^{2(\frac{3}{2}q+p+3)}} \, d\mathbb{B}_w,$$

$$= 2^s \int_{\mathbb{B}} |\psi(w,a)|^p \left( \frac{1}{|w|} - 1 \right)^s \frac{(1-|w|^2)^{\frac{3}{2}q}(1-|a|^2)^{\frac{3}{2}q+3}}{|1-\bar{a}w|^{2(\frac{3}{2}q+p+3)}} \, d\mathbb{B}_w,$$

$$= \int_{\mathbb{B}} |\psi(w,a)|^p (1-|w|^2)^{\frac{3}{2}q+s} \frac{(1-|a|^2)^{\frac{3}{2}q+3}}{|1-\bar{a}w|^{2(\frac{3}{2}q+p+3)}} \left\{ 1 - \frac{2^s(1-|w|)^s}{|w|^s(1-|w|^2)^s} \right\} \, d\mathbb{B}_w,$$

$$= \int_{\mathbb{B}} |\psi(w,a)|^p (1-|w|^2)^{\frac{3}{2}q+s} \frac{(1-|a|^2)^{\frac{3}{2}q+3}}{|1-\bar{a}w|^{2(\frac{3}{2}q+p+3)}} \left\{ 1 - \frac{2^s}{|w|^s(1+|w|)^s} \right\} \, d\mathbb{B}_w,$$

$$\leq (2)^{3q+s+3} \int_{\mathbb{B}} |\psi(w,a)|^p (1-|w|)^s-2p-3 \left\{ 1 - \frac{2^s}{|w|^s(1+|w|)^s} \right\} \, d\mathbb{B}_w,$$

$$= (2)^{3q+s+3} \int_{0}^{1} (M_p(\overline{D}f,r))^p (1-r)^s-2p-3 \left( 1 - \frac{2^s}{r^s(1+r)^s} \right) r^2 \, dr \leq 0,$$

with

$$(M_p(\overline{D}f,r))^p = \int_{0}^{\pi} \int_{0}^{2\pi} |h(r)\overline{D}f(r,\theta_1,\theta_2)|^p \sin \theta_1 d\theta_2 d\theta_1,$$
where, $h(r)$ stands for $\frac{1}{|1 - aw|^r}$ in spherical coordinates.

Because $(M_p(\mathcal{D}f, r))^p \geq 0 \ \forall \ r \in [0, 1]$ and $(1 - r)^{s-2p-3}\left(1 - \frac{2^s}{r^2(1+r)}\right)r^2 \leq 0$ \ \forall \ r \in [0, 1], 0 < p < \frac{\pi}{2} - 1; \ 2 < s < \infty$ and $0 < q < \infty$. Hence, we deduce that

\[
\int_{\mathbb{B}} |\psi(w, a)|^p (1 - |w|^2)^{\frac{3}{2}q+s} \frac{(1 - |a|^2)^{\frac{3}{2}q+3}}{|1 - \bar{a}w|^{2(\frac{3}{2}q+p+3)}} d\mathbb{B}_w \leq \frac{1}{(4\pi)^s} C_2(s) \int_{\mathbb{B}} |\psi(w, a)|^p \left(1 - \frac{1}{|w|}\right)^s \frac{(1 - |w|^2)^{\frac{3}{2}q}(1 - |a|^2)^{\frac{3}{2}q+3}}{|1 - \bar{a}w|^{2(\frac{3}{2}q+p+3)}} d\mathbb{B}_w.
\]

**Part 2**

Let $C_1(s) = (\frac{11}{100})^s (4\pi)^s$. Then,

\[
I_2 = \int_{\mathbb{B}} |\psi(w, a)|^p (1 - |w|^2)^{\frac{3}{2}q+s} \frac{(1 - |a|^2)^{\frac{3}{2}q+3}}{|1 - \bar{a}w|^{2(\frac{3}{2}q+p+3)}} d\mathbb{B}_w
- \frac{C_1(s)}{(4\pi)^s} \int_{\mathbb{B}} |\psi(w, a)|^p \left(1 - \frac{1}{|w|}\right)^s \frac{(1 - |w|^2)^{\frac{3}{2}q}(1 - |a|^2)^{\frac{3}{2}q+3}}{|1 - \bar{a}w|^{2(\frac{3}{2}q+p+3)}} d\mathbb{B}_w
= \int_{\mathbb{B}} |\psi(w, a)|^p (1 - |w|^2)^{\frac{3}{2}q+s} \frac{(1 - |a|^2)^{\frac{3}{2}q+3}}{|1 - \bar{a}w|^{2(\frac{3}{2}q+p+3)}} d\mathbb{B}_w.
\]

where $G(|w|) = 1 - (\frac{11}{100})^s \left(\frac{1}{|w|(1+|w|)}\right)^s$. To get our estimates, the integral $I_2$ must be greater than or equal to zero. Now, we have

\[
I_2 = -\int_{\mathbb{B}} |\psi(w, a)|^p (1 - |w|^2)^{\frac{3}{2}q+s} \frac{(1 - |a|^2)^{\frac{3}{2}q+3}}{|1 - \bar{a}w|^{2(\frac{3}{2}q+p+3)}} G(|w|) d\mathbb{B}_w
+ \int_{\mathbb{B} \setminus \mathbb{B}_r} |\psi(w, a)|^p (1 - |w|^2)^{\frac{3}{2}q+s} \frac{(1 - |a|^2)^{\frac{3}{2}q+3}}{|1 - \bar{a}w|^{2(\frac{3}{2}q+p+3)}} G(|w|) d\mathbb{B}_w
+ \int_{\mathbb{B} \setminus \mathbb{B}_{\frac{9}{10}r}} |\psi(w, a)|^p (1 - |w|^2)^{\frac{3}{2}q+s} \frac{(1 - |a|^2)^{\frac{3}{2}q+3}}{|1 - \bar{a}w|^{2(\frac{3}{2}q+p+3)}} G(|w|) d\mathbb{B}_w
+ \int_{\mathbb{B} \setminus \mathbb{B}_{\frac{16}{15}r}} |\psi(w, a)|^p (1 - |w|^2)^{\frac{3}{2}q+s} \frac{(1 - |a|^2)^{\frac{3}{2}q+3}}{|1 - \bar{a}w|^{2(\frac{3}{2}q+p+3)}} G(|w|) d\mathbb{B}_w
= \int_{\mathbb{B} \setminus \mathbb{B}_{\frac{16}{15}r}} |\psi(w, a)|^p (1 - |w|^2)^{\frac{3}{2}q+s} \frac{(1 - |a|^2)^{\frac{3}{2}q+3}}{|1 - \bar{a}w|^{2(\frac{3}{2}q+p+3)}} G(|w|) d\mathbb{B}_w, \quad (4.1)
\]

where $B_r$ is the ball centered at zero with radius $r$. It is clear that the second and the fourth integrals in $(4.1)$ are greater than zero. Therefore, it is sufficient to compare the first and the third integrals in $(4.1)$. Now, since in $B_{\frac{16}{15}r}$, we have that $\frac{9}{10} \leq 1 - |w| \leq |1 - \bar{a}w|$ and in $B_{\frac{16}{15}r} \setminus B_{\frac{9}{10}r}$, we have

\[
1 - |w| \leq |1 - \bar{a}w| \leq \frac{16}{10}.
\]
Then,
\[-\left(\frac{10}{9}\right)^{2(\frac{1}{2}q+p+3)} \frac{1}{\pi} \int_0^1 \left(M_p(\overline{Df}, r) \right)^p (1 - r^2)^{\frac{3}{2}q+s} \left(1 - \left(\frac{11}{100}\right)^s \frac{1}{r^s(1 + r)^s}\right) r^2 dr\]
\[\leq \left(\frac{10}{16}\right)^{2(\frac{1}{2}q+p+3)} \frac{6}{\pi} \int_0^1 \left(M_p(\overline{Df}, r) \right)^p (1 - r^2)^{\frac{3}{2}q+s} \left(1 - \left(\frac{11}{100}\right)^s \frac{1}{r^s(1 + r)^s}\right) r^2 dr.
\]

In particular we have that $M_p(\overline{Df}, r)$ is a nondecreasing function, this because $\overline{Df}$ is harmonic in $\mathbb{B}$ and belongs to $L_p(\mathbb{B})$; $\forall 0 \leq r < 1$.
Thus, $I_2 \geq 0$, and our theorem is therefore established.

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**References**


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