The Independence Number of $\Gamma(Z_{p^n}(x))$

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Abstract.
The zero-divisor graph of a commutative ring with unity (say $R$) is a graph whose vertices are the nonzero zero-divisors of this ring, where two distinct vertices are adjacent when their product is zero. This graph is denoted by $\Gamma(R)$. In this paper, we study the structure of the zero-divisor graph $\Gamma(Z_{p^n}(x))$ where $p$ is an odd prime number, $Z_{p^n}$ is the set of integers modulo $p^n$, and $Z_{p^n}(x) = \{a + bx : a, b \in Z_{p^n} \text{ and } x^2 = 0\}$. We find the Independence number of $\Gamma(Z_{p^n}(x))$.

Key Words: Zero-divisor graph, Independence number, Integers modulo $n$.

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1 Introduction

Throughout this paper, all rings are commutative with identity. For a ring $R$, let $R(x) = R[X]/(X^2)$, where $x = \overline{X}$ and $X$ is an indeterminant over $R$. Clearly, $R(x) = \{a + bx \mid x^2 = 0, x \notin R\}$. For a ring $R$, let $Z(R)$ be the set of zero divisors of $R$ and let $Z^*(R) = Z(R) \setminus \{0\}$. Zero-divisor graph was first introduced by Beck [7], where Beck was mainly interested in graph coloring. In his work, for a ring $R$, $Z(R)$ was taken to be the vertex set and distinct vertices $x$ and $y$ are adjacent if $xy = 0$. In a subsequent work, Anderson and Livingston [5] modified the definition. In their definition, the vertex set is taken to be $Z^*(R)$ and distinct vertices $x$ and $y$ are adjacent if $xy = 0$. The zero-divisor graph of $R$, $\Gamma(Z^*(R))$, is usually written $\Gamma(R)$. The definition of zero-divisor graph given by Anderson and Livingston is the one that has been used in the recent literature. In this paper we will also use their definition.

Much work has been done on zero-divisor graphs, and the reader is advised to consult [2, 5, 6, 10] for more details. Some researchers generalized the idea to commutative semigroups, see [8, 9]. Others worked on the noncommutative case, where they studied a directed graph related to the zero-divisors of noncommutative rings. For more information see [12, 13, 15]. Recently, some researchers worked on graphs whose vertex set is the set of all non-zero and non-unit elements, see [3, 4].

An independent set in a graph $\Gamma$ is a subset $I$ of the vertex set of $\Gamma$ such that no two vertices of $I$ are adjacent, i.e., the induced subgraph on $I$ is discrete. The independence number of $\Gamma$, denoted by $\text{Indep}(\Gamma)$, is defined as the maximum of the set of cardinalities of independent sets of vertices of $\Gamma$.

In this paper, we study the structure of the graph $\Gamma(\mathbb{Z}_{p^n}(x))$, where $p$ is an odd prime number, $\mathbb{Z}_{p^n}$ is the set of integers modulo $p^n$. We find the independence number of $\Gamma(\mathbb{Z}_{p^n}(x))$.

2 The structure of $\Gamma(\mathbb{Z}_{p^n}(x))$

The goal of this section is to describe the adjacency rules among vertices of $\Gamma(\mathbb{Z}_{p^n}(x))$. Part of this was given in [1], [11] and [14]. The following lemma characterizes the zero-divisors of $R(x)$. The proof of this lemma is easy and direct, thus it will be omitted here.

Lemma 1. Let $R$ be a commutative ring. Then $a + bx$ is a zero-divisor in $R(x)$ if and only if $a$ is a zero-divisor in $R$.

According to Lemma 1, one can characterize the zero-divisors of $\mathbb{Z}_{p^n}(x)$ by finding the zero-divisors of $\mathbb{Z}_{p^n}$. But the zero divisors of $\mathbb{Z}_{p^n}$ are the set of all integers in $\mathbb{Z}_{p^n}$ that are divisible by $p$. Hence the set of the zero divisors of $\mathbb{Z}_{p^n}(x)$
is \( \{a + bx : a, b \in \mathbb{Z}_p^n, \text{ and } \gcd(a, p) > 1\} \). We want to discuss how the vertices of \( \Gamma(\mathbb{Z}_p^n(x)) \) are adjacent. To do that we divide the set of vertices of \( \Gamma(\mathbb{Z}_p^n(x)) \) into three types. The first type consists of the \( n - 1 \) sets: \( S_p, S_{p^2}, \ldots, S_{p^{n-1}} \) where \( S_i = \{ sp^i : \gcd(s, p^{n-i}) = 1 \} \). The use of Euler’s phi-function gives the sizes of the \( S_i \)'s and one will get \( |S_i| = p^{n-i} - p^{n-(i+1)} \) for \( 1 \leq i \leq n - 1 \). Note that \( S_i \) is the set of units in \( \mathbb{Z}_p^n \), i.e. the set of elements in \( \mathbb{Z}_p^n \) that are not divisible by \( p \). The second type consists of \( n \) sets: \( E_{p^i}, E_{p^{i+1}}, \ldots, E_{p^{n-1}} \) where \( E_{p^i} = \{ bx : b \in S_{p^i} \} \). We have \( |E_{p^i}| = |S_{p^i}| = p^{n-i} - p^{n-(i+1)} \) for \( 1 \leq i \leq n - 1 \) and \( |E_{p^n}| = p^n - p^{n-1} \). The third type consists of \( n(n - 1) \) sets and these sets are: \( S_{p^i, p^j}, i \in \{1, 2, \ldots, n - 1\} \) and \( j \in \{0, 1, \ldots, n - 1\} \) where \( S_{p^i, p^j} = \{ sp^i + tp^j x : \gcd(s, p^{n-i}) = 1 \) and \( \gcd(t, p^{n-j}) = 1\) \}. Again the use of Euler’s phi-function gives the sizes of the \( S_{p^i, p^j} \)'s and one will get \( |S_{p^i, p^j}| = (p^{n-i} - p^{n-(i+1))}(p^{n-j} - p^{n-(j+1)}) \) for \( 1 \leq i \leq n - 1 \) and \( 0 \leq j \leq n - 1 \).

We state how the elements of these types are adjacent. Each element of \( S_{p^i} \) is adjacent to all the elements of \( S_{p^{i'}} \) if \( i + i' \geq n \). Also, each element of \( S_{p^i} \) is adjacent to all the elements of \( E_{p^i} \) if \( i + i' \geq n \). Again, each element of \( S_{p^i} \) is adjacent to all the elements of \( S_{p^{i'}, p^j} \) if \( i + i' \geq n \) and \( i + j' \geq n \). For the \( E_{p^i} \)'s, each element of \( E_{p^i} \) is adjacent to all the elements of \( E_{p^{i'}} \) for any \( i' \) and hence \( \bigcup_{i=0}^{n-1} E_{p^i} \) forms a complete subgraph of \( \Gamma(\mathbb{Z}_p^n(x)) \). Again, each element of \( E_{p^i} \) is adjacent to all the elements of \( S_{p^i, p^{i'}} \) if \( i + i' \geq n \). Observe that no conditions are required on \( j' \).

Finally, we want to discuss how the elements of the third type are adjacent to each other. To do that, consider the two sets \( S_{p^i, p^j} \) and \( S_{p^k, p^m} \). Observe that a necessary condition in order that some elements in the set \( S_{p^i, p^j} \) are adjacent to some elements in the set \( S_{p^k, p^m} \) is that \( i + k \geq n \). Now consider the following subcases where \( i + k \geq n \).

1) If \( i + m \geq n \) and \( k + j \geq n \), then it is clear that each element of \( S_{p^i, p^j} \) is adjacent to all the elements of \( S_{p^k, p^m} \).

2) Suppose that \( i + m \geq n \) and \( k + j < n \) and take \( y_1 \in S_{p^i, p^j} \) and \( y_2 \in S_{p^k, p^m} \), say \( y_1 = a_1 p^i + b_1 p^j x \) and \( y_2 = a_2 p^k + b_2 p^m x \) where \( \gcd(a_1, p^{n-i}) = \gcd(b_1, p^{n-j}) = \gcd(a_2, p^{n-k}) = \gcd(b_2, p^{n-m}) = 1 \). We have \( y_1 \cdot y_2 = a_1 a_2 p^{i+k} + a_1 b_2 p^i + m x + a_2 b_1 p^{k+j} x = a_2 b_1 p^{k+j} x \neq 0 \) (mod \( p^n \)). Hence no element of \( S_{p^i, p^j} \) is adjacent to any element of \( S_{p^k, p^m} \). Similarly, if \( i + m < n \) and \( k + j \geq n \), then no element of \( S_{p^i, p^j} \) is adjacent to any element of \( S_{p^k, p^m} \).

3) Suppose that \( i + m < n \) and \( k + j < n \) with \( i + m \neq k + j \). We assume
that \( i + m < k + j \). Take \( y_1 \in S_{p^i, p^j} \) and \( y_2 \in S_{p^k, p^m} \), say \( y_1 = a_1 p^i + b_1 p^j x \) and \( y_2 = a_2 p^k + b_2 p^m x \) where \( \gcd(a_1, p^{n-i}) = \gcd(b_1, p^{n-j}) = \gcd(a_2, p^{n-k}) = \gcd(b_2, p^{n-m}) = 1 \). We have \( y_1 \cdot y_2 = a_1 a_2 p^{i+k} + a_1 b_2 p^{i+m} x + a_2 b_1 p^{k+j} x = p^{i+m}(a_1 b_2 + a_2 b_1) p^{k+j-i-m} x \pmod{p^n} \). If \( p^{i+m}(a_1 b_2 + a_2 b_1) p^{k+j-i-m} = 0 \pmod{p^n} \), then \( p^{n-(i+m)} \) divides \( (a_1 b_2 + a_2 b_1) p^{k+j-i-m} \) and hence \( p \) divides \( a_1 b_2 \). So we get \( p \) divides \( a_1 \) or \( p \) divides \( b_2 \). But this is impossible because \( \gcd(a_1, p^{n-i}) = \gcd(b_2, p^{n-m}) = 1 \). Thus \( y_1 \cdot y_2 \neq 0 \pmod{p^n} \). Hence no element of \( S_{p^i, p^j} \) is adjacent to any element of \( S_{p^k, p^m} \). Similarly, if \( i + m < n \) and \( k + j < n \) with \( i + m > k + j \), then no element of \( S_{p^i, p^j} \) is adjacent to any element of \( S_{p^k, p^m} \).

4) Suppose that \( i + m < n \) and \( k + j < n \) with \( i + m = k + j \). In this case, we show that some elements of \( S_{p^i, p^j} \) are adjacent to some elements of \( S_{p^k, p^m} \). To explain that, take \( y_1 \in S_{p^i, p^j} \) and \( y_2 \in S_{p^k, p^m} \), say \( y_1 = a_1 p^i + b_1 p^j x \) and \( y_2 = a_2 p^k + b_2 p^m x \) where \( \gcd(a_1, p^{n-i}) = \gcd(b_1, p^{n-j}) = \gcd(a_2, p^{n-k}) = \gcd(b_2, p^{n-m}) = 1 \). We have \( y_1 \cdot y_2 = a_1 a_2 p^{i+k} + a_1 b_2 p^{i+m} x + a_2 b_1 p^{k+j} x = p^{i+m}(a_1 b_2 + a_2 b_1) x \pmod{p^n} \). If \( a_1 b_2 + a_2 b_1 = 0 \pmod{p^{n-i-m}} \), then \( y_1 \) is adjacent to \( y_2 \) and otherwise \( y_1 \) and \( y_2 \) are not adjacent. For instance, if \( a_1 = a_2 = 1, b_1 = p^{n-i-m} + 1 \), and \( b_2 = p^{n-i-m} - 1 \), then \( a_1 b_2 + a_2 b_1 = 0 \pmod{p^{n-i-m}} \) and hence \( y_1 \) and \( y_2 \) are adjacent. On the other hand, if \( a_1 = a_2 = 1, b_1 = b_2 = 1 \), then \( a_1 b_2 + a_2 b_1 = 1 \pmod{p^{n-i-m}} \) and hence \( y_1 \) and \( y_2 \) are not adjacent.

Now, we look more closely in the last subcase. For \( S_{p^i, p^j} \) and \( S_{p^k, p^m} \), suppose that \( i + k \geq n \) and \( i + m = k + j < n \). We decompose \( S_{p^i, p^j} \) into \( p^{n-(k+j)} - p^{n-(k+j)-1} \) mutually disjoint subsets. These subsets are \( X_t = \{r p^i + p^j(s p^{n-k-j} + r t) x, \ \text{where} \ 1 \leq r \leq p^{n-i} \text{with} \ \gcd(r, p^{n-i}) = 1, \ 0 \leq s \leq p^k - 1 \} \) and \( 1 \leq t \leq p^{n-(k+j)} \text{with} \ \gcd(t, p^{n-(k+j)}) = 1 \). Observe that the set \( \{t : 1 \leq r \leq p^{n-i} \text{and} \ \gcd(r, p^{n-i}) = 1\} \) with multiplication is the group \( (U(p^{n-i}), \cdot) \) and the set \( \{t : 1 \leq t \leq p^{n-(k+j)} \text{and} \ \gcd(t, p^{n-(k+j)}) = 1\} \) with multiplication is the group \( (U(p^{n-(k+j)}), \cdot) \). The bar on \( rt \) means that we are taking \( rt \) modulo \( p^{n-(k+j)} \). Observe that when \( t \) runs over all the elements of \( U(p^{n-(k+j)}) \) and \( r \in U(p^{n-i}) \) then \( rt \) runs over all the elements of \( U(p^{n-(k+j)}) \). Hence \( S_{p^i, p^j} = \bigcup_{t \in U(p^{n-(k+j)})} X_t \) and the size of each \( X_t \) is \( p^k(p^{n-i} - p^{n-i-1}) \). Let \( X \) be the set \( \{X_t : t \in U(p^{n-(k+j)})\} \). Then we have \( |X| = p^{n-(k+j)} - p^{n-(k+j)-1} \).

Similarly, \( S_{p^k, p^m} = \bigcup_{t \in U(p^{n-(i+m)})} Y_t \) where \( Y_t = \{r' p^k + p^m(s' p^{n-i-m} + r't) x : r' \in U(p^{n-k}) \text{and} \ 0 \leq s' \leq p^i - 1 \} \) and \( |Y_t| = p^i(p^{n-k} - p^{n-k-1}) \). Let \( Y \) be the set \( \{Y_t : t \in U(p^{n-(i+m)})\} \). Then we have \( |Y| = p^{n-(i+m)} - p^{n-(i+m)-1} \).

Since \( i + m = k + j \), we get \( |X| = |Y| \). We want to see when the elements of \( X_t \) are adjacent to the elements of \( Y_t \) where \( t, t' \in U(p^{n-(k+j)}) \).
Suppose that \( y \in X_t \) and \( y' \in Y_t \), say \( y = rp^i + p^j(s p^{p_n - (j+k)} + rt)x \) and 
\( y' = r'p^k + p^m(s' p^{p_n - (j+k)} + rt'x) \). We get \( y \cdot y' = rrp^{k+i} + (rs'p^n + r'sp^n + 
rt'p^{i+m} + r'tp^{k+j})x \). Since \( i + k \geq n \) and \( i + m = k + j \), we get \( y \cdot y' = (rr't'+ 
rt'p^{k+j})x = ((\bar{r} + up^{n-(k+j)})p^n + (\bar{t'} + u'p^{p_n-(k+j)})p^n)k+jx = (rr't'+ r'tt')p^{k+j}x \) 
( mod \( p^n \)) = \((\bar{r'})p^{k+j}x \) ( mod \( p^n \)), where \( u \) and \( u' \) are nonnegative integers. Since \( r \in U(p^{n-i}) \) and \( r' \in U(p^{n-k}) \), we get \( y \cdot y' = 0 \) (mod \( p^n \)) if 
and only if \( p^{n-(k+j)} \) divides \( t + t' \). Hence, \( y \) and \( y' \) are adjacent if and only if 
\( p^{n-(k+j)} \) divides \( t + t' \). So, each element of \( X_t \) is adjacent to all the elements 
of \( Y_t \) if and only if \( p^{n-(k+j)} \) divides \( t + t' \). Thus each element of \( X_t \) is adjacent 
to all the elements of \( Y_{p^{n-(k+j)-t}} \), where \( t \in U(p^{-(k+j)}) \).

Let us see how the elements of \( S_{p^u,p^v} \) are adjacent to each other when 
\( u \geq \lfloor (n+1)/2 \rfloor \) and \( u + v \leq n - 1 \). As above, 
\( S_{p^u,p^v} = \bigcup_{t \in U(p^{-(u+v)})} W_t \) with 
\( W_t = \{ rp^u + p^v(s p^{n-u-v} + rt)x , \text{ where } r \in U(p^{n-u}) \text{ and } 0 \leq s \leq p^{u} - 1 \} \) and 
\( |W_t| = p'(p^{n-u} - p^{n-v}) \). As explained above, each element of \( W_t \) is adjacent 
to all the elements of \( W_{t'} \) if and only if \( p^{n-(u+v)} \) divides \( t + t' \). Thus each 
element of \( W_t \) is adjacent to all the elements of \( W_{p^{n-(u+v)-t}} \) and no element of \( W_t \) 
is adjacent to any element of \( S_{p^u,p^v} - W_{p^{n-(u+v)-t}} \) where \( t \in U(p^{-(u+v)}) \). Hence 
the induced subgraph on each pair \( W_t, W_{p^{n-(u+v)-t}} \), where \( t \in U(p^{-(u+v)}) \), 
is isomorphic to the complete bipartite graph \( K_{(p^{n-(u+v)} - p^{n-(u+v)-1})p^u,(p^{n-(u+v)} - p^{n-(u+v)-1})p^v} = K_{(p-1)p^u-1,(p-1)p^v-1} \). So, the induced subgraph on \( S_{p^u,p^v} \) is isomorphic to 
\( (p^{n-(u+v)} - p^{n-(u+v)-1})/2 \) disjoint copies of \( K_{(p-1)p^u-1,(p-1)p^v-1} \). We summarize 
this in the following lemma.

**Lemma 2.** For \( S_{p^u,p^v} \) and \( S_{p^k,p^m} \), suppose that \( i + k \geq n \) and \( i + m = k + j < n \). 
Consider \( \{ X_t : t \in U(p^{n-(k+j)}) \} \) and \( \{ Y_t : t \in U(p^{n-(i+m)}) \} \), where \( X_t \) and \( Y_t \) 
are defined above. Then each element of \( X_t \) is adjacent to all the elements 
of \( Y_t \) if and only if \( p^{n-(k+j)} \) divides \( t_1 + t'_1 \). Moreover, suppose that \( 2u \geq n \) 
and \( u + v \leq n - 1 \) hold in \( S_{p^u,p^v} \). Then the induced subgraph on \( S_{p^u,p^v} \) is 
isomorphic to \( (p^{n-(u+v)} - p^{n-(u+v)-1})/2 \) disjoint copies of the complete bipartite 
graph \( K_{(p-1)p^u-1,(p-1)p^v-1} \).

The previous work gives an algorithm to construct the zero divisor graph 
\( \Gamma(Z_{p^n}(x)) \). The following example explains how to construct the zero divisor 
graph \( \Gamma(Z_{5^7}(x)) \).

**Example 1.** Consider the zero divisor graph \( \Gamma(Z_{5^7}(x)) \). For \( 1 \leq i \leq 6 \), each 
element \( y \) of \( S_{5^i} \) is adjacent to all the elements of 
\( \bigcup_{j=7-i}^{6} \bigcup_{t=7-i}^{6} (S_{5^j \cup E_{5^j} \cup S_{5^j \cup 5^j}} - \) 
\{ y \}. Also, for \( 0 \leq i \leq 6 \), each element of \( E_{5^i} \) is adjacent to all the elements
and no element of $S_{5^{i},5^{j}}$ is adjacent to any element of $S_{5^{k},5^{m}}$.

**Case 3)** $i + m$ and $j + k$ are both less than 7. Here there are two subcases. (i) $i + m$ is not equal to $j + k$. In this subcase, no element of $S_{5^{i},5^{j}}$ is adjacent to any element of $S_{5^{k},5^{m}}$. (ii) $i + m = j + k$. In this subcase, we use Lemma 2 to determine the adjacency between the elements of $S_{5^{i},5^{j}}$ and $S_{5^{k},5^{m}}$. For instance, take $S_{5^{3},5^{2}}$ and $S_{5^{4},5^{3}}$. According to Lemma 2 we divide the set $S_{5^{3},5^{2}}$ into $5^{7−(3+3)} − 5^{7−(3+3)−1} = 5 − 1 = 4$ subsets. These subsets are $X_{t} = \{r5^{3} + 5^{2}(s5^{3} + r17) : r \in U(5^{4}) \text{ and } 0 \leq s \leq 5^{4}−1\}$, where $t \in U(5)$ and $|X_{t}| = 5^{4}(5^{4}−3)$.

Also, we divide $S_{5^{4},5^{3}}$ into $5^{7−(3+3)} − 5^{7−(3+3)−1} = 5 − 1 = 4$ subsets. These subsets are $Y_{t} = \{r'5^{4} + 5^{3}(s'5^{1} + r'77) : r' \in U(5^{3}) \text{ and } 0 \leq s' \leq 5^{3}−1\}$, where $t' \in U(5)$ and $|Y_{t'}| = 5^{3}(5^{3}−2)$.

We use Lemma 2 to get, that each element of $X_{t}$ is adjacent to all the elements of $Y_{5−t}$, where $t \in U(5) = \{1, 2, 3, 4\}$. For instance, each element of $X_{1}$ is adjacent to all the elements of $Y_{4}$ and no element of $X_{1}$ is adjacent to any element of the sets $Y_{1}, Y_{2}$, or $Y_{3}$. Now, let us see how the elements of $S_{5^{i},5^{j}}$ are adjacent to each other. According to Lemma 2, we divide the elements of the set $S_{5^{i},5^{j}}$ into $5^{2} − 5 = 20$ subsets. These subsets are $W_{t} = \{r5^{4} + 5^{2}(s5^{2} + r17) : r \in U(5^{3}) \text{ and } 0 \leq s \leq 5^{4}−1\}$, where $t \in U(5^{2})$. Using Lemma 2, every element of $W_{t}$ is adjacent to all the elements of $W_{5^{2}−t}$ where $t \in U(5^{2})$. And no element of $W_{t}$ is adjacent to any element of $S_{5^{3},5^{1}}$.

Hence, the induced subgraph on each pair $W_{t}, W_{5^{2}−t}$ is a complete bipartite graph that is isomorphic to $K_{5^{4}(5^{3}−5^{2})}, 5^{4}(5^{3}−5^{2})$. This way we are able to construct the zero divisor graph $\Gamma(Z_{5^{7}}(x))$.

3 **The Independence number of $\Gamma(Z_{p^{n}}(x))$**

This section is devoted to compute the independence number of the zero divisor graph $\Gamma(Z_{p^{n}}(x))$. In finding the independence number, we will study the structure of $\Gamma(Z_{p^{n}}(x))$ thoroughly. Suppose that $I$ is an independent set with maximum cardinality in $\Gamma(Z_{p^{n}}(x))$. We present a series of lemmas that determine the elements of the set $I$.

In the following lemma, we show that $I$ contains exactly one element of the
Lemma 3. Suppose that $I$ is an independent set with maximum cardinality in $\Gamma(\mathbb{Z}_p^n(x))$. Then $I$ contains exactly one element of the set $\bigcup_{i=0}^{n-1} E_{p^i}$. Moreover, $I$ can be chosen so that this element sits in $E_{p^0}$.

Proof. Suppose that $I$ is an independent set with maximum cardinality in $\Gamma(\mathbb{Z}_p^n(x))$. Since $\bigcup_{i=0}^{n-1} E_{p^i}$ is a complete subgraph of $\Gamma(\mathbb{Z}_p^n(x))$, the set $I$ contains at most one element of $\bigcup_{i=0}^{n-1} E_{p^i}$. Suppose $I$ contains an element $w$ in $\bigcup_{i=1}^{n-1} E_{p^i}$ and suppose $w_0 \in E_{p^0}$. The element $w_0$ is not adjacent to any element in $V(\Gamma(\mathbb{Z}_p^n(x))) - \bigcup_{i=0}^{n-1} E_{p^i} = \bigcup_{i=1}^{n-1} S_{p^i} \cup \bigcup_{i=1}^{n-1} \bigcup_{j=0}^{n-1} S_{p^i, p^j}$. Hence $I_0 = (I \cup \{w_0\}) - \{w\}$ is an independent set with maximum cardinality. If the set $I$ contains no elements of $\bigcup_{i=1}^{n-1} E_{p^i}$, then $I$ must contain an element of $E_{p^0}$. This is because $I$ has a maximum cardinality and none of the elements of $E_{p^0}$ are adjacent to any element of $\bigcup_{i=1}^{n-1} S_{p^i} \cup \bigcup_{i=1}^{n-1} \bigcup_{j=0}^{n-1} S_{p^i, p^j}$. So either $I$ contains an element of $E_{p^0}$ or we can replace $I$ by a independent set with maximum cardinality that contains an element of $E_{p^0}$.

\[\square\]

In the following, we always assume that $I$ is taken as in Lemma 3. In the following two lemmas, we show that $I$ contains no elements of the set

$$\bigcup_{i=\lfloor \frac{n}{2} \rfloor + 1}^{n-1} S_{p^i} \cup \bigcup_{i=\lfloor \frac{n}{2} \rfloor + 1}^{n-1} \bigcup_{j=\lfloor \frac{n}{2} \rfloor + 1}^{n-1} S_{p^i, p^j}.$$ 

Lemma 4. Suppose that $I$ is an independent set with maximum cardinality in $\Gamma(\mathbb{Z}_p^n(x))$, where $n \geq 3$. Then $I$ can not contain any element of the set $\bigcup_{i=\lfloor \frac{n}{2} \rfloor + 1}^{n-1} S_{p^i}$.

Proof. Since $\bigcup_{i=\lfloor \frac{n}{2} \rfloor + 1}^{n-1} S_{p^i}$ is a complete subgraph of $\Gamma(\mathbb{Z}_p^n(x))$, then $I$ contains
at most one element of \( \bigcup_{i=\lceil n/2 \rceil+1}^{n-1} S_{p^i} \). Suppose that \( x_0 \in S_{p^0} \cap I \), where \( i_0 \geq \lceil n/2 \rceil + 1 \). Since all the elements of \( \bigcup_{i=i_0}^{n-1} S_{p^i} \cup \bigcup_{i=i_0}^{n-1} S_{p^{j,i}} \cup \bigcup_{i=i_0}^{n-1} E_{p^i} \) are adjacent to \( x_0 \), then \( I \cap \left( \bigcup_{i=i_0}^{n-1} S_{p^i} \cup \bigcup_{i=i_0}^{n-1} S_{p^{j,i}} \cup \bigcup_{i=i_0}^{n-1} E_{p^i} \right) = \emptyset \). Let \( I_0 = (I \cup S_{p^{n-i_0}}) - \{x_0\} \). Note that the induced graph on the vertices \( S_{p^{n-i_0}} \) in \( \Gamma(Z_{p^n}(x)) \) is the null graph. Also, the elements of \( S_{p^{n-i_0}} \) are not adjacent to any element of \( I - \{x_0\} \) because \( I \cap \left( \bigcup_{i=i_0}^{n-1} S_{p^i} \cup \bigcup_{i=i_0}^{n-1} S_{p^{j,i}} \cup \bigcup_{i=i_0}^{n-1} E_{p^i} \right) = \emptyset \). Hence \( I_0 \) is an independent set. But \( |I_0| = |I| - 1 + p^{i_0} - p^{i_0-1} > |I| \). So, \( I \) contains no elements of \( \bigcup_{i=\lceil n/2 \rceil+1}^{n-1} S_{p^i} \).

Lemma 5. Suppose that \( I \) is an independent set with maximum cardinality in \( \Gamma(Z_{p^n}(x)) \), where \( n \geq 3 \). Then \( I \) contains no elements of the set \( \bigcup_{i=\lceil n/2 \rceil+1}^{n-1} \bigcup_{j=\lceil n/2 \rceil+1}^{n-1} S_{p^i,p^j} \).

Proof. The proof is similar to that of the last lemma and so we skip this proof.

In the following lemma, we show that \( \bigcup_{i=1}^{\lceil n/2 \rceil} S_{p^i} \) is a subset of \( I \).

Lemma 6. Suppose that \( I \) is an independent set with maximum cardinality in \( \Gamma(Z_{p^n}(x)) \), where \( n \geq 3 \). Then \( \bigcup_{i=1}^{\lceil n/2 \rceil} S_{p^i} \) is a subset of \( I \).

Proof. The induced graph on the vertices \( \bigcup_{i=1}^{\lceil n/2 \rceil} S_{p^i} \) in \( \Gamma(Z_{p^n}(x)) \) is the null graph and the adjacency set of these vertices is a subset of the vertices \( \bigcup_{i=\lceil n/2 \rceil+1}^{n-1} S_{p^i} \cup \bigcup_{i=\lceil n/2 \rceil+1}^{n-1} S_{p^{j,i}} \cup \bigcup_{i=\lceil n/2 \rceil+1}^{n-1} E_{p^i} \). Using Lemmas 3, 4 and 5, \( I \) contains
no elements of $\bigcup_{i=[\frac{n}{2}]+1}^{n-1} S_{p} \cup \bigcup_{i=[\frac{n}{2}]+1}^{n-1} S_{p,n-2i} \cup \bigcup_{i=[\frac{n}{2}]+1}^{n-1} E_{p}$. Hence an independent set with maximum cardinality contains all the elements of $\bigcup_{i=1}^{\lceil \frac{n-1}{2} \rceil} S_{p}$.

In the following lemma, we show that $\bigcup_{i=1}^{\lceil \frac{n-1}{2} \rceil} \bigcup_{j=0}^{n-1} S_{p,i} \cup S_{p,i,j}$ is a subset of $I$ and

$$I \cap \left( \bigcup_{i=[\frac{n}{2}]+1}^{n-1} \bigcup_{j=i-[\frac{n-1}{2}]}^{n-1} S_{p,i,j} \right) = \emptyset$$

**Lemma 7.** Suppose that $I$ is an independent set with maximum cardinality in $\Gamma(\mathbb{Z}_{p}(x))$, where $n \geq 3$. Then $I \cap \left( \bigcup_{i=[\frac{n}{2}]+1}^{n-1} \bigcup_{j=i-[\frac{n-1}{2}]}^{n-1} S_{p,i,j} \right) = \emptyset$ and $\bigcup_{i=1}^{\lceil \frac{n-1}{2} \rceil} \bigcup_{j=0}^{n-1} S_{p,i} \cup S_{p,i,j}$ is a subset of $I$.

**Proof.** Take $S_{p,i,j}$ with $i \leq \lfloor (n+1)/2 \rfloor$. Using Lemma 4 and Lemma 6, there are no elements in $\bigcup_{i=1}^{n-1} S_{p} \cap I$ adjacent to $S_{p,i,j}$. For $S_{p,k,p}^{n}$ with $i+k \geq n$, $i+m$, $k+j \geq n$, we get $k \geq \lceil n/2 \rceil + 1$ and $m \geq \lceil n/2 \rceil + 1$. Hence, using Lemma 5, none of the elements of $S_{p,k,p}^{n}$ are in $I$, where $i\leq k \geq n$, $i+m$, $k+j \geq n$.

So, in this case the elements of $S_{p,i} \cup S_{p,i,j}$ are adjacent to some elements that are outside $I$.

Using Lemma 2, every element of $S_{p,i,j}$ is adjacent to some of the elements of $S_{p,n-i,p-n-2i}$ provided that $n - 2i + j \leq n - 1$. In case $n - 2i + j \geq n$, the induced subgraph on $S_{p,i,j}$ is a set of null vertices whose elements are adjacent to some elements that are outside $I$ and hence $S_{p,i,j}$ is a subset of $I$. For the case where $n - 2i + j \leq n - 1$, every element of $S_{p,i,j}$ is adjacent to some of the elements of $S_{p,n-i,p-n-2i}, S_{p,n-i+1,p-n-2i+j+1}, \ldots, S_{p,n-i+s,p-n-2i+j+s}$, where $n - i + s = n - 1$, $n - 2i + j + s \leq n - 1$, or $n - i + s \leq n - 1$, $n - 2i + j + s = n - 1$. Hence, if $n - 2i + j \leq n - 1$, then every element of $S_{p,i,j}$ is adjacent to some elements of $A = \bigcup_{r=0}^{n-1} S_{p,n-i+r,p-n-2i+j+r}$. Note that if $(n - i) + (n - 2i + j) \geq n$, then the induced subgraph on $A$ is complete. Since the induced graph on the vertices $S_{p,i,j}$ in $\Gamma(\mathbb{Z}_{p}(x))$ is the null graph, we get $I \cap A = \emptyset$ and $S_{p,i,j} \subseteq I$. Also, if $j \geq i$, then $(n - i) + (n - 2i + j) \geq n$.
because $i \leq [(n-1)/2]$. Hence the induced subgraph on $A$ is complete. Since $S_{p', p'}$ is a set of null vertices, we get $I \cap A = \emptyset$ and $S_{p', p'} \subseteq I$. So, we can assume that $j < i$ and $(n-i)+(n-2i+j) \leq n-1$. Our goal is to show that $I \cap A = \emptyset$. Assume on the contrary that $I \cap A \neq \emptyset$. For $0 \leq r \leq i - j - 1$, Lemma 2 gives $S_{p', p'} = \bigcup_{t \in U(p')} X'_t$ where $X'_t = \{a_r p^i + p^j (b_r b^{-j-r} + a_r t) : a_r \in U(p') \}$, and $S_{p^n-i+r, p^n-2i+j+r} = \bigcup_{t' \in U(p')}$, $Y'_{t'}$ where $Y'_{t'} = \{a_r' p^n-i+r + p^n-2i+j+r (b_{r'} b^{-j-r} + a_r t') : a_r' \in U(p') \}$, and since the induced subgraph on $Y'_{t'}$ is complete, then $I$ contains at most $T \sum_{r=0}^s (p^r - p^{i-r-1}) p^i$ elements of $\bigcup_{r=0}^s S_{p^n-i+r, p^n-2i+j+r}$ and since the induced subgraph on $\bigcup_{r=0}^s S_{p^n-i+r, p^n-2i+j+r}$ is complete, then $I$ contains at most one element of $\bigcup_{r=0}^s S_{p^n-i+r, p^n-2i+j+r}$. Hence $I$ contains at most $T \sum_{r=0}^{i-j} (p^r - p^{i-r-1}) p^i + 1$ elements of $A$. Take $I_0 = (I - A) \cup \bigcup_{t \in T^0} X'_{t_0}$. Note that, the induced subgraph on $\bigcup_{t \in T^0} X'_{t_0}$ is a set of null vertices and the elements of $\bigcup_{t \in T^0} X'_{t_0}$ are either adjacent to some elements that are outside $I$ or to some elements of $A$. Hence $I_0$ is an independent set. Since $I$ contains at least one element from each $Y'_{t_0}$ where $t' \in T^0$ and this element is adjacent to all the elements of $X'_{t_0}$, then $I \cap \bigcup_{t \in T^0} X'_{t_0} = \emptyset$. So, $|I_0| \geq |I| - (T \sum_{r=0}^{i-j} (p^r - p^{i-r-1}) p^i + 1) + T(p^n - p^{n-i}) p^i + 1) + T(p^n - p^{n-i}) p^n > |I| - (T(p^i - p^{i-1}) p^i + 1) + T(p^n - p^{n-i}) p^n > |I|$. The last greater than follows because $i \leq [(n-1)/2] < n - i$. This contradicts the fact that $I$ is an independent set with maximum cardinality. Hence $I \cap A = \emptyset$. As above by taking $S_{p^i, p^j} = S_{p^{[n-1]/2}, p^j}$, $0 \leq j \leq n - 1$, we get $I \cap \left( \bigcup_{i=\lceil \frac{n-1}{2} \rceil + 1}^{n-1} \bigcup_{j=\lceil \frac{n-1}{2} \rceil}^{n-1} S_{p^i, p^j} \right) = \emptyset$ and $\bigcup_{i=1}^{n-1} \bigcup_{j=0}^{[n-1]/2} S_{p^i, p^j} \subseteq I$. $\square$
We summarize what we have about an independent set $I$ with maximum cardinality in $\Gamma(\mathbb{Z}_{p^\nu}(x))$, where $n \geq 3$. Using Lemmas 3 to 7 to get
\[
\bigcup_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} \bigcup_{j=0}^{n-1} S_{p^i,j} \cup \{w_0\} \subseteq I \quad \text{and} \quad I \cap \bigg( \bigcup_{i=\lceil \frac{n}{2} \rceil+1}^{\lfloor \frac{n-1}{2} \rfloor} S_{p^i} \bigcup \bigg( \bigcup_{i=\lfloor \frac{n}{2} \rfloor+1}^{\lfloor \frac{n}{2} \rfloor-1} S_{p^i,p^j} \bigcup \bigcup_{i=\lfloor \frac{n}{2} \rfloor}^{n-1} (E_{p^i} - \{w_0\}) \bigg) \bigg) = \emptyset,
\]
where $w_0$ is an element of $E_{p^0}$. Now, $| \bigcup_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} S_{p^i} \bigcup \bigcup_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} S_{p^i,p^j} \bigcup \bigcup_{i=\lceil \frac{n}{2} \rceil+1}^{\lfloor \frac{n}{2} \rfloor} S_{p^i,p^j} \bigcup \bigcup_{i=\lfloor \frac{n}{2} \rfloor}^{n-1} (E_{p^i} - \{w_0\}) | = \sum_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} (p^{n-i} - p^{n-i-1}) + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=0}^{n-1} (p^{n-i} - p^{n-i-1})(p^{n-j} - p^{n-j-1}) + 1 = p^{2n-\lfloor \frac{n-1}{2} \rfloor - 1}(p^{1-\lfloor \frac{n-1}{2} \rfloor} - 1) + 1. We state this in the following corollary.

**Corollary 1.** Suppose that $I$ is an independent set with maximum cardinality in $\Gamma(\mathbb{Z}_{p^\nu}(x))$, where $n \geq 3$. Then $I$ contains exactly $p^{2n-\lfloor \frac{n-1}{2} \rfloor - 1}(p^{1-\lfloor \frac{n-1}{2} \rfloor} - 1) + 1$ elements from the set
\[
\bigcup_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} S_{p^i} \bigcup \bigcup_{i=1}^{\lfloor \frac{n}{2} \rfloor} S_{p^i,p^j} \bigcup \bigcup_{i=\lfloor \frac{n}{2} \rfloor}^{n-1} S_{p^i} \bigcup \bigcup_{i=\lfloor \frac{n}{2} \rfloor}^{n-1} \bigcup_{j=0}^{n-1} S_{p^i,p^j} \bigcup \bigcup_{i=\lfloor \frac{n}{2} \rfloor}^{n-1} (E_{p^i} - \{w_0\}).
\]

To determine $I$ completely we have to determine which elements of $\bigcup_{i=\lceil \frac{n}{2} \rceil+1}^{n-1} S_{p^i,j} \cup S_{p^i,p^j}$ are in $I$ when $n$ is an odd integer and which elements of $\bigcup_{i=\lceil \frac{n}{2} \rceil+1}^{n-1} S_{p^i,j} \cup S_{p^i,p^j}$ are in $I$ when $n$ is an even integer. Observe that
\[
\bigcup_{i=\lceil \frac{n}{2} \rceil+1}^{n-1} S_{p^i,j} = \bigcup_{i=0}^{\lfloor \frac{n}{2} \rfloor-1} \bigcup_{j=0}^{\lfloor \frac{n-1}{2} \rfloor-i-1} S_{p^{i+j},p^j}.
\]
First, we consider the case where $n = 4n_1 + 3$ with $n_1$ is a positive integer. So, for any $i$ with $0 \leq i \leq \lfloor (n-1)/2 \rfloor - 1 = \lfloor (4n_1 + 3 - 1)/2 \rfloor - 1 = 2n_1$, we want to determine which elements of $\bigcup_{j=0}^{2n_1+1-i-1} S_{p^{i+j},p^j} = \bigcup_{j=0}^{2n_1+1-i-1} S_{p^{2n_1+1-i+j},p^{j}}$ are in $I$. 

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First, take \( i = 0 \), so we want to determine which elements of \( \bigcup_{j=0}^{2n_1} S_{p^{2n_1+2+j},p} \) are in \( I \). Observe that \( \bigcup_{j=0}^{2n_1} S_{p^{2n_1+2+j},p} \) is a complete subgraph of \( \bigcup_{j=0}^{2n_1} S_{p^{2n_1+2+j},p} \) and hence \( I \) can contain at most one element from it. We want to see which elements of \( \bigcup_{j=0}^{2n_1} S_{p^{2n_1+2+j},p} \) are in \( I \). To do that, we divide \( S_{p^{2n_1+2+0},p}, S_{p^{2n_1+2+1},p} \), \( \ldots, S_{p^{2n_1+2+j},p} \), \( \ldots, S_{p^{2n_1+2+n_1},p^{n_1}} \) into \( X^0_{t_0} = \{r_0 p^{2n_1+2} + p^j(s_0 p^{2n_1+1} + r_0 t_0)x : r_0 \in U(p^{2n_1+1}) \text{ and } 0 \leq s_0 \leq p^{2n_1+2} - 1 \} \) where \( t_0 \in U(p^{2n_1+1}) \), \( X^1_{t_1} = \{r_1 p^{2n_1+3} + p^j(s_1 p^{2n_1+1} + r_1 t_1)x : r_1 \in U(p^{2n_1}) \text{ and } 0 \leq s_1 \leq p^{2n_1+2} - 1 \} \) where \( t_1 \in U(p^{2n_1}) \), \( \ldots \), \( X^j_{t_j} = \{r_j p^{2n_1+2+j} + p^j(s_j p^{2n_1+1+j} + r_j t_j)x : r_j \in U(p^{2n_1+1+j}) \text{ and } 0 \leq s_j \leq p^{2n_1+2} - 1 \} \) where \( t_j \in U(p^{2n_1+1+j}) \), \( \ldots \), \( X^{n_1}_{t_{n_1}} = \{r_{n_1} p^{2n_1+2+n_1} + p^j(s_{n_1} p^{2n_1+1+n_1} + r_{n_1} t_{n_1})x : r_{n_1} \in U(p^{2n_1+1+n_1}) \text{ and } 0 \leq s_{n_1} \leq p^{2n_1+2} - 1 \} \) where \( t_{n_1} \in U(p^{2n_1+1+n_1}) \) respectively.

The bar on \( r_j t_j \) means that we are taking \( r_j t_j \) mod \( p^{2n_1+1+j} \). Observe that when \( t_j \) runs over all the elements of \( U(p^{2n_1+1+j}) \) and \( r_j \in U(p^{2n_1+1+j}) \), then \( r_j t_j \) runs over all the elements of \( U(p^{2n_1+1+j}) \). So, for any \( j \) with \( 0 \leq j \leq n_1 \), we have \( S_{p^{2n_1+2+j},p} = \bigcup_{t_j \in U(p^{2n_1+1+j})} X^j_{t_j} \) i.e. \( S_{p^{2n_1+2+j},p} \) has been divided into \( \{U(p^{2n_1+1+j})\} \) subsets and these subsets are \( X^j_{t_j} \) where \( t_j \in U(p^{2n_1+1+j}) \). We want to see how the elements of these subsets are adjacent. Suppose that \( y_1 \in X^j_{t_j} \) and \( y_2 \in X^{j'}_{s_{j'}} \), where \( 0 \leq j, j' \leq n_1 \), \( t_j \in U(p^{2n_1+1+j}) \), and \( s_{j'} \in U(p^{2n_1+1+j'}) \). As we have explained for Lemma 2, we get \( y_1 \) and \( y_2 \) are adjacent if and only if \( p^{2n_1+1-j-j'} \) divides \( t_j + s_{j'} \). So, each element of \( X^j_{t_j} \) is adjacent to all the elements of \( X^{j'}_{s_{j'}} \) if and only if \( p^{2n_1+1-j-j'} \) divides \( t_j + s_{j'} \). Note that the exponent \( 2n_1 + 1 - j - j' \) is positive and this is because \( j \) and \( j' \) are less than or equal to \( n_1 \). By taking \( j = j' \), we get each element of \( X^j_{t_j} \) is adjacent to every element of \( X^{j'}_{s_{j'}} \) if and only if \( t_j + s_j \equiv 0 \pmod{p^{2n_1+1-2j}} \). Note that both \( t_j \) and \( s_j \) are elements of \( U(p^{2n_1+1-j}) \). Hence, if \( l \in U(p^{2n_1+1-2j}) \), then each element of \( X^j_{t_j} \) is adjacent to every element of \( X^j_{(p^{2n_1+1-2(j-c)}+c)p^{2n_1+1-2j}} \) where \( 0 \leq c \leq p^j - 1 \). So, if \( l \in U(p^{2n_1+1-2j}) \), then every element of \( \bigcup_{b=0}^{p^j-1} X^j_{l+bp^{2n_1+1-2j}} \) is adjacent to every el-
element of $\bigcup_{c=0}^{p^j-1} X^j_{(p^{2n_1+1-2j-c}+\ell)} \cup p^{2n_1+1-2j}$, whereas no element of $\bigcup_{b=0}^{p^j-1} X^j_{l+b+p^{2n_1+1-2j}}$ is adjacent to any element of $S_{p^{2n_1+2+j}} - \bigcup_{c=0}^{p^j-1} X^j_{(p^{2n_1+1-2j-c}+\ell)} \cup p^{2n_1+1-2j}$. So, if $l \in U(p^{2n_1+1-2j})$, then the induced subgraph on each pair $\bigcup_{b=0}^{p^j-1} X^j_{l+b+p^{2n_1+1-2j}}$ is isomorphic to the complete bipartite graph $K_{(p^{2n_1+1-j}+\ell)} \cup p^{2n_1+2+j}$. The induced subgraph on $S_{p^{2n_1+2+j}}$ is isomorphic to the disjoint union of $|U(p^{2n_1+1-2j})|/2 = (p^{2n_1+1-2j} - p^{2n_1-2j})/2$ copies of the complete bipartite graph $K_{p^{2n_1+2+j}}$. Recall that $I$ is an independent set with maximum cardinality of $\Gamma(Z_{p^n}(x))$. So, for $0 \leq j \leq n_1$, $I$ can contain at most $p^{4n_1+2}(p-1)(p^{2n_1+1-2j} - p^{2n_1-2j})/2$ elements of $S_{p^{2n_1+2+j}}$. Thus, $I$ can contain at most $\sum_{j=0}^{n_1} p^{4n_1+2}(p-1)(p^{2n_1+1-2j} - p^{2n_1-2j})/2 = p^{6n_1+2}(p-1)^2/2 \sum_{j=0}^{n_1} (1/p^2)^j = p^{4n_1+2}(p-1)(p^{2n_1+2} - 1)/(2(p+1))$ elements of $\bigcup_{j=0}^{n_1} S_{p^{2n_1+2+j}}$.

We summarize this in the following lemma.

**Lemma 8.** If $0 \leq j \leq n_1$, then the induced subgraph on $S_{p^{2n_1+2+j}}$ is isomorphic to the disjoint union of $(p^{2n_1+1-2j} - p^{2n_1-2j})/2$ copies of the complete bipartite graph $K_{p^{2n_1+2+j}}$. For $n_1 + 1 \leq j \leq 2n_1$, the induced subgraph on $S_{p^{2n_1+2+j}}$ is complete. Moreover, $I$ contains at most $p^{4n_1+2}(p-1)(p^{2n_1+2} - 1)/(2(p+1))$ elements of $\bigcup_{j=0}^{n_1} S_{p^{2n_1+2+j}}$.

If $0 \leq j \leq n_1$, then define $X_j = \{ X_t^j : t \in \{1, 2, \ldots, (p-1)/2\} \mod p \}$, $Y_j = \{ Y_t^j : t \in \{(p+1)/2, (p+3)/2, \ldots, p-1\} \mod p \}$, $X_0 = \bigcup_{j=0}^{n_1} X_j$, and $Y_0 = \bigcup_{j=0}^{n_1} Y_j$. We have the following lemma about $X_0$ and $Y_0$.

**Lemma 9.** $X_0$ and $Y_0$ form a partition of $\bigcup_{j=0}^{n_1} S_{p^{2n_1+2+j}}$. Moreover, the induced subgraph on each one of $X_0$ and $Y_0$ is a set of null vertices and $|X_0| = |Y_0| =
\[
\left| \bigcup_{j=0}^{n_1} S_{p^2n_1+2+j,p} \right| / 2 = p^{4n_1+2}(p-1)(p^{2n_1+2}-1)/(2(p+1)).
\]

**Proof.** The proof follows immediately from the definition of \(X_0\), \(Y_0\) and the \(X^j_i\)’s.

We want to show that \(I\) contains exactly \(p^{4n_1+2}(p-1)(p^{2n_1+2}-1)/(2(p+1))\) elements of \(\bigcup_{j=0}^{2n_1} S_{p^2n_1+2+j,p}\). Suppose that \(C = \bigcup_{j=0}^{n_1} S_{p^2n_1+2+j,p}\) and \(B\) is a subset of \(\Gamma(\mathbb{Z}_{p^n}(x))\) with \(B \cap \left( \bigcup_{i=\lceil \frac{n}{2} \rceil +1}^{n-1} \bigcup_{j=\lceil \frac{n-1}{2} \rceil}^{n-1} S_{p^i,p^j} \cup \bigcup_{i=\lceil \frac{n}{2} \rceil +1}^{n-1} S_{p^i} \cup \bigcup_{i=1}^{n-1} E_{p^i} \right) = \emptyset\), \(B \cap C = \emptyset\), and any element of \(B\) is adjacent to some element of \(C\). Our goal is to show that \(I\) does not contain any elements of \(B\). It is straightforward to check that \(B\) does not contain any elements of \(\bigcup_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} \bigcup_{j=0}^{n_1} S_{p^i,p^n} \cup \bigcup_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} S_{p^i} \cup E_{p^i}\) and \(\bigcup_{j=n_1+1}^{2n_1} S_{p^2n_1+2+i,p^j}\) is a subset of \(C\). Hence \(B\) is a subset of \(\bigcup_{i=0}^{n_1} \bigcup_{j=0}^{2n_1} S_{p^2n_1+2+i,j,p^j} - C\). So, it is straightforward to check that the induced subgraph on \(B\) is complete and any element of \(B\) is adjacent to all the elements of \(S_{p^2n_1+2+n_1}\). We state this in the following lemma.

**Lemma 10.** For the set \(B\) that is defined above, the induced subgraph on \(B\) is complete and any element of \(B\) is adjacent to all the elements of \(S_{p^2n_1+2+n_1}\).

In the following lemma, we show that \(I\) does not contain any element of \(B\).

**Lemma 11.** Suppose that \(I\) is an independent set with maximum cardinality in \(\Gamma(\mathbb{Z}_{p^n}(x))\), where \(n = 4n_1 + 3\) and \(n_1\) is a positive integer. Then \(I \cap B = \emptyset\).

**Proof.** Since \(B\) is complete, \(I\) can contain at most one element of \(B\). Suppose that \(y \in I \cap B\). Since any element of \(B\) is adjacent to all the elements of \(S_{p^i,p^n}\), then \(I\) contains no elements of \(S_{p^2n_1+2+n_1,p^n}\). So, using Lemma 8, \(I\) can contain at most \(\left\lfloor p^{4n_1+2}(p-1)(p^{2n_1+2}-1)/(2(p+1)) \right\rfloor - \left| S_{p^2n_1+2+n_1,p^n} \right| + 1\) elements from the set \(\bigcup_{j=0}^{n_1} S_{p^2n_1+2+j,p} \cup B\). Define \(I_0 = (I - \bigcup_{j=0}^{n_1} S_{p^2n_1+2+j,p} \cup B) \cup X_0\), where \(X_0\) is defined in Lemma 9. Using Lemma 9 and Lemma 10, we get
the induced subgraph on \( I_0 \) is a set of null vertices. So, \( I_0 \) is an independent set with \( |I_0| > |I| \). This contradicts the fact that \( I \) is an independent set with maximum cardinality. Hence \( I \) does not contain any element of \( B \).

\[ \square \]

**Lemma 12.** Suppose that \( I \) is an independent set with maximum cardinality in \( \Gamma(\mathbb{Z}_p^n(x)) \), where \( n = 4n_1 + 3 \) and \( n_1 \) is a positive integer. Then \( I \) contains exactly \( \sum_{j=0}^{2n_1-1} |S_{p^{2n_1+2+j},p^j}| / 2 = p^{4n_1+2}(p - 1)(p^{2n_1+2} - 1)/(2(p + 1)) \) elements of \( \bigcup_{j=0}^{2n_1} S_{p^{2n_1+2+j},p^j} \).

**Proof.** The result is an immediate consequence of Lemma 8, Lemma 9, and Lemma 11.

\[ \square \]

Note that, \( I \) might contain the elements of \( X_0 \) from \( \bigcup_{j=0}^{2n_1} S_{p^{2n_1+2+j},p^j} \), where \( X_0 \) is defined in Lemma 9. For \( i = 0 \), we determined how many elements \( I \) contains from \( \bigcup_{j=0}^{2n_1} S_{p^{2n_1+2+i+j},p^j} \). This was done in Lemma 12. Similar to the case where \( i = 0 \), one can show that \( I \) contains exactly \( \sum_{j=0}^{n_1-1} |S_{p^{2n_1+2+1+i},p^j}| / 2 \) elements from \( \bigcup_{j=0}^{n_1-1} S_{p^{2n_1+1+1+i},p^j} \) and one element from \( S_{p^{2n_1+2+1+1},p^{n_1}} \). Observe that the induced subgraph on \( S_{p^{2n_1+2+1+1},p^{n_1}} \) is complete and hence \( I \) contains exactly one element of it. Continuing this way for \( i = 2, 3, \ldots, 2n_1 \) and we get that \( I \) contains exactly \( \sum_{i=0}^{2n_1-1} \sum_{j=0}^{2n_1-1-i} |S_{p^{2n_1+2+i+j},p^j}| / 2 + n_1 \) elements from \( \bigcup_{i=0}^{2n_1-1} \bigcup_{j=0}^{2n_1-1-i} S_{p^{2n_1+2+i+j},p^j} \). This will determine the independence number of \( \Gamma(\mathbb{Z}_p^n(x)) \), where \( n = 4n_1 + 3 \) and \( n_1 \) is a positive integer. We state this in the following theorem.

**Theorem 1.** Suppose that \( n = 4n_1 + 3 \) where \( n_1 \) is a positive integer. Then the independence number of \( \Gamma(\mathbb{Z}_p^n(x)) \) is equal to \( (1 - p) \binom{4n_1+3}{2} / (2(1 + p)) \left[ (n_1 + 1)/p + n_1 - (1 - p^{2n_1+1})p/(1 - p) \right] + p^{6n_1+4}(p^{2n_1+1} - 1) + n_1 + 1. \)
Proof. If $I$ is an independent set with maximum cardinality in $\Gamma(\mathbb{Z}_{p^n}(x))$, then $I$ contains

$$\sum_{i=0}^{2n_1} \sum_{j=0}^{\lfloor \frac{2n_1-1}{2} \rfloor} \frac{1}{2} |S_{p^{2n_1+2+i+j},p^i}| + n_1 \text{ elements from } \bigcup_{i=0}^{2n_1} \bigcup_{j=0}^{2n_1-i} S_{p^{2n_1+2+i+j},p^i}.$$ 

We have

$$\sum_{i=0}^{2n_1} \sum_{j=0}^{\lfloor \frac{2n_1-1}{2} \rfloor} |S_{p^{2n_1+2+i+j},p^i}| = \sum_{i=0}^{2n_1} \sum_{j=0}^{\lfloor \frac{2n_1-1}{2} \rfloor} p^{6n_1+2}(p^{-i-j+1} - p^{-i-j})(p^{-j+1} - p^{-j})$$

$$= p^{6n_1+2}(p-1)^2 \sum_{i=0}^{2n_1} \sum_{j=0}^{\lfloor \frac{2n_1-1}{2} \rfloor} \left( \frac{1}{p} \right)^i \left( \frac{1}{p} \right)^{2j}$$

$$= p^{6n_1+2}(p-1)^2 \sum_{i=0}^{2n_1} \left( \frac{1}{p} \right)^i \frac{\left( \left( \frac{1}{p} \right)^{\lfloor \frac{2n_1-1}{2} \rfloor+1} \right) - 1}{\left( \frac{1}{p} \right) - 1}$$

$$= \frac{(1-p)p^{n_1+4}}{1+p} \left[ \left( \frac{n_1+1}{p^{2n_1+2}} \right) + \left( \frac{n_1}{p^{2n_1+1}} \right) - \sum_{i=0}^{2n_1} \frac{1}{p} \right]$$

Using Corollary 1, $I$ contains $p^{6n_1+4}(p^{2n_1+1} - 1) + 1$ elements from $\Gamma(\mathbb{Z}_{p^n}(x)) - \bigcup_{i=0}^{2n_1} \bigcup_{j=0}^{2n_1-i} S_{p^{2n_1+2+i+j},p^i}$. Now, the result follows immediately.

We give the following example to explain Theorem 1.

**Example 2.** Consider the zero divisor graph $\Gamma(\mathbb{Z}_{5^7}(x))$. Partition the sets $S_{5^4,5^0}$ and $S_{5^5,5^1}$ into $X_0^t = \{r_05^4 + (s_05^2 + r_0t)x : r_0 \in U(5^3) \text{ and } 0 \leq s_0 \leq 5^4 - 1 \}$ where $t \in U(5^3)$ and $X_1^t = \{r_15^5 + 5^1(s_15^2 + r_1t)x : r_1 \in U(5^2) \text{ and } 0 \leq s_1 \leq 5^4 - 1 \}$ where $t' \in U(5^2)$. Let $X^0 = \{X_0^t : t \in \{1,2\} \mod 5\}$ and $X^1 = \{X_1^t : t' \in \{1,2\} \mod 5\}$. Also, we partition the sets $S_{5^5,5^0}$ and $S_{5^6,5^1}$ into $Y_0^t = \{r_05^5 + (s_05^2 + r_0t)x : r_0 \in U(5^2) \text{ and } 0 \leq s_0 \leq 5^5 - 1 \}$ where $t \in U(5^2)$ and $Y_1^t = \{r_15^6 + 5^1(s_15^2 + r_1t)x : r_1 \in U(5^1) \text{ and } 0 \leq s_1 \leq 5^5 - 1 \}$ where $t' \in U(5^1)$. Take $Y^0 = \{Y_0^t : t \in \{1,2\} \mod 5\}$ and $Y^1 = \{Y_1^t : t' \in \{1,2\} \mod 5\}$. Lastly, we partition $S_{5^6,5^0}$ into $Z_0^t = \{r_05^6 + (s_05^1 + r_0t)x : r_0 \in U(5^1) \text{ and } 0 \leq s_0 \leq 5^6 - 1 \}$ where $t \in U(5^1)$ and take $Z^0 = \{Z_0^t : t \in \{1,2\} \mod 5\}$. If $I = \bigcup_{i=0}^3 \bigcup_{j=0}^3 S_{p^i,p^j} \cup \bigcup_{i=1}^3 S_{p^i} \cup X^0 \cup X^1 \cup Y^0 \cup Z^0 \cup \{5^6 + 5(5+1)x\} \cup \{w_0\}$, where $5^6 + 5(5+1)x$ is an element of $Y^1$ and $w_0 \in E_{p^0}$, then $I$ is an independent set with maximum cardinality in $\Gamma(\mathbb{Z}_{5^7}(x))$. Hence the independence number
of $\Gamma(\mathbb{Z}_{g7}(x))$ is equal to $|I| = 5^{10}(5^3 - 1) + 5^6(5 - 1)^2(5^2 + 5 + 2)/2 + 2$.

Similarly, one can find the independence number of $\Gamma(\mathbb{Z}_{p^n}(x))$ where $n = 4n_1 + 1$ and $n_1$ is a positive integer. We state the following.

**Theorem 2.** Suppose that $n = 4n_1 + 1$ where $n_1$ is a positive integer. Then the independence number of $\Gamma(\mathbb{Z}_{p^n}(x))$ is equal to

$$
\sum_{i=0}^{2n_1-1} \sum_{j=0}^{n_1-1} |S_{p^{2n_1+1+i+j,p^n}}|/2 + n_1 + \left| \bigcup_{i=1}^{\lfloor n/2 \rfloor - 1} S_{p,i}^{n_1} \right| \left| \bigcup_{i=1}^{n_1-1} \bigcup_{j=0}^{n_1-1} S_{p,i,p^n} \cup \{w_0\} \right| = p^{4n_1+1}(p-1)/(2(p+1))(p^{2n_1} - 1) + 1.
$$

We want to determine the independence number of $\Gamma(\mathbb{Z}_{p^n}(x))$ when $n$ is an even integer. If $I$ is an independence set with maximum cardinality in $\Gamma(\mathbb{Z}_{p^n}(x))$ and $n$ is an even integer, then the use of Lemma 3, Lemma 4, Lemma 5, Lemma 6, and Lemma 7 gives

$$
\sum_{i=1}^{\lfloor n/2 \rfloor - 1} S_{p,i}^{n_1} \cup \bigcup_{i=1}^{\lfloor n/2 \rfloor - 1} \bigcup_{j=0}^{n_1-1} S_{p,i,p^n} \cup \{w_0\} \subseteq I
$$

and $I \cap \left( \bigcup_{i=1}^{\lfloor n/2 \rfloor - 1} S_{p,i}^{n_1} \cup \bigcup_{i=1}^{n_1-1} \bigcup_{j=0}^{n_1-1} S_{p,i,p^n} \cup \{w_0\} \right) = \emptyset$, where $w_0$ is an element of $E_{p,n}$. To determine $I$ completely, we have to determine which elements of $S_{p,i}^{n_1} \cup \bigcup_{i=1}^{\lfloor n/2 \rfloor - 1} \bigcup_{j=0}^{n_1-1} S_{p,i,p^n} \cup \bigcup_{j=0}^{n_1-1} S_{p,i,j,p^n}$ are in $I$. For $1 \leq k \leq \lfloor n/2 \rfloor - 1$, the use of Lemma 2 gives that the induced subgraph on $S_{p,i}^{n_1} \cup S_{p,i,j,p^n}$ is isomorphic to $(p^{n-\lfloor n/2 \rfloor - k} - p^{n-\lfloor n/2 \rfloor - k-1})/2$ disjoint copies of the complete bipartite graph $K_{(p-1)p^{n-1},(p-1)p^{n-1}}$. Observe that, for $1 \leq k \leq \lfloor n/2 \rfloor - 1$, no element of $S_{p,i}^{n_1} \cup S_{p,i,j,p^n}$ is adjacent to any of the elements of $S_{p,i}^{n_1} \cup \bigcup_{i=1}^{\lfloor n/2 \rfloor - 1} \bigcup_{j=0}^{n_1-1} S_{p,i,p^n} \cup \bigcup_{j=0}^{n_1-1} S_{p,i,j,p^n}$. Hence, for $1 \leq k \leq \lfloor n/2 \rfloor - 1$, $I$ contains $(p^{n-\lfloor n/2 \rfloor - k} - p^{n-\lfloor n/2 \rfloor - k-1})(p-1)p^{n-1}/2$ elements of $S_{p,i}^{n_1} \cup S_{p,i,j,p^n}$. So, $I$ contains

$$
\sum_{k=1}^{\lfloor n/2 \rfloor - 1} (p^{n-\lfloor n/2 \rfloor - k} - p^{n-\lfloor n/2 \rfloor - k-1})(p-1)p^{n-1}/2 = p^{n-1}(p-1)(p^{\lfloor n/2 \rfloor - 1})/2
$$

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elements of \( \bigcup_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} S_{p^{i}2^{j}p^{k}} \). We state this result in the following lemma.

**Lemma 13.** Suppose that \( I \) is an independent set with maximum cardinality in \( \Gamma(\mathbb{Z}_{p^{n}}(x)) \), where \( n \) is a positive even integer. Then \( I \) contains exactly \( p^{n-1}(p-1)(p^{\lfloor (n-1)/2 \rfloor} - 1)/2 \) elements of \( \bigcup_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} S_{p^{i}2^{j}p^{k}} \).

Now, we determine which elements of \( \bigcup_{i=\lfloor \frac{n}{2} \rfloor+1}^{n-1} \bigcup_{j=0}^{\lfloor \frac{n-1}{2} \rfloor - 1} S_{p^{i}p^{j}} \cup S_{p^{\lfloor \frac{n}{2} \rfloor}p^{j}} \) are in \( I \). Observe that \( \bigcup_{i=\lfloor \frac{n}{2} \rfloor+1}^{n-1} \bigcup_{j=0}^{\lfloor \frac{n-1}{2} \rfloor - 1} S_{p^{i}p^{j}} \cup S_{p^{\lfloor \frac{n}{2} \rfloor}p^{j}} = \bigcup_{i=0}^{\lfloor \frac{n}{2} \rfloor-1} \bigcup_{j=0}^{\lfloor \frac{n-1}{2} \rfloor - i} S_{p^{i}p^{\lfloor \frac{n}{2} \rfloor + i + j}p^{j}}. \) Similar to the case where \( n \) is odd, we determine the number of elements of \( \bigcup_{i=0}^{\lfloor \frac{n}{2} \rfloor} \bigcup_{j=0}^{\lfloor \frac{n-1}{2} \rfloor - i} S_{p^{i}p^{\lfloor \frac{n}{2} \rfloor + i + j}p^{j}} \) that are in \( I \). We state that in the following lemma.

**Lemma 14.** Suppose that \( I \) is an independent set with maximum cardinality in \( \Gamma(\mathbb{Z}_{p^{n}}(x)) \). If \( n = 4n_{1} + 2 \), where \( n_{1} \) is a positive integer, then \( I \) contains exactly \( (p-1)p^{4n_{1}+2}/(2(p+1)) \left( \left(- (n_{1} + 1)/p - n_{1} \right) + (p^{2n_{1}+1} - 1)p/(p-1) \right) + n_{1} \) elements of \( \bigcup_{i=0}^{\lfloor \frac{n}{2} \rfloor} \bigcup_{j=0}^{\lfloor \frac{n-1}{2} \rfloor - i} S_{p^{i}p^{\lfloor \frac{n}{2} \rfloor + i + j}p^{j}}. \) If \( n = 4n_{1} \), where \( n_{1} \) is a positive integer, then \( I \) contains exactly \( (p-1)p^{4n_{1}+2}/(2(p+1)) \left( \left(- n_{1}/p - n_{1} \right) + (p^{2n_{1}-1} - 1)p/(p-1) \right) + n_{1} \) elements of \( \bigcup_{i=0}^{\lfloor \frac{n}{2} \rfloor} \bigcup_{j=0}^{\lfloor \frac{n-1}{2} \rfloor - i} S_{p^{i}p^{\lfloor \frac{n}{2} \rfloor + i + j}p^{j}}. \)

We state the independence number of \( \Gamma(\mathbb{Z}_{p^{n}}(x)) \), where \( n \) is a positive even integer in the following two theorems.

**Theorem 3.** Suppose that \( n = 4n_{1} + 2 \) where \( n_{1} \) is a positive integer. Then the independence number of \( \Gamma(\mathbb{Z}_{p^{n}}(x)) \) is equal to \( (p-1)p^{4n_{1}+2}/(2(p+1)) \left( \left(- (n_{1} + 1)/p - n_{1} \right) + (p^{2n_{1}+1} - 1)p/(p-1) \right) + n_{1} + p^{6n_{1}+3}(p^{2n_{1}} - 1) + 1 + p^{4n_{1}+1}(p^{2n_{1}} - 1)(p-1)/2 + 1. \)

**Proof.** Suppose that \( I \) is an independent set with maximum cardinality in \( \Gamma(\mathbb{Z}_{p^{n}}(x)) \). Using Lemma 13 and Lemma 14, \( I \) contains \( (p-1)p^{4n_{1}+2}/(2(p+1)) \left( \left(- (n_{1} + 1)/p - n_{1} \right) + (p^{2n_{1}+1} - 1)p/(p-1) \right) + n_{1} + p^{4n_{1}+1}(p^{2n_{1}} - 1)(p-1)/2 \)
Proof. The proof is similar to that one of last theorem.

Our theorems do not cover the cases where $n = 1, 2, 3$. We cover these cases in the following theorem.

**Theorem 4.** Suppose that $n = 4n_1$ where $n_1$ is a positive integer. Then the independence number of $\Gamma(\mathbb{Z}_{p^n}(x))$ is equal to $(p-1)p^{4n_1+2}/(2(p+1))(-n_1/p-n_1)+(p^{2n_1}-1)p/(p-1)+n_1+p^{6n_1}(p^{2n_1}-1)+1+p^{4n_1-1}(p^{2n_1-1}-1)(p-1)/2+1$.

**Proof.** The proof is similar to that one of last theorem.

**Theorem 5.** The independence number of $\Gamma(\mathbb{Z}_{p^n}(x))$ where $n = 1, 2, 3$ is 1, $(p^2 - p)(p-1)/2 + 1$, $(p^2 - p)p^3 + (p^3 - p^2)(p-1)/2 + 1$ respectively.

**Proof.** The independence number of $\Gamma(\mathbb{Z}_{p^1}(x))$ is equal to 1 and this is because the graph $\Gamma(\mathbb{Z}_{p^1}(x))$ is complete. The set $I = \{ap+(bp+c)x : a \in \{1, 2, \ldots, p-1\}, b \in \{0, 1, \ldots, p-1\} \text{ and } c \in \{1, \ldots, (p-1)/2\}\} \cup \{w_0\}$, where $w_0 \in E_{p^0}$, is an independent set with maximum cardinality in $\Gamma(\mathbb{Z}_{p^1}(x))$ and hence the independence number of $\Gamma(\mathbb{Z}_{p^1}(x))$ is equal to $|I| = (p^2 - p)(p-1)/2 + 1$. The set $I = \bigcup_{j=0}^{2} S_{p^j,p^0} \cup S_{p^1} \cup \{ap^2+(bp+c)x : a \in \{1, 2, \ldots, p-1\}, b \in \{0, 1, \ldots, p^2-1\} \text{ and } c \in \{1, \ldots, (p-1)/2\}\} \cup \{w_0\}$, where $w_0 \in E_{p^0}$, is an independent set with maximum cardinality in $\Gamma(\mathbb{Z}_{p^3}(x))$. Hence the independence number of $\Gamma(\mathbb{Z}_{p^3}(x))$ is equal to $|I| = (p^2 - p)p^3 + (p^3 - p^2)(p-1)/2 + 1$.

**References**


