ON THE RADICAL BANACH ALGEBRAS RELATED TO SEMIGROUP ALGEBRAS

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Abstract. Let $S$ be a compactly cancellative foundation semigroup with identity. It is well-known that $L^\infty_0(S; M_a(S))^*$ can be equipped with a multiplication that extends the original multiplication on $M_a(S)$ and makes $L^\infty_0(S; M_a(S))^*$ a Banach algebra. In this paper, among the other things, it is shown that if $S$ is a nondiscrete compactly cancellative foundation semigroup with an identity, then the radical of $L^\infty_0(S; M_a(S))^*$ is infinite-dimensional.

1. Introduction and Notations

Let $S$ be a locally compact, Hausdorff topological semigroup with identity $e$. Let $M(S)$ be the space of all complex Borel measures on $S$. Then $\tilde{M}(S)$ is the continuous dual of $C_0(S)$, the space of all continuous functions on $S$ vanishing at infinity. The set of all measures $\mu \in M(S)$ for which both the mappings $x \mapsto \delta_x * |\mu|$ and $x \mapsto |\mu| * \delta_x$ are weakly continuous will denoted by $M_a(S)$, where $\delta_x$ denotes the Dirac measure at $x$. A topological semigroup $S$ is called a foundation semigroup if $S$ coincides with the closure of $\cup\{\text{supp}(\mu); \mu \in M_a(S)\}$. If $S$ is a foundation topological semigroup, then $M_a(S)$ is a closed $L$-ideal of $M(S)$ called the semigroup algebra $S$ [4]. More information on this matter can be found in [1], [4] and [5].

A complex-valued function $f$ on $S$ is said to be $M_a(S)$-measurable if it is $\mu$-measurable for all $\mu \in M_a(S)$. Denote by $L^\infty(S; M_a(S))$ the space of all bounded $M_a(S)$-measurable functions on $S$ formed by identifying functions that agree $\mu$-almost every where for all $\mu \in M_a(S)$. Observe that $L^\infty(S; M_a(S))$ with complex cojugation as involution, the pointwise operations and the norm $\|\cdot\|$ is a commutative $C^*$-algebra.

It is well-known from [11] that if $S$ is a foundation semigroup with an identity, then $L^\infty(S; M_a(S))$ can be identified with $M_a(S)^*$. We say that a function $f \in L^\infty(S; M_a(S))$ vanishes at infinity if for each $\epsilon > 0$, there is a compact subset $K$ of $S$ for which $\|f \chi_{S\setminus K}\| < \epsilon$, that is, for

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each \( \mu \in M_a(S) \), \( |f(x)| < \epsilon \) for \( \mu \)-almost all \( x \in S \setminus K \) (\( \mu \in M_a(S) \)).

Let \( \mathcal{L}^0_S(S; M_a(S)) \) be the \( C^* \)-algebra of all \( M_a(S) \)-measurable functions \( f \) on \( S \) such that \( f \) vanishes at infinity. Finally, let us recall that \( S \) is said to be compactly cancellative if \( C^{-1}D \) and \( CD^{-1} \) are compact subsets of \( S \) for all compact subsets \( C \) and \( D \) of \( S \) [8]. Compactly cancellative foundation semigroups form a large class of locally compact semigroups which includes locally compact groups as elementary examples. As another example, consider the semigroup

\[
S = \{0\} \cup \left\{ \frac{1}{n} ; \ n \geq 1 \right\} \cup \left\{ \frac{1}{2} + \frac{1}{n} ; \ n \geq 1 \right\}
\]

and set

\[
\mathcal{B} = \left\{ \{x\}; \ x \neq 0 \right\} \cup \left\{ \{0\} \cup \left\{ \frac{1}{n} ; \ n \geq k \right\} ; \ k \geq 1 \right\}.
\]

Then \( S \) with \( \mathcal{B} \) as a base of the topology and the operation \( xy = \max\{x, y\} \) defines a compactly cancellative foundation semigroup with identity. For an extensive study of \( \mathcal{L}^\infty_S(S; M_a(S)) \) in the compactly cancellative foundation semigroup case of \( S \), see [7], [8] and [9].

2. Main results

Let \( S \) be a compactly cancellative foundation semigroup with identity. Given any \( \mu \in M_a(S) \) and \( f \in \mathcal{L}^\infty_S(S; M_a(S)) \), define the complex-valued functions \( f \mu \) and \( \mu f \) on \( S \) by \( f \mu(x) = \mu(L_x f) \) and \( \mu f(x) = \mu(R_x f) \), where \( L_x f(y) = f(xy) \) and \( R_x f(y) = f(yx) \) for all \( x, y \in S \). It is known that \( f \mu \) and \( \mu f \) are in \( \mathcal{L}^\infty_S(S; M_a(S)) \) with \( \| f \mu \| \leq \| f \| \| \mu \| \) and \( \| \mu f \| \leq \| f \| \| \mu \| \). For \( f \in \mathcal{L}^\infty_S(S; M_a(S)) \) and \( F \in \mathcal{L}^\infty_S(S; M_a(S))^* \) we define \( Ff \in \mathcal{L}^\infty_S(S; M_a(S)) \) as a linear functional on \( M_a(S) \) by \( (Ff, \mu) = \langle F, \mu f \rangle \), see Proposition 3.2 in [8]. We define the Arens product of \( G \) and \( F \), denoted by \( G.F \) to be the functional defined by \( (G.F, f) = \langle G, Ff \rangle \) for \( f \in \mathcal{L}^\infty_S(S; M_a(S)) \). Equipped with this multiplication, \( \mathcal{L}^\infty_S(S; M_a(S))^* \) is a Banach algebra and this multiplication agrees on \( M_a(S) \) with the given product [8].

**Theorem 1.** Let \( S \) be a compactly cancellative foundation semigroup with an identity. Then \( C_0(S)^\perp \) is a closed two-sided ideal of \( \mathcal{L}^\infty_S(S; M_a(S))^* \) and \( \mathcal{L}^\infty_S(S; M_a(S))^*/C_0(S)^\perp \) is isometrically isomorphic as an algebra to \( M(S) \).

**Proof.** By Theorem 3.6 in [9], \( C_0(S)^\perp \) is a weak* closed two-sided ideal of \( \mathcal{L}^\infty_S(S; M_a(S))^* \), and so \( C_0(S)^\perp \) is a norm closed ideal of \( \mathcal{L}^\infty_S(S; M_a(S))^* \). From Banach space theory, there is an isometric linear space isometric between \( \mathcal{L}^\infty_S(S; M_a(S))^*/C_0(S)^\perp \) and \( C_0(S)^\perp \) [10]. In addition, there is an isometric linear space isomorphism between \( C_0(S)^* \) and \( M(S) \). The
composite isometric isomorphism $T$ is defined by $T(F + C_0(S)^\perp) = \mu$, where $\langle F, f \rangle = \int f(x)d\mu(x)$ for all $f \in C_0(S)$. It remains for us to see that $T$ is an algebra isomorphism when $L_0^\infty(S; M_a(S))^*/C_0(S)^\perp$ is given the quotient space multiplication induced from the multiplication in $L_0^\infty(S; M_a(S))^*$ and multiplication in $M(S)$ is convolution. For $F_1, F_2 \in L_0^\infty(S; M_a(S))^*$, we put $\mu_1 = T(F_1 + C_0(S)^\perp)$ and $\mu_2 = T(F_2 + C_0(S)^\perp)$. Let $\mu_3 = T(F_1, F_2 + C_0(S)^\perp)$. Then for each $f \in C_0(S)$, $F_2 f \in C_0(S)$. Indeed, any $f \in C_0(S)$ can be written in the form $f = \mu oh$ with $\mu \in M_a(S)$ and $h \in L_0^\infty(S; M_a(S))$, see Proposition 2.6 in [8]. On the other hand, $F_2 f = F_2 \mu oh = \mu F_2 h$. By Proposition 3.1 in [9], $L_0^\infty(S; M_a(S))$ is a left introverted subspace of $L_0^\infty(S; M_a(S))$. This shows that $F_2 h \in L_0^\infty(S; M_a(S))$. Hence $F_2 f = \mu F_2 h \in C_0(S)$ again by Proposition 2.6 in [8]. It is easy to see that $F_2 f(x) = \langle F_2, L_x f \rangle$ for all $x \in S$. Now, let $f \in C_0(S)$. We have

$$
\int f(z)d\mu_3(z) = \langle F_1, F_2, f \rangle = \langle F_1, F_2 f \rangle = \int \langle F_2, L_x f \rangle d\mu_1(x) = \int \int f(xy)d\mu_1(x)d\mu_2(y) = \int f(z)d\mu_1 \ast \mu_2(z).
$$

Since this holds for all $f \in C_0(S)$, we conclude that $\mu_3 = \mu_1 \ast \mu_2$ and so $T$ defines an isometric algebra isomorphism from $L_0^\infty(S; M_a(S))^*/C_0(S)^\perp$ onto $M(S)$.

**Theorem 2.** Let $S$ be a nondiscrete and compactly cancellative foundation semigroup with an identity. Then $L_0^\infty(S; M_a(S))^*$ is not semisimple and is not commutative.

**Proof.** Since $S$ is not discrete, it is an immediate consequence of the Hahn-Banach theorem that $C_0(S)^\perp \neq \{0\}$. Now if $F \in L_0^\infty(S; M_a(S))^*$, let $F'$ be an extension of $F$ to $L^\infty(S; M_a(S))^*$ such that $\|F\| = \|F'\|$. By [11], $L^\infty(S; M_a(S))$ can be identified with $M_a(S)^*$. Since $M_a(S)$ is weak* dense in $M_a(S)^{**}$ [10], so that we can find a net $\{\mu_\alpha\} \in M_a(S)$ such that $\mu_\alpha \rightarrow F'$ in the weak* topology of $M_a(S)^{**}$. We conclude that $\mu_\alpha \rightarrow F$ in the weak* topology of $L_0^\infty(S; M_a(S))^*$. For $G \in C_0(S)^\perp$ and $f \in C_0(S)$, we have

$$
\langle F, G, f \rangle = \langle F, Gf \rangle = \lim_\alpha (\mu_\alpha, Gf) = \lim_\alpha (G, \mu_\alpha of) = 0,
$$

since $\mu_\alpha of \in C_0(S)$ for all $\alpha$, see Proposition 2.1 in [8]. This shows that $L_0^\infty(S; M_a(S))^* C_0(S)^\perp = \{0\}$. By Proposition 1.5.6 in [3], we have $0 \neq C_0(S)^\perp \subseteq \text{rad}(L_0^\infty(S; M_a(S))^*)$ and consequently $L_0^\infty(S; M_a(S))^*$ is not semisimple.

It remains for us to see that $L_0^\infty(S; M_a(S))^*$ is not commutative. Suppose that $L_0^\infty(S; M_a(S))^*$ is commutative. Let $F \in L_0^\infty(S; M_a(S))^*$.
Clearly, the map \( G \mapsto F.G = G.F \) is weak* weak* continuous on \( L_0^\infty(S; M_a(S))^* \). This says that \( L_0^\infty(S; M_a(S))^* \) is Arens regular. By Theorem 4.3 in [8], \( S \) is discrete which is contradiction. \( \square \)

**Corollary 1.** Let \( S \) be a nondiscrete compactly cancellative foundation semigroup with an identity. Then the radical of \( L_0^\infty(S; M_a(S))^* \) is infinite-dimensional.

**Proof.** For any integer \( n \), there are \( n \) mutually disjoint relatively compact open subsets \( U_1, \ldots, U_n \) in \( S \), whose union is not all of \( S \). For \( 1 \leq i \leq n \), \( 1_{U_i} \) denotes the characteristic function of \( U_i \). Since \( 1_{U_i} \) is not in the closure of \( C_0(S) \), there exists \( F_i \in L_0^\infty(S; M_a(S))^* \) such that \( \langle F_i, 1_{U_i} \rangle \neq 0 \) for every \( f \in C_0(S) \). For \( 1 \leq i \leq n \), \( \{1_{U_1}, \ldots, 1_{U_i}\} \oplus C_0(S) \) is a closed subspace of \( L_0^\infty(S; M_a(S)) \), see Theorem 1.42 in [10]. Theorem 3.5 in [10] furnishes then a \( F_{i+1} \in C_0(S)^\perp \) such that \( \langle F_{i+1}, 1_{U_{i+1}} \rangle \neq 0 \) and \( \langle F_{i+1}, 1_{U_j} \rangle = 0 \) for all \( 1 \leq j \leq i \). Clearly \( \{F_1, \ldots, F_n\} \) is a linearly independent subset of \( C_0(S)^\perp \). By Theorem 2 and its proof, the radical \( L_0^\infty(S; M_a(S))^* \) is an infinite-dimensional subspace of \( L_0^\infty(S; M_a(S))^* \). \( \square \)

By a semicharacter on \( S \) we mean a non-zero function \( \chi \) in \( B(S) \) such that \( \chi(xy) = \chi(x)\chi(y) \) for all \( x, y \in S \). We denote the set of all continuous semicharacters on \( S \) by \( \hat{S} \). Let \( A \) be a closed subalgebra of \( M(S) \). By a multiplicative linear functional on \( A \) we mean a non-zero functional \( h \in A^* \) such that \( \langle h, \mu * \nu \rangle = \langle h, \mu \rangle \langle h, \nu \rangle \) for all \( \mu, \nu \in A \). The set of all multiplicative linear functionals on \( A \) is denoted by \( \hat{A} \). There exists a one-to-one mapping \( \tau \) of \( \hat{S} \) onto \( M(S) \) such that \( \hat{\chi}(\mu) = \int \chi(x)d\mu(x) \) for all \( \hat{\chi} \in M_a(S) \) where \( \tau(\chi) = \hat{\chi} \) is in \( \hat{S} \), see Theorem 5.3 in [4].

**Example 1.** Let \( S \) be the additive semigroup \( \mathbb{Z}^+ \) of all nonnegative integer numbers. Then \( S \) with the discrete topology is a compactly cancellative foundation semigroup with identity. A character \( \chi \) of \( \mathbb{Z}^+ \) is plainly determined by the number \( \chi(1) \), since \( \chi(n) = \chi(1)^n (n \in \mathbb{Z}^+) \), and \( \chi(1) \) can be any number in \( \mathbb{T} \). Then clearly \( \hat{S} \) separates the points of \( S \).

Let \( S \) be a compactly cancellative foundation semigroup with an identity. A function \( f \in L_0^\infty(S; M_a(S)) \) is said to be almost periodic if the set \( \{L_x f \mid x \in S\} \) of left translates of \( f \) is norm relatively compact in \( L_0^\infty(S; M_a(S)) \). The set of all almost periodic functions on \( S \) is denoted by \( AP(S) \).
Theorem 3. Let $\mathcal{S}$ be a compactly cancellative foundation semigroup with an identity which is not compact. Further, suppose that $\mathcal{S}$ is commutative and $\hat{\mathcal{S}}$ separates the points of $\mathcal{S}$. Then

$$AP(\mathcal{S})^\perp \not\subseteq \text{rad}(L^\infty_0(\mathcal{S}; M_a(\mathcal{S}))^*) .$$

Proof. Assume that $AP(\mathcal{S})^\perp \subseteq \text{rad}(L^\infty_0(\mathcal{S}; M_a(\mathcal{S}))^*)$. By Theorem 5.9 in [4], $M(\mathcal{S})$ is semisimple. It follows from Theorem 1 and Theorem 1.5.21 in [3] that $\text{rad}(L^\infty_0(\mathcal{S}; M_a(\mathcal{S}))^*) \subseteq C_0(\mathcal{S})^\perp$. We conclude that $AP(\mathcal{S})^\perp \subseteq C_0(\mathcal{S})^\perp$, and consequently $C_0(\mathcal{S}) \subseteq AP(\mathcal{S})$. However, since $\mathcal{S}$ is not compact, $C_0(\mathcal{S}) \cap AP(\mathcal{S}) = \{0\}$ is a consequence of the theory of almost periodic functions on semigroups [2]. \hfill \Box

Remark 1. (i): Let $\mathcal{S}$ be a compactly cancellative foundation semigroup with an identity. By Theorem 3.3 in [9], $M_a(\mathcal{S})$ is a closed ideal in $L^\infty_0(\mathcal{S}; M_a(\mathcal{S}))^*$. Further, suppose that $\mathcal{S}$ is commutative and $\hat{\mathcal{S}}$ separates the points of $\mathcal{S}$. By Theorem 5.9 in [4], $M_a(\mathcal{S})$ is semisimple. Since $\text{rad}(M_a(\mathcal{S})) = M_a(\mathcal{S}) \cap \text{rad}(L^\infty_0(\mathcal{S}; M_a(\mathcal{S}))^*)$, see Theorem 1.5.4 in [3], we conclude that $M_a(\mathcal{S}) \cap \text{rad}(L^\infty_0(\mathcal{S}; M_a(\mathcal{S}))^*) = \{0\}$. Now, let $\mathcal{S}$ be a compact abelian group. Then $\hat{\mathcal{S}}$ separates the points of $\mathcal{S}$ [6]. Consequently, if $\mathcal{S}$ is a compact abelian group, then

$$\text{rad}(L^1(\mathcal{S})^{**}) \cap L^1(\mathcal{S}) = \{0\} .$$

(ii): Let $\mathcal{S}$ be a nondiscrete and compactly cancellative foundation semigroup with an identity. By Theorem 2 and its proof, it is easy to see that

$$\text{rad}(L^\infty_0(\mathcal{S}; M_a(\mathcal{S}))^*) = \{F; L^\infty_0(\mathcal{S}; M_a(\mathcal{S}))^* F = \{0\}\} .$$

(iii) Let $\mathcal{S}$ be a compact foundation semigroup with identity. Let $f \in M_a(\mathcal{S})$, $\mu \in M_a(\mathcal{S})$. Clearly $\mu f \in M_a(\mathcal{S})$. It follows that $L^\infty_0(\mathcal{S}; M_a(\mathcal{S}))^* M_a(\mathcal{S})^\perp = \{0\}$, and so $M_a(\mathcal{S})^\perp \subseteq \text{rad}(L^\infty_0(\mathcal{S}; M_a(\mathcal{S}))^*)$. But $L^\infty_0(\mathcal{S}; M_a(\mathcal{S}))^*/M_a(\mathcal{S})^\perp$ is semisimple. We conclude that

$$M_a(\mathcal{S})^\perp = \text{rad}(L^\infty_0(\mathcal{S}; M_a(\mathcal{S}))^*) .$$

Theorem 4. Let $\mathcal{S}$ be a compactly cancellative foundation semigroup with an identity. Further, suppose that $M_a(\mathcal{S})$ is a semisimple Banach algebra. Then $L^\infty_0(\mathcal{S}; M_a(\mathcal{S}))^*/\text{rad}(L^\infty_0(\mathcal{S}; M_a(\mathcal{S}))^*) \cong M_a(\mathcal{S})$ if and only if $\mathcal{S}$ is a discrete semigroup.

Proof. Let $\mathcal{S}$ be a discrete semigroup. By Proposition 3.4 in [8], we have $L^\infty_0(\mathcal{S}; M_a(\mathcal{S}))^* \cong M_a(\mathcal{S})$. It follows that

$$L^\infty_0(\mathcal{S}; M_a(\mathcal{S}))^*/\text{rad}(L^\infty_0(\mathcal{S}; M_a(\mathcal{S}))^*) \cong M_a(\mathcal{S}) .$$
Suppose \( S \) is not discrete. Let \( U \) denote the family of relatively compact neighbourhoods of \( e \) and regard \( U \) as a directed set in the usual way: \( U \succeq V \) if \( U \subseteq V \). Since \( S \) is a foundation semigroup, we can find a probability measure \( e_U \in M_a(S) \) such that \( e_U(U) = 1 \) for all \( U \in U \). It is easy to see that \( \{e_U\}_{U \in U} \) is a bounded approximate identity for \( M_a(S) \) \[4\]. By the Banach-Alaoglu’s theorem, without loss of generality, we may assume that \( e_a \to E \) in the weak* topology of \( L_0^\infty(S; M_a(S))^* \). It is known that \( E \) is a right identity for \( L_0^\infty(S; M_a(S))^* \) \[3\]. We conclude that \( E.F - F \in rad(L_0^\infty(S; M_a(S))^*) \) for all \( F \in L_0^\infty(S; M_a(S))^* \). Thus \( E + rad(L_0^\infty(S; M_a(S))^*) \) is an identity for \( L_0^\infty(S; M_a(S))^*/rad(L_0^\infty(S; M_a(S))^*) \). By assumption, \( M_a(S) \) has an identity, say \( \mu \). Since \( \{e_U\}_{U \in U} \) is a bounded approximate identity for \( M_a(S) \), \( e_U = e_U * \mu \to \mu \) in the norm topology. It is not hard to see that \( e_U \to \delta_e \) in the \( \sigma(M(S), C_0(S)) \) topology of \( M(S) \). It follows that \( \delta_e = \mu \in M_a(S) \). This is a contradiction, see Exercise 3.10 in \[4\].

Let \( S \) be a locally compact foundation semigroup with an identity. If \( i : C_0(S) \to L_0^\infty(S; M_a(S)) \) is the inclusion map, then the restriction \( i^*(F) \) of \( F \in L_0^\infty(S; M_a(S))^* \) to the subspace \( C_0(S) \) of \( L_0^\infty(S; M_a(S))^* \) determines a quotient mapping \( i^* : L_0^\infty(S; M_a(S))^* \to M(S) \). Notice that \( i^* \) is the identity on \( M_a(S) \).

**Theorem 5.** Let \( S \) be a compactly cancellative foundation semigroup with identity. Then \( S \) is compact if there is a finite-dimensional right ideal \( I \) in \( L_0^\infty(S; M_a(S))^* \) such that \( i^*(I) \cap M(S) \neq \{0\} \).

**Proof.** Suppose that \( S \) is non-compact. Assume towards a contradiction that \( I \) is a finite-dimensional right ideal in \( L_0^\infty(S; M_a(S))^* \) such that \( i^*(I) \cap M(S) \neq \{0\} \). If \( x \in S \), let \( G \) be an extension of \( \delta_x \) (regarded as a functional on \( C_0(S) \)) to \( L_0^\infty(S; M_a(S)) \) such that \( \|G\| = \|\delta_x\| \) \[10\]. Then, for every \( F \in I \), we have \( F \delta_x = F i^*(G) = F.G \in I \). This shows that \( I \) is a right translation invariant subspace of \( L_0^\infty(S; M_a(S))^* \). Take \( F \in I \) such that \( i^*(F) \neq 0 \) and \( \|i^*(F)\| = 1 \). Take \( \nu \in M_a(S) \) such that \( i^*(F) * \nu \neq 0 \). Otherwise, \( i^*(F) = 0 \). Thus, without loss of generality, we may assume that \( i^*(F) \in M_a(S) \). Since \( I \) is a finite-dimensional subspace of \( L_0^\infty(S; M_a(S))^* \), \( \mathcal{X} := \{i^*(F) * \delta_x; \ x \in S\} \) is finite-dimensional. Let \( \dim(\mathcal{X}) = n \). Let \( i^*(F) * \delta_{x_1}, ..., i^*(F) * \delta_{x_n} \) generate \( \mathcal{X} \) as a subspace of \( M_a(S) \). It is evident that the mapping \( \varphi : \mathbb{C}^n \to \mathcal{X} \) defined by \( \varphi(c_1, ..., c_n) = \sum_{j=1}^n c_j i^*(F) * \delta_{x_j} \) is a homeomorphism \[10\]. Hence, there is a constant \( c > 0 \) such that each \( \mu \in \mathcal{X} \) can be written as \( \sum_{j=1}^n c_j i^*(F) * \delta_{x_j} \) with \( c_1, ..., c_n \in \mathbb{C} \) and \( \sum_{j=1}^n |c_j| \leq c \|\mu\| \). Choose \( \epsilon \in (0, 1) \) with \( \epsilon(1 + c) < 1 \). Let \( K \) be a
compact subset of $S$ such that $|i^*(F)|(K) > 1 - \epsilon$. Since the semigroup is non-compact, there exists $x \in S$ such that $Kx$ is disjoint from $Kx_1 \cup \ldots \cup Kx_n$. Clearly

$$1 - \epsilon < |i^*(F) \ast \delta_x|(Kx) \leq \sum_{j=1}^{n} |\alpha_j||i^*(F) \ast \delta_{x_j}|(Kx) < c\epsilon.$$ 

We conclude that $\epsilon(1 + c) > 1$ which is contradiction. \hfill $\square$

**Theorem 6.** Let $S$ be a compactly cancellative foundation semigroup with an identity. Let $I$ be a right ideal of $M_a(S)$ of dimension $n \geq 1$. Then $\mathcal{I}rad(L_0^\infty(S; M_a(S))^*) \subset I$.

**Proof.** Let $F \in rad(L_0^\infty(S; M_a(S))^*)$, $\mu \in I$ and $\{e_{\alpha}\}_{\alpha \in I}$ be a bounded approximate identity for $M_a(S)$ [4]. For $\alpha \in J$ we have $e_{\alpha} \cdot F \in M_a(S)$, since $M_a(S)$ is an ideal in $L_0^\infty(S; M_a(S))^*$ (see Proposition 3.3 in [8]). Since $I$ is finite-dimensional, $I$ is a closed right ideal in $M_a(S)$, see Theorem 1.21 in [10]. Clearly $\|\mu * e_{\alpha} \cdot F - \mu \cdot F\| \to 0$ and $\mu * e_{\alpha} \cdot F \in I$ for all $\alpha \in J$. We conclude that $\mu \cdot F \in I$ and so $\mathcal{I}rad(L_0^\infty(S; M_a(S))^*) \subset I$.

We assume that a contrario that $\mathcal{I} = \mathcal{I}$$L_0^\infty(S; M_a(S))^*$. If $I$ is cyclic, say $\mathcal{I} = \mu M_a(S)$, then

$$\mathcal{I} = \mathcal{I}rad(L_0^\infty(S; M_a(S))^*) = \mu rad(L_0^\infty(S; M_a(S))^*).$$

We must have $\mu = \mu \cdot F$ for some $F \in rad(L_0^\infty(S; M_a(S))^*)$. By Corollary 1.5.3 in [3], we have $\mu = 0$ and thus $\mathcal{I} = 0$ which is a contradiction. Now suppose that $\mathcal{I} = \mu_1 M_a(S) + \ldots + \mu_n M_a(S)$, and that the Theorem holds for right $L_0^\infty(S; M_a(S))^*$-modules with $n - 1$ generators. Since $\mathcal{I}/\mu_1 M_a(S)$ has $n - 1$ generators, and since

$$\mathcal{I}/\mu_1 M_a(S)rad(L_0^\infty(S; M_a(S))^*) = \mathcal{I}/\mu_1 M_a(S),$$

it follows that $\mathcal{I}/\mu_1 M_a(S) = \{0\}$. This shows that $\mathcal{I} = \mu_1 M_a(S)$. Therefore, by cyclic case, $\mathcal{I} = 0$. This is contradiction. \hfill $\square$

**References**


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