Abstract

Let \( \frac{\partial}{\partial t} + (-\Delta)^2 + V^2 \) be a higher order parabolic Schrödinger operator on \( \mathbb{R}^{n+1} (n \geq 5) \), where the nonnegative potential \( V \) belongs to the reverse Hölder class \( B_{q_1}(\mathbb{R}^n) \) for some \( q_1 > n/2 \). In this paper we obtain the \( L^p(\mathbb{R}^{n+1}) \) estimates for the operator \( \nabla^4 (\frac{\partial}{\partial t} + (-\Delta)^2 + V^2)^{-1} \).

**Keywords:** \( L^p \) estimates, Parabolic Schrödinger operators, fundamental solution, reverse Hölder class.

**AMS Mathematics Subject Classification:** 35J10, 35K25.

1 Introduction

In this paper we consider the higher order parabolic Schrödinger operator

\[
\frac{\partial}{\partial t} + (-\Delta)^2 + V^2 \quad \text{on} \quad \mathbb{R}^{n+1}, \quad n \geq 5,
\]

where \( (-\Delta)^2 \) is the bilaplacian on \( \mathbb{R}^n \) and the nonnegative potential \( V(x) \) is independent of variable \( t \). The studies of Schrödinger operators with nonnegative potentials have attracted much attention, see for example [5, 14, 17, 9, 1, 8, 10, 19]. In recent years, on the one hand, some scholars generalize the results for Schrödinger operator to the case of higher order Schrödinger operators (cf. [13, 11, 12]). On the other hand, some one study similar results for the parabolic

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Schrödinger operators (cf. [3, 6, 7, 15]). Motivated by the above papers, we continue this line to study the higher order parabolic Schrödinger operators and obtain the \(L^p\) boundedness of \(\nabla^4(\partial/\partial t + (-\Delta)^2 + V^2)^{-1}\).

Note that a nonnegative locally \(L^q\) integrable function \(V\) on \(\mathbb{R}^n\) is said to belong to \(B_q(\mathbb{R}^n)\) \((1 < q < \infty)\) if there exists \(C > 0\) such that the reverse Hölder inequality

\[
\left( \frac{1}{|B|} \int_B V(x)^q \, dx \right)^{\frac{1}{q}} \leq C \left( \frac{1}{|B|} \int_B V(x) \, dx \right)
\]

holds for every ball \(B\) in \(\mathbb{R}^n\). Moreover, if there exists a constant \(C > 0\) such that

\[
\| V \|_{L^\infty(B)} \leq C \left( \frac{1}{|B|} \int_B V(x) \, dx \right)
\]

holds for every ball \(B\) in \(\mathbb{R}^n\), we say \(V \in B_{\infty}(\mathbb{R}^n)\).

It follows from [14] that the \(B_q\) class has a property of self improvement; that is, if \(V \in B_q\), then \(V \in B_{q+\varepsilon}\) for some \(\varepsilon > 0\). For \(1 < p < \infty\), it is easy to see that \(B_{\infty}(\mathbb{R}^n) \subseteq B_p(\mathbb{R}^n)\). If \(V \in B_{\infty}(\mathbb{R}^n)\), then there is a positive constant \(C\) such that \(V(x) \leq Cm(x,V)^2\) a.e. on \(\mathbb{R}^n\) (Remark 2.9, [14]).

We are now in a position to give the main results in this paper.

**Theorem 1.** Suppose \(V(x) \in B_{q_1}(\mathbb{R}^n), q_1 > n/2\). Then, for \(1 < p \leq q_1/2\),

\[
\| \nabla^4(\partial/\partial t + (-\Delta)^2 + V^2)^{-1}f \|_{L^p(\mathbb{R}^{n+1})} \leq C \| f \|_{L^p(\mathbb{R}^{n+1})},
\]

where \(\nabla^4 = \partial^4/\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}, \alpha_1 + \cdots + \alpha_n = 4\).

If the potential \(V\) satisfies stronger condition, we can get the following result which removes the restriction of the range of \(p\).

**Corollary 1.** Suppose \(V(x) \in B_{\infty}(\mathbb{R}^n)\). Then, for \(1 < p < \infty\),

\[
\| \nabla^4(\partial/\partial t + (-\Delta)^2 + V^2)^{-1}f \|_{L^p(\mathbb{R}^{n+1})} \leq C \| f \|_{L^p(\mathbb{R}^{n+1})},
\]

where \(\nabla^4 = \partial^4/\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}, \alpha_1 + \cdots + \alpha_n = 4\).

We also obtain the \(L^p\) boundedness of the operator \(V^{2\alpha}(\partial/\partial t + (-\Delta)^2 + V^2)^{-\alpha}\) for \(0 < \alpha < 1\). See Theorem 3 in the last section.

This paper is organized as follows. In Section 2 we recall some basic facts for the auxiliary function \(m(x,V)\) and give some estimates on the fundamental solution to \(\partial u/\partial t + (-\Delta)^2 u + V^2(x)u = 0\) in \(\mathbb{R}^{n+1}\). In Section 3 we recall some basic facts for \(L^p(\mathbb{R}^{n+1})\) multipliers. Section 4 shows that Theorem 1 holds true. In the last section we prove the \(L^p\) boundedness of the operator \(V^{2\alpha}(\partial/\partial t + (-\Delta)^2 + V^2)^{-\alpha}\) for \(0 < \alpha < 1\).

Throughout this paper the letter \(C\) stands for a constant and is not necessarily the same at each occurrence. By \(B_1 \sim B_2\), we mean that there exists a constant \(C > 1\) such that \(1/C \leq B_1/B_2 \leq C\).
2 The auxiliary function $m(x, V)$ and estimates of fundamental solutions

In the first part of this section we recall the definition of the auxiliary function $m(x, V)$ and some lemmas about the auxiliary function $m(x, V)$ which have been proved in [14]. We always assume $V \in B_{q_i}$ for $q_i > n/2$ throughout this section.

The auxiliary function $m(x, V)$ is defined by

$$
\frac{1}{m(x, V)} \triangleq \sup_{r > 0} \left\{ r : \frac{1}{r^{n-2}} \int_{B(x,r)} V(y) \, dy \leq 1 \right\}, \quad x \in \mathbb{R}^n.
$$

**Lemma 1.** The measure $V(x) \, dx$ satisfies the doubling condition, that is, there exists a constant $C > 0$ such that

$$
\int_{B(x,2r)} V(y) \, dy \leq C \int_{B(x,r)} V(y) \, dy
$$

holds for all balls $B(x, r)$ in $\mathbb{R}^n$.

**Lemma 2.** There exists a constant $C > 0$ such that, for $0 < r < R < \infty$,

$$
\frac{1}{r^{n-2}} \int_{B(x,r)} V(y) \, dy \leq C \left( \frac{r}{R} \right)^{2-q_1} \frac{1}{R^{n-2}} \int_{B(x,R)} V(y) \, dy.
$$

**Lemma 3.** If $r = \frac{1}{m(x, V)}$, then

$$
\frac{1}{r^{n-2}} \int_{B(x,r)} V(y) \, dy = 1.
$$

Moreover,

$$
\frac{1}{r^{n-2}} \int_{B(x,r)} V(y) \, dy \sim 1 \quad \text{if and only if} \quad r \sim \frac{1}{m(x, V)}.
$$

**Lemma 4.** There exists $l_0 > 0$ such that, for any $x$ and $y$ in $\mathbb{R}^n$,

$$
\frac{1}{C} \left( 1 + m(x, V) | x - y | \right)^{-l_0} \leq \frac{m(x, V)}{m(y, V)} \leq C \left( 1 + m(x, V) | x - y | \right)^{l_0/l_0 + 1}.
$$

In particular, $m(x, V) \sim m(y, V)$ if $| x - y | < \frac{C}{m(x, V)}$.

**Lemma 5.** There exists $l_1 > 0$ such that

$$
\int_{B(x,R)} \frac{V(y)}{|x - y|^{n-2}} \, dy \leq \frac{C}{R^{n-2}} \int_{B(x,R)} V(y) \, dy \leq C \left( 1 + Rm(x, V) \right)^{l_1}.
$$

The next lemma has been proved by Tao and Wang in [16].

**Lemma 6.** Let $q > s \geq 0$, $q \geq \max\{1, sn/\alpha\}$, $\alpha > 0$, and $k$ sufficiently large, then there are positive constants $k_0, C$ and $C_k$ such that

$$
\int_{|x - y| < r} \frac{V(y)^s}{|x - y|^{n-\alpha}} \, dy \leq C r^{\alpha - 2s} \{1 + rm(x, V)\}^{s k_0} \quad \text{(3)}
$$

and

$$
\int_{\mathbb{R}^n} \frac{V(y)^s}{\{1 + m(x, V)|x - y|\}^k |x - y|^{n-\alpha}} \, dy \leq C_k m(x, V)^{2s - \alpha} \quad \text{(4)}
$$

for any $r > 0, x \in \mathbb{R}^n$ and $V \in B_q(\mathbb{R}^n)$.
Next we recall some fundamental properties of functions in the reverse Hölder class (cf. [18]).

**Lemma 7.** If \( V(x) \in B_q(1 < q \leq \infty) \), \( \lambda \) is a nonnegative constant, then \( V(x) + \lambda \in B_q \).

**Lemma 8.** If \( V(x) \in B_q(q \geq n/2) \), \( \lambda \) is a nonnegative constant, then \( m(x, V) \leq m(x, V + \lambda) \).

Similar to the proof of above lemmas, we easily obtain the following lemma.

**Lemma 9.** If \( V(x) \in B_q(1 < q \leq \infty) \), \( \lambda \) is a nonnegative constant, then \( \sqrt{V^2(x) + \lambda} \in B_q \).

**Lemma 10.** If \( V(x) \in B_q(q \geq n/2) \), \( \lambda \) is a nonnegative constant, then \( m(x, V) \leq m(x, \sqrt{V^2 + \lambda}) \).

**Remark 1.** It is not difficult to check that if \( \lambda \) is a nonnegative constant, then \( m(x, \lambda) = C\sqrt{\lambda} \), where \( C \) is a positive constant and is independent of \( \lambda \).

In this paper we endow the space \( \mathbb{R}^{n+1} \) with the following parabolic metric which is different from the usual Euclidean metric:

\[
d((x, t), (y, s)) = \max(|x - y|, |t - s|^{1/4}),
\]
for any \((x, t), (y, s) \in \mathbb{R}^{n+1}\).

Next we give some estimates on the fundamental solution of higher order parabolic Schrödinger operator.

Let \( \Gamma(x, t; y, s; \lambda) \) be the fundamental solution to \( \partial u / \partial t + (-\Delta)^{2}u + V^2(x)u + \lambda u = 0 \) in \( \mathbb{R}^{n+1} \), where \( \lambda \in [0, \infty) \). Especially, we denote \( \Gamma(x, t; y, s; 0) = \Gamma(x, t; y, s) \). By Lemma 2.5 in [2] we easily get the following lemma.

**Lemma 11.** Let \( V \in B_q \) for \( q > n/2 \). For every \( N \in \mathbb{N} \), there exist positive constants \( C_N \) and \( \tilde{C} \) such that for all \((x, t), (y, s) \in \mathbb{R}^{n+1} \) and \( t > s \),

\[
|\Gamma(x, t; y, s)| \leq \frac{C_N}{[1 + (t-s)^{1/2}m^2(x, V)]^N} (t-s)^{-n/4} \exp\left\{-\frac{C N}{(t-s)^{1/3}} \right\}. \tag{6}
\]

**Lemma 12.** Let \( V \in B_q \) for \( q > n/2 \). For every \( N \in \mathbb{N} \), there exist positive constants \( C_N \) and \( \tilde{C}_1 \) such that for all \((x, t), (y, s) \in \mathbb{R}^{n+1} \) and \( t > s \),

\[
|\Gamma(x, t; y, s)| \leq \frac{C_N}{[1 + |x-y|^2 m^2(x, V)]^N} (t-s)^{-n/4} \exp\left\{-\frac{\tilde{C}_1 |x-y|^{4/3}}{(t-s)^{1/3}} \right\}. \tag{7}
\]

**Proof.** For any \((x, t), (y, s) \in \mathbb{R}^{n+1} \) and \( t > s \), it is easy to deduce that the inequality (7) holds true when \((t-s)^{1/2} > |x-y|^2 \). Now, we assume that \((t-s)^{1/2} \leq |x-y|^2 \). \( \forall N > 0 \), by (6) we have

\[
[|x-y|^2 m^2(x, V)]^N |\Gamma(x, t; y, s)| \leq \frac{C_N[|x-y|^2 m^2(x, V)]^N}{[1 + (t-s)^{1/2}m^2(x, V)]^N} (t-s)^{-n/4} \exp\left\{-\frac{\tilde{C}_1 |x-y|^{4/3}}{(t-s)^{1/3}} \right\}
\]

\[
\leq C_N \left(\frac{|x-y|^{4/3}}{(t-s)^{1/3}}\right)^{3N/2} \exp\left\{-\frac{\tilde{C}_1 |x-y|^{4/3}}{(t-s)^{1/3}} \right\}
\leq C_N (t-s)^{-n/4} \exp\left\{-\frac{\tilde{C}_1 |x-y|^{4/3}}{(t-s)^{1/3}} \right\}
\]

\[
= C_N (t-s)^{-n/4} \exp\left\{-\frac{\tilde{C}_1 |x-y|^{4/3}}{(t-s)^{1/3}} \right\},
\]
where $\tilde{C}_1 = \tilde{C} - \varepsilon$ and $0 < \varepsilon < \tilde{C}$. Therefore,

$$\left| [x-y]^{2m} \Gamma(x, t; y, s) \right|^{1/N} \leq \left( C_N(t-s)^{-n/4} \exp \left\{ -\tilde{C}_1 \frac{|x-y|^{4/3}}{(t-s)^{1/3}} \right\} \right)^{1/N}.$$  

Furthermore, applying (6) again we have

$$\left| \Gamma(x, t; y, s) \right| \leq C_N(t-s)^{-n/4} \exp \left\{ -\tilde{C}_1 \frac{|x-y|^{4/3}}{(t-s)^{1/3}} \right\}.$$  

Combining the above inequalities, we deduce that (7) is valid.

From Lemma 10, Lemma 11, Lemma 12 and Remark 1, we deduce the following corollary.

**Corollary 2.** Let $V \in B_{q_1}$ for $q_1 > n/2$. For every $N \in \mathbb{N}$, there exist positive constants $C_N$ and $\tilde{C}_1$ such that for all $(x, t), (y, s) \in \mathbb{R}^{n+1}$ and $t > s$,

$$\left| \Gamma(x, t; y, s; \lambda) \right| \leq \frac{C_N}{[1 + \lambda^{1/4}d((x, t), (y, s))]^N[1 + d((x, t), (y, s))m(x, V)]^N} \exp \left\{ -\tilde{C}_1 \frac{|x-y|^{4/3}}{(t-s)^{1/3}} \right\}.  \quad (8)$$

### 3 $L^p(\mathbb{R}^{n+1})$ Multipliers

In this section we recall some results for $L^p(\mathbb{R}^{n+1})$ multipliers in order to prove Theorem 1 (cf. [4]).

Let $a = (1, \cdots , 1, 4)$. For $\beta = (\beta_1, \cdots , \beta_{n+1})$ define

$$(a, \beta) = \sum_{j=1}^{n+1} a_j \beta_j = \sum_{j=1}^{n} \beta_j + 4\beta_{n+1} \text{ and } |\beta| = \sum_{j=1}^{n+1} \beta_j.$$  

For $x' = (x, t) = (x_1, \cdots , x_n, t)$ define

$$\lambda^a x' = (\lambda x_1, \cdots , \lambda x_n, \lambda^4 t) \text{ and } (x')^{\beta} = x_1^{\beta_1} \cdots x_n^{\beta_n} t^{\beta_{n+1}}.$$  

For a fixed $x' \in \mathbb{R}^{n+1}$, defined $\rho(x')$ as the unique solution of $F(x', \rho) = \sum_{j=1}^{n} x_j^2/\rho^2 + t^2/\rho^8 = 1$. It follow from [4] that $\rho(x')$ is a non-isotropic norm on $\mathbb{R}^{n+1}$ and has a dilation invariance property $\rho(\lambda^a x') = \lambda \rho(x')$. Note that the metric induced by $\rho(x')$ is equivalent to the parabolic metric introduced in (5).

The function $h(x)$ is said to be a multiplier when

$$\| T\varphi \|_{L^p(\mathbb{R}^{n+1})} \leq A_p \| \varphi \|_{L^p(\mathbb{R}^{n+1})} \text{ for every } p, 1 < p < \infty,$$

where $T\varphi = F^{-1}(hF(\varphi))$ and $F$ is the Fourier transform operator.

The following proposition has been proved in [4].

**Proposition 1.** Let $h(x, t) \in L^\infty(\mathbb{R}^{n+1})$, and assume $h(x, t)$ is $N$ times continuously differentiable, where $N > |a|/2 = (n+4)/2$; moreover, assume that

$$\int_{R/2 \leq \rho(x,t) \leq 3R} \left| R^{(a, \beta)}(\frac{\partial}{\partial x})^\beta h(x, t) \right|^2 \frac{dxdt}{R^{|a|}} \leq C, \ |h(x, t)| \leq C \ a.e.,  \quad (9)$$

5
where $C$ is independent of $R$, say $C \geq 1$ and $(\frac{\partial}{\partial x})^{\beta} = (\frac{\partial}{\partial x_1})^{\beta_1} \cdots (\frac{\partial}{\partial x_n})^{\beta_n}(\frac{\partial}{\partial t})^{\beta_{n+1}}$. Then
\[ \| T\varphi \|_{L^p(R^{n+1})} = \| F^{-1}(hF(\varphi)) \|_{L^p(R^{n+1})} \leq A_pC \| \varphi \|_{L^p(R^{n+1})}, \varphi \in C_0^\infty(R^{n+1}), \]
where $A_p$ depends only on $a$ and $p$.

We define an operator $T_j$ by
\[ F(T_jf)(x,t) = \frac{ix_j}{(it + |x|^4)^{1/4}}F(f)(x,t), 1 \leq j \leq n, f \in C_0^\infty(R^{n+1}). \]

By simple computation we conclude that the function $h(x,t) = \frac{ix_j}{(it + |x|^4)^{1/4}}$ satisfies the condition (9) in Proposition 1. Therefore,
\[ \| T_jf \|_{L^p(R^{n+1})} \leq C \| f \|_{L^p(R^{n+1})}, j = 1, 2, \cdots, n. \]

Then for multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$ with $|\alpha| = 4$,
\[ F(\nabla^4 f)(x,t) = \langle ix \rangle^\alpha F(f)(x,t) = \frac{(ix)^\alpha}{it + |x|^4}(it + |x|^4)F(f)(x,t) \]
\[ = \frac{ix_1^{\alpha_1}ix_2^{\alpha_2} \cdots ix_n^{\alpha_n}}{it + |x|^4}(it + |x|^4)F(f)(x,t) \]
\[ = \frac{ix_1^{\alpha_1}ix_2^{\alpha_2} \cdots ix_n^{\alpha_n}}{(it + |x|^4)^{\alpha_1/4}} \frac{(it + |x|^4)^{\alpha_2/4}}{(it + |x|^4)^{\alpha_3/4}} \cdots \frac{(it + |x|^4)^{\alpha_n/4}}{(it + |x|^4)^{\alpha_1/4}}(it + |x|^4)F(f)(x,t) \]
\[ = F(T_1^{\alpha_1} \cdots T_n^{\alpha_n}(\partial/\partial t + (-\Delta)^2)f)(x,t). \]

By the $L^p$ boundedness of $T_j$, we have
\[ \| \nabla^4 f \|_{L^p(R^{n+1})} \leq C \| T_1^{\alpha_1} \cdots T_n^{\alpha_n}(\partial/\partial t + (-\Delta)^2)f \|_{L^p(R^{n+1})} \leq C \| (\partial/\partial t + (-\Delta)^2)f \|_{L^p(R^{n+1})}. \]

Therefore,
\[ \| \nabla^4(\partial/\partial t + (-\Delta)^2)^{-1}f \|_{L^p(R^{n+1})} \leq C \| f \|_{L^p(R^{n+1})}. \]
(10)

4 The Proof of Theorem 1

In this section we devote to the proof of Theorem 1. Before completing the proof of Theorem 1, we first give the following theorem.

Theorem 2. Suppose $V(x) \in B_{q_1}(R^n), q_1 > n/2$. Then, for $1 < p < q_1/2$,
\[ \| V^2(\frac{\partial}{\partial t} + (-\Delta)^2 + V^2)^{-1}f \|_{L^p(R^{n+1})} \leq C \| f \|_{L^p(R^{n+1})}. \]

Proof.

Let $f \in L^p(R^{n+1})$ for $1 < p \leq q_1/2$ and
\[ u(x,t) = (\frac{\partial}{\partial t} + (-\Delta)^2 + V^2)^{-1}f(x,t) = \int_{-\infty}^{t} \int_{R^n} \Gamma(x,t; y,s)f(y,s)dyds. \]
write
\[ u(x, t) = \int_{-\infty}^{t} \int_{|x-y|<r} \Gamma(x, t; y, s)f(y, s)dyds + \int_{-\infty}^{t} \int_{|x-y|\geq r} \Gamma(x, t; y, s)f(y, s)dyds \]
\[ = u_1(x, t) + u_2(x, t), \]
where \( r = 1/m(x, V). \)

Because of self improvement of the \( B_{q_1} \) class, \( V \in B_{q_0} \) for some \( q_0 > q_1 \). For convenience, we denote
\[ G_1(x, t; y, s) = \frac{C_N}{[1 + (t-s)^{1/2}m^2(x, V)]^N[1 + |x-y|m(x, V)]^N} \]
and
\[ G_2(x, t; y, s) = (t-s)^{-n/4} \exp \left\{ -\tilde{C}_1 \frac{|x-y|^{4/3}}{(t-s)^{1/3}} \right\}. \]

At first, we have
\[
\left[ \int_{-\infty}^{t} \int_{|x-y|<r} |G_2(x, t; y, s)|^q dyds \right]^{1/q} \\
\leq \left[ \int_{|x-y|<r} \left( \int_{0}^{\infty} t^{-qn/4} \exp \left\{ -q\tilde{C}_1 \frac{|x-y|^{4/3}}{t^{1/3}} \right\} dt \right) dy \right]^{1/q} \\
\leq \left[ \int_{|x-y|<r} \frac{1}{|x-y|^{qn-4}} \left( \int_{0}^{\infty} s^{3qn/4-4} e^{-q\tilde{C}_1 s} ds \right) dy \right]^{1/q} \\
\leq C(\int_{0}^{r} t^{-qn+4+n-1} dt)^{1/q} = Cr^{n/q-n+4/q} = Cr^{-2n/q_0+4/q},
\]
where \( 1/q + 2/q_0 = 1 \).

Then using Hölder inequality,
\[
|u_1(x, t)| \leq \left[ \int_{-\infty}^{t} \int_{|x-y|<r} |G_1(x, t; y, s)||f(y, s)|^{q_0/2} dyds \right]^{2/q_0} \left[ \int_{-\infty}^{t} \int_{|x-y|<r} |G_2(x, t; y, s)|^q dyds \right]^{1/q} \\
\leq Cm(x, V)^{2n/q_0-4/q} \left[ \int_{-\infty}^{t} \int_{|x-y|<r} |G_1(x, t; y, s)||f(y, s)|^{q_0/2} dyds \right]^{2/q_0}.
\]

Note that \( |x-y| < 1/m(x, V) \) and \( m(x, V) \sim m(y, V) \). Denote \( R = 1/m(y, V) \). Therefore, by Lemma 5,
\[
\int_{\mathbb{R}^{n+1}} (V^2(x)|u_1(x, t)|)^{q_0/2} dxdt \\
\leq C \int_{\mathbb{R}^{n+1}} V^{q_0}(x)|m(x, V)|^{n-2q_0/q} \left[ \int_{-\infty}^{t} \int_{|x-y|<r} |G_1(x, t; y, s)||f(y, s)|^{q_0/2} dyds \right] dxdt \\
\leq \int_{\mathbb{R}^{n+1}} |f(y, s)|^{q_0/2} \left[ \int_{|x-y|<C_1 R} V^{q_0}(x)|m(y, V)|^{n-2q_0/q}|G_1(x, t; y, s)| dxdt \right] dyds
\]
\[
\begin{align*}
&\leq \int_{\mathbb{R}^{n+1}} |f(y, s)|^{q_0/2} \left[ \int_{|x-y|<C_1 R} \int_{:\mathbb{R}}^{\infty} C_N V^{q_0}(x)[m(y, V)]^{n-2q_0/q} \frac{1}{[1 + |x-y|m(y, V)]^N} \frac{1}{[1 + (t-s)^{1/2} m^2(x, V)]^N} dx dt \right] dy ds \\
&\leq \int_{\mathbb{R}^{n+1}} |f(y, s)|^{q_0/2} \left[ \int_{|x-y|<C_1 R} C_N V^{q_0}(x)[m(y, V)]^{n-2q_0/q} \frac{1}{[1 + |x-y|m(y, V)]^N} \frac{1}{[1 + (t-s)^{1/2} m^2(x, V)]^N} dx \right] dy ds \\
&\leq CC_N \int_{\mathbb{R}^{n+1}} |f(y, s)|^{q_0/2} \left[ \int_{|x-y|<C_1 R} V^{q_0}(x) dx \right] dy ds \\
&\leq CC_N \int_{\mathbb{R}^{n+1}} |f(y, s)|^{q_0/2} (m(y, V))^{n-2q_0} \left[ \int_{|x-y|<C_1 R} V^{q_0}(x) dx \right] dy ds \\
&\leq CC_N \int_{\mathbb{R}^{n+1}} |f(y, s)|^{q_0/2} (m(y, V))^{n-2q_0} \left[ \int_{|x-y|<C_1 R} V(x) dx \right]^{q_0} dy ds \\
&\leq CC_N \int_{\mathbb{R}^{n+1}} |f(y, s)|^{q_0/2} (m(y, V))^{n-2q_0} R^{n-2q_0} dy ds \\
&\leq CC_{k_0} \int_{\mathbb{R}^{n+1}} |f(y, s)|^{q_0/2} dy ds.
\end{align*}
\]

Moreover, we have
\[
\begin{align*}
&\int_{\mathbb{R}^{n+1}} V^2(x)|u_1(x, t)|dx dt \\
&\leq C \int_{\mathbb{R}^{n+1}} V^2(x) \left[ \int_{-\infty}^t \int_{|x-y|<r} |\Gamma(x, t; y, s)||f(y, s)|dy ds \right] dx dt \\
&\leq \int_{\mathbb{R}^{n+1}} |f(y, s)| \left[ \int_{|x-y|<C_1 R} \int_{:\mathbb{R}}^{\infty} V^2(x) |\Gamma(x, t; y, s)| dx dt \right] dy ds \\
&\leq \int_{\mathbb{R}^{n+1}} |f(y, s)| \left[ \int_{|x-y|<C_1 R} \int_{:\mathbb{R}}^{\infty} C_N V^2(x) \frac{\exp \left\{-\tilde{C}_1 \frac{|x-y|^{4/3}}{(t-s)^{1/3}}\right\}}{[1 + |x-y|m(x, V)]^N} \frac{1}{(t-s)^{n/4}} dx dt \right] dy ds \\
&\leq \int_{\mathbb{R}^{n+1}} |f(y, s)| \left[ \int_{|x-y|<C_1 R} \frac{C_N V^2(x)}{[1 + |x-y|m(y, V)]^N} \int_0^\infty t^{-n/4} \exp \left\{-\tilde{C}_1 \frac{|x-y|^{4/3}}{t^{1/3}}\right\} dt \right] dx dy ds \\
&\leq C \int_{\mathbb{R}^{n+1}} |f(y, s)| \left[ \int_{|x-y|<C_1 R} \frac{V^2(x)}{[1 + |x-y|m(y, V)]^N|x-y|^{n-4}} dx \right] dy ds \\
&\leq CC_{k_0} \int_{\mathbb{R}^{n+1}} |f(y, s)| dy ds,
\end{align*}
\]

where we use (3) in Lemma 6 in the last inequality.

Therefore, by using interpolation,
\[\| V^2 u_1 \|_{L^p(\mathbb{R}^{n+1})} \leq C \| f \|_{L^p(\mathbb{R}^{n+1})} \text{ for } 1 \leq p \leq q_0/2.\]
To finish the proof, using (4) in Lemma 6, we first have

\[
\int_{\mathbb{R}^{n+1}} \Gamma(x, t; y, s) \ dyds \leq \int_{-\infty}^{t} \int_{\mathbb{R}^{n}} \frac{C_{N}}{[1 + |x - y|m(x, V)]^{N}} (t - s)^{-n/4} \exp \left\{ -\tilde{C}_{1} \frac{|x - y|^{4/3}}{(t - s)^{1/3}} \right\} dyds \\
\leq \int_{\mathbb{R}^{n}} \frac{C}{[1 + |x - y|m(x, V)]^{N}} |x - y|^{-n/4} \int_{0}^{\infty} s^{n/4 - 4} e^{-C_{0} s} \ ds \\
\leq C m(x, V)^{-4}.
\]

For \(1 < p \leq q_{0}/2\), we obtain

\[
|u_{2}(x, t)| \leq \left[ \int_{-\infty}^{t} \int_{|x - y| < r} |\Gamma(x, t; y, s)||f(y, s)|^{p} \ dyds \right]^{1/p} \left[ \int_{-\infty}^{t} \int_{|x - y| < r} |\Gamma(x, t; y, s)| \ dyds \right]^{1/p'} \\
\leq C m(x, V)^{-4/p'} \left[ \int_{-\infty}^{t} \int_{|x - y| < r} |\Gamma(x, t; y, s)||f(y, s)|^{p} \ dyds \right]^{1/p},
\]

where \(1/p + 1/p' = 1\).

Let \(R = 1/m(y, V)\). By Lemma 4,

\[
\int_{\mathbb{R}^{n+1}} (V^{2}(x)|u_{2}(x, t)|)^{p} \ dx \ dt \\
\leq C \int_{\mathbb{R}^{n+1}} V^{2p}(x)[m(x, V)]^{-4p/p'} \left[ \int_{-\infty}^{t} \int_{|x - y| \geq r} |\Gamma(x, t; y, s)||f(y, s)|^{p} \ dyds \right] \ dx \ dt \\
\leq \int_{\mathbb{R}^{n+1}} |f(y, s)|^{p} \left[ \int_{|x - y| \geq r} \int_{s}^{\infty} V^{2p}(x)[m(y, V)]^{-\frac{2p}{p'}} |\Gamma(x, t; y, s)| \ dyds \right] \ dx \ dt \\
\leq \int_{\mathbb{R}^{n+1}} |f(y, s)|^{p} \left[ \int_{|x - y| \geq C_{1} R} \int_{s}^{\infty} \frac{C_{N} V^{2p}(x)[m(y, V)]^{-4p/p'} \exp \left\{ -\tilde{C}_{1} \frac{|x - y|^{4/3}}{(t - s)^{1/3}} \right\}}{[1 + |x - y|m(y, V)]^{N}} \ dx \ dyds \right] \\
\leq \int_{\mathbb{R}^{n+1}} |f(y, s)|^{p} \left[ \int_{|x - y| \geq C_{1} R} \frac{V^{2p}(x)[m(y, V)]^{-4p/p'}}{[1 + |x - y|m(y, V)]^{N}} \ dx \ dt dyds \right] \\
\leq C \int_{\mathbb{R}^{n+1}} |f(y, s)|^{p} \left[ \int_{|x - y| \geq C_{1} R} \frac{V^{2p}(x)[m(y, V)]^{-4p/p'}}{[1 + |x - y|m(y, V)]^{N}} \ dx \ dyds \right] \\
\leq C \int_{\mathbb{R}^{n+1}} |f(y, s)|^{p} \left[ \sum_{j=0}^{\infty} \int_{2^{j} R \leq |x - y| < 2^{j+1} R} \frac{V^{2p}(x)}{[1 + |x - y|m(y, V)]^{N}} \ dx \ dyds \right] \\
\leq C \int_{\mathbb{R}^{n+1}} |f(y, s)|^{p} \left[ \sum_{j=0}^{\infty} \frac{1}{[1 + 2^{j}]^{N}} \left( \frac{1}{[2^{j+1}]^{n}} \int_{|x - y| < 2^{j+1} R} V^{2p}(x) \ dx \right) \ dyds \right] \\
\leq C \int_{\mathbb{R}^{n+1}} |f(y, s)|^{p} \left[ \sum_{j=0}^{\infty} \frac{(2^{j} R)^{4}}{[1 + 2^{j}]^{N}} \left( \frac{1}{[2^{j+1}]^{n}} \int_{|x - y| < 2^{j+1} R} V^{2p}(x) \ dx \right) \ dyds \right].
\]
Suppose \( \mathcal{L} \) is a linear operator on \( \mathbb{R}^n \).

Let \( z \) be a point in \( \mathbb{R}^n \) and \( R > 0 \) be sufficiently large.

It follows from (10) that

\[
\| \mathcal{L}(z) \|_{L^p(\mathbb{R}^n)} \leq C \| z \|_{L^p(\mathbb{R}^n)}
\]

Thus the theorem is proved.

**Proof of Theorem 1:** By Theorem 3, we have

\[
\| V^2(\partial/\partial t + (-\Delta)^2 + V^2)^{-1}f \|_{L^p(\mathbb{R}^{n+1})} \leq C \| f \|_{L^p(\mathbb{R}^{n+1})} \quad \text{for} \quad 1 \leq p \leq q_0/2.
\]

It follows from (10) that

\[
\| \nabla^4(\partial/\partial t + (-\Delta)^2 + V^2)^{-1}f \|_{L^p(\mathbb{R}^{n+1})} \leq C \| \nabla^4(\partial/\partial t + (-\Delta)^2)^{-1}(\partial/\partial t + (-\Delta)^2)(\partial/\partial t + (-\Delta)^2 + V^2)^{-1}f \|_{L^p(\mathbb{R}^{n+1})}
\]

\[
\leq \| \nabla^4(\partial/\partial t + (-\Delta)^2)(\partial/\partial t + (-\Delta)^2 + V^2)^{-1}f \|_{L^p(\mathbb{R}^{n+1})}
\]

\[
\leq \| f \|_{L^p(\mathbb{R}^{n+1})} + \| V^2(\partial/\partial t + (-\Delta)^2 + V^2)^{-1}f \|_{L^p(\mathbb{R}^{n+1})}
\]

\[
\leq C \| f \|_{L^p(\mathbb{R}^{n+1})}.
\]

The proof of Theorem 1 is finished.

**5 The \( L^p \) boundedness of the operator \( V^{2\alpha}(\partial/\partial t + (-\Delta)^2 + V^2)^{-\alpha} \)**

**Theorem 3.** Suppose \( V(x) \in B_{q_1}(\mathbb{R}^n), q_1 > n/2 \). Then, for \( 1 < p \leq q_1/2 \),

\[
\| V^{2\alpha}(\partial/\partial t + (-\Delta)^2 + V^2)^{-\alpha}f \|_{L^p(\mathbb{R}^{n+1})} \leq C \| f \|_{L^p(\mathbb{R}^{n+1})},
\]

**Proof.** By the functional calculus, we may write, for all \( 0 < \alpha < 1 \),

\[
(\partial/\partial t + (-\Delta)^2 + V^2)^{-\alpha} = \frac{1}{\pi} \int_0^\infty \lambda^{-\alpha}(\partial/\partial t + (-\Delta)^2 + V^2)^{-1}d\lambda.
\]

Let \( f \in C_0^\infty(\mathbb{R}^{n+1}) \). From \((\partial/\partial t + (-\Delta)^2 + V^2)^{-1}f(x,t) = \int_{\mathbb{R}^{n+1}} \Gamma(x,t;y,s)\lambda f(y,s)dyds\), it follows that

\[
V^{2\alpha}(\partial/\partial t + (-\Delta)^2 + V^2)^{-\alpha}f(x,t) = \int_{\mathbb{R}^{n+1}} K(x,t;y,s)V(x)^{2\alpha}f(y,s)dyds,
\]

where we choose \( N \) sufficiently large.

Hence,

\[
\int_{\mathbb{R}^{n+1}} |V^2(x)u_2(x,t)|^p dxdt \leq \int_{\mathbb{R}^{n+1}} |f(x,t)|^p dxdt \quad \text{for} \quad 1 \leq p \leq q_0/2.
\]

The proof of Theorem 1 is finished.
where
\[ K(x, t; y, s) = \frac{1}{\pi} \int_0^{\infty} \lambda^{-\alpha} \Gamma(x, t; y; \lambda) d\lambda \quad \text{for} \quad 0 < \alpha < 1, \]

Let \( f \in C_0^\infty(\mathbb{R}^{n+1}) \). The adjoint of \( V^{2\alpha}(\partial/\partial t + (-\Delta)^2 + V^2)^{-\alpha} \) is given by
\[
(V^{2\alpha}(\partial/\partial t + (-\Delta)^2 + V^2)^{-\alpha})^* f(x, t) = \int_{\mathbb{R}^{n+1}} K(y, s; x, t) V(y)^{2\alpha} f(y, s) dy ds.
\]

Note that for all \((x, t), (y, s) \in \mathbb{R}^{n+1}\) and \( t > s \),
\[
(t-s)^{-n/4} \exp \left\{ -\tilde{C}_1 \frac{|x-y|^{4/3}}{(t-s)^{1/3}} \right\} \leq \frac{1}{d((x, t), (y, s))^n}.
\]

By Corollary 2 we conclude that for every \( N \in \mathbb{N} \), there exists positive constants \( C_N \) and \( \tilde{C}_1 \) such that for all \((x, t), (y, s) \in \mathbb{R}^{n+1}\) and \( s > t \),
\[
|\tilde{K}(y, s; x, t)| \leq \frac{C_N}{[d((x, t), (y, s))]^{4-4\alpha}[1 + d((x, t), (y, s))m(x, V)]^N d((x, t), (y, s))^n}.
\]  
(11)

Let \( r = \frac{1}{m(x, V)} \). It follows from Hölder’s inequality and (11) that
\[
\left| (V^{2\alpha}(\partial/\partial t + (-\Delta)^2 + V^2)^{-\alpha})^* f(x, t) \right|
\leq \int_{\mathbb{R}^{n+1}} \left[ d((x, t), (y, s))]^{4-4\alpha}[1 + d((x, t), (y, s))m(x, V)]^N \right] \frac{V(y)^{2\alpha}}{d((x, t), (y, s))^n} |f(y, s)| dy ds
\leq C_N \sum_{j=-\infty}^{\infty} \int_{2^{j-1}r < d((x, t), (y, s)) \leq 2^j r} \frac{1}{(1 + 2^{j-1})^N (2^{j-r})^{n+4}} V(y)^{2\alpha} |f(y, s)| dy ds
\leq C_{CN} \sum_{j=-\infty}^{\infty} \frac{(2^j r)^{4\alpha}}{(1 + 2^{j-1})^N} \left\{ \frac{1}{(2^{j-1})^{n+4}} \int_{|x-y| \leq 2^j r} V(y)^{q_1} dy \right\}^{2\alpha} \left\{ \frac{1}{(2^{j-1})^{n+4}} \int_{d((x, t), (y, s)) \leq 2^j r} |f(y, s)|^{q_2} dy ds \right\}^{1/q_2}
\leq C_{CN} \sum_{j=-\infty}^{\infty} \frac{(2^j r)^{4\alpha}}{(1 + 2^{j-1})^N} \left\{ \frac{1}{(2^{j-1})^{n+4}} \int_{|x-y| \leq 2^j r} V(y)^{2\alpha} dy \right\} \left\{ \frac{1}{(2^{j-1})^{n+4}} \int_{d((x, t), (y, s)) \leq 2^j r} |f(y, s)|^{q_2} dy ds \right\}^{1/q_2}
= C_{CN} \sum_{j=-\infty}^{\infty} \frac{1}{(1 + 2^{j-1})^N} \left\{ \frac{1}{(2^{j-1})^{n+2}} \int_{|x-y| \leq 2^j r} V(y)^{2\alpha} dy \right\} \left\{ \frac{1}{(2^{j-1})^{n+2}} \int_{d((x, t), (y, s)) \leq 2^j r} |f(y, s)|^{q_2} dy ds \right\}^{1/q_2}.
\]

By Lemma 5 we conclude that for the case \( j \geq 1 \) there exists a constant \( C_0 \) such that
\[
\frac{(2^j r)^2}{[(x, 2^j r)^n]} \int_{|x-y| \leq 2^j r} V(y) dy \leq C_0 (2^j)^{1}.
\]
For the case \( j \leq 0 \), by using Lemma 2 we see that
\[
\frac{(2^j r)^2}{[(x, 2^j r)^n]} \int_{|x-y| \leq 2^j r} V(y) dy \leq C \left( \frac{r}{2^j r} \right)^{n/q_1-2} \left( \frac{1}{(2^j r)^{n/2}} \int_{|x-y| \leq r} V(y) dy \right)
= C (2^j)^{2-n/q_1}.
\]
Thus,
\[
\left|(V^{2a}(\partial/\partial t + (-\Delta)^2 + V^2)^{-\alpha})^* f(x, t)\right| \leq CC_N \{\mathcal{M}(|f|^q_2)(x, t)\}^{1/q_2} \sum_{j=-\infty}^{\infty} \left\{ \frac{(2^j)^{l_1}}{(1 + 2^j)^N} + (2^j)^{2-n/q_1} \right\}
\]
\[
\leq C \{\mathcal{M}(|f|^q_2)(x, t)\}^{1/q_2},
\]
where $\mathcal{M}$ is the Hardy-Littlewood maximal operator on $\mathbb{R}^{n+1}$ and we take $N$ sufficiently large.

Then
\[
\| (V^{2a}(\partial/\partial t + (-\Delta)^2 + V^2)^{-\alpha})^* f \|_{L^p_{\mathbb{R}^{n+1}}} \leq C \| f \|_{L^p_{\mathbb{R}^{n+1}}}, \ (q_1/2\alpha)' \leq p < \infty.
\]

Therefore,
\[
\| (V^{2a}(\partial/\partial t + (-\Delta)^2 + V^2)^{-\alpha}) f \|_{L^p_{\mathbb{R}^{n+1}}} \leq C \| f \|_{L^p_{\mathbb{R}^{n+1}}}, \ 1 < p \leq q_1/2\alpha.
\]

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References


