On quasi pseudo-GP-injective rings and modules

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Abstract

In 2010, Sanh et al. introduced a class of pseudo-M-gp-injective modules, following this, a right \( R \)-module \( N \) is called pseudo-M-gp-injective if for any homomorphism \( 0 \neq \alpha \in \text{End}(M) \), there exists \( a \in \mathbb{N} \) such that \( \alpha^a \neq 0 \) and every monomorphism from \( \alpha^n(M) \) to \( N \) can be extended to a homomorphism from \( M \) to \( N \). In this paper, we give more properties of pseudo-gp-injective modules.

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1 Introduction

Throughout the paper, \( R \) is an associative ring with identity \( 1 \neq 0 \) and all modules are unitary \( R \)-modules. We write \( M_R \) (resp., \( _RM \)) to indicate that \( M \) is a right (resp., left) \( R \)-module. Let \( J \) (resp., \( Z_r, S_r \)) be the Jacobson radical (resp. the right singular ideal, the right socle) of \( R \) and \( E(M_R) \) the injective hull of \( M_R \). If \( X \) is a subset of \( R \), the right (resp. left) annihilator of \( X \) in \( R \) is denoted by \( r_R(X) \) (resp., \( l_R(X) \)) or simply \( r(X) \) (resp. \( l(X) \)). If \( N \) is a submodule of \( M \) (resp., proper submodule) we write \( N \leq M \) (resp. \( N < M \)). Moreover, we write \( N \leq^e M, N \ll M, N \leq^0 M \) and \( N \leq^{max} M \) to indicate that \( N \) is an essential
submodule, a small submodule, a direct summand and a maximal submodule of $M$, respectively. A module $M$ is called uniform if $M \neq 0$ and every non-zero submodule of $M$ is essential in $M$. A module $M$ is finite dimensional (or has finite rank) if $E(M)$ is a finite direct sum of indecomposable submodules. A right $R$-module $N$ is called $M$-generated if there exists an epimorphism $M^{(1)} \to N$ for some index set $I$. If the set $I$ is finite, then $N$ is called finitely $M$-generated. In particular, $N$ is called $M$-cyclic if it is isomorphic to $M/L$ for some submodule $L$ of $M$. Hence, any $M$-cyclic submodule $X$ of $M$ can be considered as the image of an endomorphism of $M$.

Following Nicholson, Yousif (see [15]), a ring $R$ is called right $P$-injective if every $R$-homomorphism from a principal right ideal of $R$ to $R$ is a left multiplication. They studied some properties of these rings and their applications. In [18], Sanh et al. transferred this notion to modules. A right $R$-module $N$ is called $M$-principally injective (briefly, $M$-p-injective) if every homomorphism from an $M$-cyclic submodule of $M$ to $N$ can be extended to one from $M$ to $N$. A right $R$-module $M$ is called quasi-principally injective (briefly, quasi $p$-injective) if $M$ is $M$-principally injective. Quasi-$p$-injective modules were defined first by Wisbauer in [24] under the terminology of semi-injective modules, but there are no details. Following [13], a module $M$ is called principally quasi-injective if every homomorphism from a cyclic submodule of $M$ to $M$ can be extended to an endomorphism of $M$. Since an $M$-cyclic submodule of $M$ needs not to be cyclic, the notion of quasi-$p$-injective modules is different from that was defined in [13].

As a generalization of injective modules, the class of pseudo injective modules have been studied by Singh and Jain in 1967 [11], Teply (1975)[22], Jain and Singh (1975)[11], Wakamatsu (1979)[23]. Recently, Hai Quang Dinh ([6]) introduced the notion of pseudo $M$-injective modules (the original terminology is $M$-pseudo-injective). A right $R$-module $N$ is called pseudo $M$-injective if for every submodule $A$ of $M$, any monomorphism $\alpha : A \to N$ can be extended to a homomorphism $M \to N$. A right $R$-module $N$ is called pseudo-injective if $N$ is pseudo-$N$-injective.

In 2009, Sanh et al., introduced the notion of pseudo-$M$-$p$-injective modules and studied the endomorphism rings of quasi pseudo-$p$-injective modules. A right $R$-module $N$ is called pseudo-$M$-$p$-injective if every monomorphism from an $M$-cyclic submodule of $M$ to $N$ can be extended to a homomorphism from $M$ to $N$, or equivalently, for any homomorphism $\alpha \in \text{End}(M)$, every monomorphism from $\alpha(M)$ to $N$ can be extended to a homomorphism from $M$ to $N$ (see [16]). A module $M$ is called quasi pseudo-$p$-injective if $M$ is pseudo-$M$-$p$-injective. A ring $R$ is called right pseudo $P$-injective if $R_R$ is quasi pseudo-$p$-injective. Following [8], a right $R$-module $M$ is said to be generalized principally injective (briefly gp-injective), if for any $0 \in x \in R$, there exists an $n \in \mathbb{N}$ such that $x^n \neq 0$ and any $R$-homomorphism from $x^n R$ into $M$ can be extended to one from $R_R$ to $M$. A ring $R$ is called right GP-injective if $R_R$ is GP-injective. The concept of
GP-injective modules was introduced in [12] to study the class of von Neumann regular rings, V-rings, self-injective rings and their generalizations. In [2], Chen et al. studied some properties of GP-injective rings. In particular, they gave some characterizations of GP-injective ring with special chain conditions. In 2009, Sanh et al. introduced the notion of pseudo-M-gp-injective modules. A right $R$-module $N$ is called for pseudo-$M$-gp-injective if for each homomorphism $0 \neq \alpha \in \text{End}(M)$, there exists $n \in \mathbb{N}$ such that $\alpha^n \neq 0$ and every monomorphism from $\alpha^n(M)$ to $N$ can be extended to a homomorphism from $M$ to $N$ ([17]). A module $M$ is called quasi-pseudo-gp-injective if $M$ is pseudo-M-gp-injective. A ring $R$ is called right pseudo GP-injective if $R_R$ is quasi-pseudo-gp-injective. In this paper, we continue studying more properties of pseudo-p-injective modules, pseudo-gp-injective modules and the endomorphism rings of pseudo-p-injective modules.

2 On pseudo-$M$-gp-injective

Firstly, we give a new characterization of pseudo-M-gp-injective modules.

**Theorem 2.1** Let $M$, $N$ be right $R$-modules. Then following conditions are equivalent:

1. $N$ is pseudo-$M$-gp-injective.
2. For each $0 \neq s \in \text{End}(M)$, there exists $n \in \mathbb{N}$ such that $s^n \neq 0$ and
   \[ \{ f \in \text{Hom}(M, N) | \text{Ker} f = \text{Ker}s^n \} \subseteq \text{Hom}(M, N)s^n. \]
3. For each $0 \neq s \in \text{End}(M)$, there exists $n \in \mathbb{N}$ such that $s^n \neq 0$ and
   \[ \{ f \in \text{Hom}(M, N) | \text{Ker} f = \text{Ker}s^n \} = \{ f \in \text{Hom}(M, N) | \text{Ker} f \cap \text{Im} s^n = 0 \} s^n. \]

**Proof.** $(1) \Rightarrow (2)$. Suppose that $0 \neq s \in \text{End}(M)$. Since $N$ is pseudo-$M$-gp-injective, there exists $n \in \mathbb{N}$ such that $s^n \neq 0$ and every monomorphism from $s^n(M)$ to $N$ can be extended to a homomorphism from $M$ to $N$. Let $f \in \text{Hom}(M, N)$ such that $\text{Ker} f = \text{Ker}s^n$. We consider homomorphism
   \[ \varphi : s^n(M) \rightarrow N \text{ via } \varphi(s^n(m)) = f(m), \forall m \in M. \]
It is easy to see that $\varphi$ is a monomorphism. By our assumption, there exists a homomorphism $h : M \rightarrow N$ such that $hu = \varphi$, where $\iota$ is the inclusion map from $s^n(M) \rightarrow M$, which implies that $f = hs^n \in \text{Hom}(M, N)s^n$.

$(2) \Rightarrow (3)$. It is clear that
   \[ \{ f \in \text{Hom}(M, N) | \text{Ker} f \cap \text{Im} s^n = 0 \} s^n \subseteq \{ f \in \text{Hom}(M, N) | \text{Ker} f = \text{Ker}s^n \}. \]
Let \( g \in \text{Hom}(M, N) \) such that \( \text{Ker}g = \text{Kers}^n \). Then by (2), there exists a homomorphism \( h : M \to N \) such that \( g = hs^n \). It follows that \( \text{Ker}h \cap \text{Im}s^n = 0 \). Hence, \( g \in \{ f \in \text{Hom}(M, N) \mid \text{Ker}f \cap \text{Im}s^n = 0 \} s^n \).

(3) \( \Rightarrow \) (1). For each \( 0 \neq s \in \text{End}(M) \), by (3), there exists \( n \in \mathbb{N} \) such that \( s^n \neq 0 \) and

\[
\{ f \in \text{Hom}(M, N) \mid \text{Ker}f = \text{Kers}^n \} = \{ f \in \text{Hom}(M, N) \mid \text{Ker}f \cap \text{Im}s^n = 0 \} s^n.
\]

Assume that \( \phi : s^n(M) \to N \) is a monomorphism. Then \( \text{Ker}(\phi s^n) = \text{Kers}^n \). Hence there is \( h \in \text{Hom}(M, N) \) such that \( \phi s^n = hs^n \). It gives \( h\iota = \phi \), where \( \iota \) is the inclusion map, proving that \( N \) is pseudo-\( M \)-gp-injective. \( \square \)

From the above theorem, we get some characterizations of quasi-pseudo-gp-injective modules.

**Corollary 2.2** Let \( M \) be right \( R \)-module and \( S = \text{End}(M) \). The following conditions are equivalent:

1. \( M \) is quasi-pseudo-gp-injective;
2. For each \( 0 \neq s \in S \), there exists \( n \in \mathbb{N} \) such that \( s^n \neq 0 \) and
   \[
   \{ f \in S \mid \text{Ker}f = \text{Kers}^n \} \subseteq Ss^n;
   \]
3. For each \( 0 \neq s \in S \), there exists \( n \in \mathbb{N} \) such that \( s^n \neq 0 \) and
   \[
   \{ f \in S \mid \text{Ker}f = \text{Kers}^n \} = \{ f \in S \mid \text{Ker}f \cap \text{Im}s^n = 0 \} s^n.
   \]

**Corollary 2.3** Let \( M, N \) be right \( R \)-modules. The following conditions are equivalent:

1. \( N \) is pseudo-\( M \)-p-injective;
2. For each \( s \in \text{End}(M) \),
   \[
   \{ f \in \text{Hom}(M, N) \mid \text{Ker}f = \text{Kers} \} \subseteq \text{Hom}(M, N)s;
   \]
3. For each \( s \in \text{End}(M) \),
   \[
   \{ f \in \text{Hom}(M, N) \mid \text{Ker}f = \text{Kers} \} = \{ f \in \text{Hom}(M, N) \mid \text{Ker}f \cap \text{Im}s = 0 \} s.
   \]

**Proposition 2.4** Let \( N \) be pseudo-\( M \)-p-injective. Then for any elements \( s, \alpha \in \text{End}(M) \), we have:

\[
\{ \beta \in \text{Hom}(M, N) \mid \text{Ker} \beta \cap \text{Im}s = \text{Ker} \alpha \cap \text{Im}s \} =
\{ \gamma \in \text{Hom}(M, N) \mid \text{Ker} \gamma \cap \text{Im}(\alpha s) = 0 \} \alpha + \{ \delta \in \text{Hom}(M, N) \mid \delta s = 0 \}.
\]
Proof. Let 
\[ \mathcal{A} = \{ \beta \in \text{Hom}(M, N) | \text{Ker} \beta \cap \text{Im}s = \text{Ker} \alpha \cap \text{Im}s \} \]
\[ \mathcal{B} = \{ \gamma \in \text{Hom}(M, N) | \text{Ker} \gamma \cap \text{Im}(\alpha s) = 0 \} \]
\[ \mathcal{C} = \{ \delta \in \text{Hom}(M, N) | \delta s = 0 \} \]

It is easy to see that \( \mathcal{B} \alpha + \mathcal{C} \subseteq \mathcal{A} \). Conversely, let \( \beta \in \text{Hom}(M, N) \) such that \( \text{Ker} \beta \cap \text{Im}s = \text{Ker} \alpha \cap \text{Im}s \) (\( \beta \in \mathcal{A} \)). It follows that \( \text{Ker}(\alpha s) = \text{Ker}(\beta s) \). By Corollary 2.3, there exists \( \gamma \in \mathcal{B} \) such that \( \beta s = \gamma \alpha s \) or \( (\beta - \gamma \alpha)s = 0 \). It means \( \beta - \gamma \alpha \in \mathcal{C} \), which implies that \( \beta \in \mathcal{B} \alpha + \mathcal{C} \). □

Proposition 2.5 If \( M = M_1 \oplus M_2 \) is quasi-pseudo-\( p \)-injective, then \( M_1 \) is \( M_2 \)-\( p \)-injective.

Proof. Let \( M = M_1 \oplus M_2 \) be quasi-pseudo-\( p \)-injective and \( s \in \text{End}(M_2) \). Let \( f : s(M_2) \to M_1 \) be a homomorphism. Consider homomorphism \( g : s(M_2) \to M \) defined by \( g(a) = f(a) + a \) for all \( a \in s(M_2) \). Then \( g \) is a monomorphism. By [16, Proposition 1.3], \( M \) is pseudo-\( M_2 \)-\( p \)-injective, whence \( g \) extends to a homomorphism \( \bar{g} : M_2 \to M \). Let \( \pi : M \to M_1 \) be the canonical projection. Then \( \pi \bar{g} : M_2 \to M \) extends \( f \). Thus \( M_1 \) is \( M_2 \)-\( p \)-injective, as required. □

Corollary 2.6 For any integer \( n \geq 2 \), if \( M^n \) is quasi-pseudo-\( p \)-injective, then \( M \) is quasi-\( p \)-injective.

Proposition 2.7 Let \( M \) and \( N \) be modules and \( X = M \oplus N \). The following conditions are equivalent:

(1) \( N \) is pseudo-\( M \)-\( p \)-injective.

(2) For each \( M \)-cyclic submodule \( K \) of \( X \) with \( K \cap M = K \cap N = 0 \), there exists \( C \leq X \) such that \( K \leq C \) and \( N \oplus C = X \).

Proof. (1) \( \Rightarrow \) (2). Let \( K \) be a submodule of \( X \) which is \( M \)-cyclic with \( K \cap M = K \cap N = 0 \), and \( \pi_M : M \oplus N \to M \) and \( \pi_N : M \oplus N \to N \) be the canonical projections. We can check that \( N \oplus K = N \oplus \pi_M(K) \) and hence \( \pi_M(K) \simeq K \), proving that \( \pi_M(K) \) is a \( M \)-cyclic submodule of \( M \). Let \( \varphi : \pi_M(K) \to \pi_N(K) \) be a homomorphism defined as follows: for \( k = m + n \in K \) (with \( m \in M, n \in N \)), \( \varphi(m) = n \). It is easy to see that \( \varphi \) is a monomorphism. Since \( N \) is pseudo-\( M \)-\( p \)-injective, there is a homomorphism \( \bar{\varphi} : M \to N \) extending \( \varphi \). Let \( C = \{ m + \bar{\varphi}(m) | m \in M \} \). Then \( X = N \oplus C \) and \( K \leq C \).

(2) \( \Rightarrow \) (1). Let \( s \in \text{End}(M) \) and \( \varphi : s(M) \to N \) be a monomorphism. Put \( K = \{ s(m) - \varphi(s(m)) | m \in M \} \). Then \( K \cap M = 0 \) and \( N \oplus K = N \oplus \pi_M(K) = N \oplus s(M) \). It is easy to see that \( K \) is \( M \)-cyclic. By assumption, there exists a submodule \( C \) of \( X \) containing \( K \) with \( N \oplus C = X \). Let \( \pi : N \oplus C \to N \) be the natural projection. Then the restriction \( \pi|_M \) extends \( \varphi \), proving (1). □
3 On quasi-pseudo-gp-injective rings and modules

From Corollary 2.3, we have some characterizations of quasi-pseudo-p-injective modules.

**Theorem 3.1** The following conditions are equivalent for module $M$ with $S = \text{End}(M)$:

1. $M$ is quasi-pseudo-p-injective;
2. If $\text{Ker} f = \text{Ker} g$ with $f, g \in S = \text{End}(M)$, then $Sf = Sg$;
3. If $f \in S = \text{End}(M)$ and $\alpha, \beta : f(M) \to M$ is monomorphisms, then $\alpha = s\beta$ for some $s \in S$.

**Proof.** (1) $\Rightarrow$ (2). By Corollary 2.3.

(2) $\Rightarrow$ (3). Assume that $0 \neq f \in S$ satisfies (2). Let $\alpha, \beta : f(M) \to M$ be monomorphisms. Then $\text{Ker}(\alpha f) = \text{Ker}(\beta f)$. By our assumption, there exists $s \in S$ such that $\alpha f = s\beta f$, which implies that $\alpha = s\beta$.

(3) $\Rightarrow$ (1). Let $s \in S$ and $\varphi : s(M) \to M$ be a monomorphism. Let $\iota : s(M) \to M$ be the inclusion. By (3), there exists $\bar{\varphi} \in S$ such that $\varphi = \bar{\varphi}\iota$ showing that $\bar{\varphi}$ extends $\varphi$. Thus $M$ is quasi-pseudo p-injective. $\square$

**Corollary 3.2** The following conditions are equivalent for ring $R$:

1. $R$ is right pseudo $P$-injective;
2. If $r(x) = r(y)$ with $x, y \in R$, then $Rx = Ry$.

We have the following relations:

quasi-p-injective $\Rightarrow$ quasi-pseudo-p-injective $\Rightarrow$ quasi-pseudo-gp-injective.

**Example 3.3** i) Let $F$ be an algebraically closed field and $x, y$ be indeterminates. Let $R = F(y)[x]$ such that $xf - fx = df/dy$, $f \in F(y)$ (see [20, Example]). Then the $R$-module $M = R/(x(x + y)(x + y - 1/y))R$ is quasi-pseudo-p-injective but not quasi-p-injective by [20, Example].

ii) Let $K = F(y_1, y_2, ...)$ and $L = F(y_2, y_3, ...)$ with $F$ a field, and $\rho : K \to L$ be an isomorphism via $\rho(y_i) = y_{i+1}$ and $\rho(c) = c$ for all $c \in F$ (see [4, Example 1]. Let $K[x_1, x_2; \rho]$ be the ring of twisted left polynomials over $K$ where $x_i k = \rho(k)x_i$ for all $k \in K$ and for $i = 1, 2$. Set $R = K[x_1, x_2; \rho]/(x_1^2, x_2^2)$. Then $R_R$ is quasi-pseudo-gp-injective which is not quasi-pseudo-p-injective.

Next we study some properties of quasi-pseudo-gp-injective, self-generator modules and their endomorphism rings.
Theorem 3.4 Let $M$ be a right $R$-module with $S = \text{End}(M)$. Then

1. If $S$ is a right pseudo GP-injective ring, then $M$ is quasi-pseudo-gp-injective.

2. If $M$ is quasi-pseudo-gp-injective and self-generator, then $S$ is a right pseudo GP-injective ring.

Proof. (1). Let $f \in S$. Since $S$ is right pseudo GP-injective, there exists $n \in \mathbb{N}$ such that $f^n \neq 0$ and if $r_S(f^n) = r_S(g)$ for some $g \in S$, then $g \in Sf^n$ by Corollary 2.2. Assume that $Ker f^n = Ker g$ with $g \in S$. Then $r_S(f^n) = r_S(g)$ and hence $g \in Sf^n$. Thus $M$ is quasi-pseudo-gp-injective by Corollary 2.2.

(2). Let $0 \neq f \in S$. Since $M$ is quasi-pseudo-gp-injective, there exists $n \in \mathbb{N}$ such that $f^n \neq 0$ and if $Ker(f^n) = Ker(g)$ with $g \in S$, then $g \in Sf^n$. Let $g \in S$ with $r_S(f^n) = r_S(g)$. Since $M$ is a self-generator, we get $Ker f^n = Ker g$. By our assumption, $g \in Sf^n$ and so $S$ is right pseudo GP-injective.

Corollary 3.5 Let $M$ be a right $R$-module with $S = \text{End}(M)$. Then

1. If $S$ is a right pseudo $P$-injective ring, then $M$ is quasi-pseudo-p-injective.

2. If $M$ is a quasi-pseudo-p-injective module which is a self-generator, then $S$ is a right pseudo $P$-injective ring.

For a right $R$-module $M$, $S = \text{End}(M)$ we denote:

$$W(S) = \{ s \in S | \text{Ker}(s) \text{ is essential in } M \}.$$

Lemma 3.6 Let $M_R$ be a quasi-pseudo-gp-injective module which is a self-generator, $S = \text{End}(M)$. If $a \notin W(S)$, then Ker$(a) < \text{Ker}(a - ata)$ for some $t \in S$.

Proof. If $a \notin W(S)$, then Ker$(a)$ is not an essential submodule of $M$. Hence there exists $0 \neq m \in M$ such that $mR \cap \text{Ker}(a) = 0$. Since $M$ is a self-generator, there exists $\lambda \in S$ such that $0 \neq \lambda(M) \leq mR$. Hence Ker$(a) \cap \lambda(M) = 0$. It follows that $a\lambda \neq 0$. Since $M$ is quasi-pseudo-gp-injective, there exists $n \in \mathbb{N}$ such that $(a\lambda)^n \neq 0$ and if Ker$(a\lambda)^n = Ker(g)$ with $g \in S = \text{End}(M)$, then $g \in S(a\lambda)^n$. From Ker$(a) \cap \lambda(M) = 0$ we also have Ker$((a\lambda)^n) = \text{Ker}(\lambda(a\lambda)^{n-1})$. Hence $\lambda(a\lambda)^{n-1} \in S(a\lambda)^n$. Therefore $\lambda(a\lambda)^{n-1} = s(a\lambda)^n$ for some $s \in S$, which implies that Im$(\lambda(a\lambda)^{n-1}) \leq \text{Ker}(a - asa)$. It follows that Ker$(a) < \text{Ker}(a - asa)$, since Im$(\lambda(a\lambda)^{n-1}) \neq \text{Ker}(a)$ and $(a\lambda)^n \neq 0$.

Lemma 3.7 Assume that $M$ is quasi-pseudo-gp-injective module which is a self-generator. Then $J(S) = W(S)$. 
Proof. Let $a \in J(S)$. If $a \notin W(S)$, then by the proof of Lemma 3.6, there exist a positive integer $n$ and $\lambda, t \in S$ such that $(a\lambda)^n \neq 0$ and $(1 - at)(a\lambda)^n = 0$. Note that $1 - at$ is left invertible, so $(a\lambda)^n = 0$, a contradiction. Conversely, let $a \in W(S)$. Then, for each $t \in S$, $ta \in W(S)$ and hence $1 - ta \neq 0$. Since $M$ is a quasi-pseudo-p-injective module, there exists $n \in \mathbb{N}$ such that $(1 - ta)^n \neq 0$ and if $Ker((1 - ta)^n)$ is left invertible, proving our lemma.

□

Corollary 3.8 If $R$ is right pseudo GP-injective, then $J(R) = Z(R_R)$.

Recall that a module $M$ is said to satisfy the generalized C2-condition (or GC2) (see [25]) if for any $N \cong M$ with $N \leq M$, $N$ is a direct summand of $M$.

Theorem 3.9 If $M$ is quasi-pseudo-gp-injective, then $M$ satisfies GC2.

Proof. Let $S = \text{End}(M)$. Assume that $Kers = 0$ with $s \in S$. We need to prove that $S = Ss$. Since $M$ is quasi-pseudo-gp-injective, there exists $n \in \mathbb{N}$ such that $s^n \neq 0$ and $Kers^n = Ker\ g$ with $g \in S$, which would imply that $g \in Ss^n$. Note that $Kers = 0 = Ker1_S$. It follows that $1_S \in Ss^n \leq Ss$, whence $S = Ss$. Thus $M$ is GC2 by [25, Theorem 3].

□

Corollary 3.10 If $R$ is right pseudo GP-injective, then $R$ is right GC2.

Proposition 3.11 Let $M$ be a quasi-pseudo-p-injective module which is a self-generator and $S = \text{End}(M)$. If every complement submodule of $M$ is $M$-cyclic, then $S/J(S)$ is von Neumann regular.

Proof. We have $J(S) = W(S)$ by Lemma 3.7. For all $\lambda \in S$, let $L$ be a complement of $\text{Ker}\lambda$. We consider the map $\phi : \lambda(L) \twoheadrightarrow M$ defined by $\phi(\lambda(x)) = x$ for all $x \in L$. Then $\phi$ is a monomorphism and $\lambda(L) \cong L$ which implies $\lambda(L)$ is a $M$-cyclic submodule of $M$. Since $M$ is quasi-pseudo-p-injective, there exists $\theta \in S$, which is an extension of $\phi$. Then $\text{Ker}\lambda + L \leq \text{Ker}(\lambda\theta\lambda - \lambda)$, and we see that $\text{Ker}\lambda \oplus L \leq M$. Consequently $\lambda\theta\lambda - \lambda \in W(S) = J(S)$.

□

Theorem 3.12 Let $M$ be a quasi-pseudo-gp-injective module which is a self-generator and $S = \text{End}(M)$. Then the following conditions are equivalent:

1. $S$ is right perfect;
2. For any infinite sequence $s_1, s_2, \cdots \in S$, the chain
\[
\text{Ker}(s_1) \leq \text{Ker}(s_2s_1) \leq \cdots
\]
is stationary.
\textbf{Proof.} (1) $\Rightarrow$ (2). Let $s_i \in S$, $i = 1, 2, \ldots$. Since $S$ is right perfect, $S$ satisfies DCC on finitely generated left ideals. So the chain $Ss_1 \geq Ss_2 s_1 \geq \ldots$ terminates. Thus there exists $n > 0$ such that $Ss_n s_{n-1} \cdots s_1 = Ss_k s_{k-1} \cdots s_1$ for all $k > n$. It follows that $\text{Ker}(s_n s_{n-1} \cdots s_1) = \text{Ker}(s_k s_{k-1} \cdots s_1)$ for all $k > n$.

(2) $\Rightarrow$ (1). We first prove that $S/W(S)$ is a von Neumann regular ring. Let $a_1 \notin W(S)$. Then by Lemma 3.6, there is $c_1 \in S$ such that $\text{Ker}(a_1) < \text{Ker}(a_1 - a_1 c_1 a_1)$. Put $a_2 = a_1 - a_1 c_1 a_1$. If $a_2 \in W(S)$, then we have $\bar{a}_1 = \bar{a}_1 c_1 \bar{a}_1$, i.e., $\bar{a}_1$ is a regular element of $S/W(S)$. If $a_2 \notin W(S)$, there exists $a_3 \in S$ such that $\text{Ker}(a_2) < \text{Ker}(a_3)$ with $a_3 = a_2 - a_2 c_2 a_2$ for some $c_2 \in S$ by the preceding proof. Repeating the above-mentioned process, we get a strictly ascending chain

\[ \text{Ker}(a_1) < \text{Ker}(a_2) < \ldots, \]

where $a_{i+1} = a_i - a_i c_i a_i$ for some $c_i \in S$, $i = 1, 2, \ldots$. Let

\[ b_1 = a_1, b_2 = 1 - a_1 c_1, \ldots, b_{i+1} = 1 - a_i c_i, \ldots, \]

then

\[ a_1 = b_1, a_2 = b_2 b_1, \ldots, a_{i+1} = b_{i+1} b_i \ldots b_2 b_1, \ldots. \]

and we have the following strictly ascending chain

\[ \text{Ker}(b_1) < \text{Ker}(b_2 b_1) < \ldots, \]

which contradicts the hypothesis. Hence there exists a positive integer $m$ such that $a_{m+1} \in W(S)$, i.e., $a_m - a_m c_m a_m \in W(S)$. This shows that $\bar{a}_m$ is a regular element of $S/W(S)$, and hence $\bar{a}_{m-1}, \bar{a}_{m-2}, \ldots, \bar{a}_1$ are regular elements of $S/W(S)$, i.e., $S/W(S)$ is von Neumann regular. We have $J(S) = W(S)$ by Lemma 3.7, proving that $S/J(S)$ is von Neumann regular. Thus $S$ is right perfect by [5, Lemma 1.9].

\[ \square \]

\textbf{Lemma 3.13} Let $M$ be a right $R$-module and $S = \text{End}(M)$. Then

(1) $l_S(A(M)) = l_S(A)$ for all $A \subseteq S$ with $A(M) = \sum_{s \in A} s(M)$.

(2) $l_S(r_M(l_S(A))) = l_S(A)$ for all $A \subseteq S$.

\textbf{Proof.} (1). Let $a \in l_S(A)$, $a \cdot A = 0$. Therefore $a \cdot s = 0$ or $a(s(M)) = 0$ for all $s \in A$. This implies that $a \in l_S(A(M))$. Hence $l_S(A) \leq l_S(A(M))$. Conversely, for every $a \in l_S(A(M))$, we have $a.s(M) = 0$ for all $s \in A$. This implies that $a \in l_S(A)$.

(2). It is clear that $l_S(r_M(l_S(A))) \geq l_S(A)$. Conversely, for all $s \in l_S(A)$, $s.A(M) = 0$. This implies that $A(M) \leq r_M(l_S(A))$. Thus
\[ l_S(A(M)) \geq l_S(r_M(l_S(A))). \]

By (1) we get the result. \qed

Let \( \emptyset \neq A \subset S = \text{End}(M) \). Put

\[ \text{Ker} A = \bigcap_{f \in A} \text{Ker} f = \{ m \in M | f(m) = 0 \ \forall f \in A \}. \]

If \( X \leq M \) and \( X = \text{Ker} A \) for some \( \emptyset \neq A \subset S \), \( X \) is called an \( M \)-annihilator.

**Proposition 3.14** Let \( M_R \) be a quasi-pseudo-gp-injective, self-generator module and \( S = \text{End}(M_R) \). If \( M_R \) satisfies ACC on \( M \)-annihilators, then \( S \) is semiprimary.

**Proof.** Now we will claim that \( S \) satisfies ACC on right annihilators or DCC on left annihilators. Indeed, we consider the descending chain

\[ l_S(A_1) \geq l_S(A_2) \geq \ldots \text{ where } A_i \subseteq S, \]

then

\[ r_M(l_S(A_1)) \leq r_M(l_S(A_2)) \leq \ldots. \]

By our assumption, there exists \( n \in \mathbb{N} \) such that \( r_M(l_S(A_n)) = r_M(l_S(A_k)) \) for all \( k > n \), and so \( l_S r_M(l_S(A_n)) = l_S r_M(l_S(A_k)) \). By Lemma 3.13, \( l_S(A_n) = l_S(A_k) \) for all \( k > n \). This shows that \( S \) satisfies DCC on left annihilators or ACC on right annihilators. Therefore \( J(S) \) is nilpotent by [14, Lemma 3.29] and Lemma 3.7. It follows that \( S \) is semiprimary by Theorem 3.12. \qed

**Corollary 3.15** If \( R \) is right pseudo GP-injective and satisfies ACC on right annihilators, then \( R \) is semiprimary.

For quasi-pseudo-p-injective modules, we have

**Theorem 3.16** Let \( M_R \) be a quasi-pseudo-p-injective module and \( S = \text{End}(M_R) \). If \( M \) satisfies ACC on \( M \)-annihilators, then \( S \) is semiprimary.

**Proof.** Consider the chain \( S f_1 \geq S f_2 \geq \cdots \) of cyclic left ideals of \( S \). Then we have \( \text{Ker} f_1 \leq \text{Ker} f_2 \leq \cdots \). By hypothesis, there exists \( n \in \mathbb{N} \) such that \( \text{Ker} f_n = \text{Ker} f_{n+k}, \ \forall k \in \mathbb{N} \). It follows that \( S f_n = S f_{n+k} \ \forall k \in \mathbb{N} \). Thus \( R \) is right perfect.

Consider the ascending chain \( r_M(J(S)) \leq r_M(J(S)^2) \leq \cdots \). By assumption, there is \( n \in \mathbb{N} \) such that \( r_M(J(S)^n) = r_M(J(S)^{n+k}) \) for all \( k \in \mathbb{N} \). Let \( B = J(S)^n \).
Then we get \( r_M(B) = r_M(B^2) \). Assume \( J(S) \) is not nilpotent. Then \( B^2 \neq 0 \) and the non-empty set

\[
\{ \text{Ker} g \mid g \in B \text{ and } Bg \neq 0 \}
\]

has a maximal element \( \text{Ker} g_0, g_0 \in B \). The relation \( BBg_0 = 0 \) would imply that \( \text{Im} g_0 \leq r_M(B^2) = r_M(B) \) and hence \( Bg_0 = 0 \), contradicting to the choice of \( g_0 \). Therefore we can find an \( h \in B \) with \( Bhg_0 \neq 0 \). However, since \( \text{Ker} g_0 \leq \text{Ker}(hg_0) \), the maximality of \( \text{Ker} g_0 \) implies that \( \text{Ker} g_0 = \text{Ker} hg_0 \). Since \( M \) is quasi-pseudo-p-injective, this implies that \( Sg_0 = Shg_0 \), i.e. \( g_0 = shg_0 \) for some \( s \in S \) or \( g_0(1 - sh) = 0 \). Since \( sh \in B \leq J(S) \), this gives \( g_0 = 0 \), a contradiction. Thus \( J(S) \) must be nilpotent. □

Following [14], a ring \( R \) is called directly finite if \( ab = 1 \) in \( R \) implies that \( ba = 1 \).

**Proposition 3.17** A right pseudo \( P \)-injective ring \( R \) is directly finite if and only if all monomorphisms \( R_R \to R_R \) are isomorphisms.

**Proof.** Assume that \( \varphi : R_R \to R_R \) is a monomorphism. Let \( a = \varphi(1) \). Then \( r(a) = 0 = r(1) \) and so \( Ra = R \) by Corollary 2.2. Hence \( ba = 1 \) for some \( b \in R \), so \( ab = 1 \) by hypothesis, and so \( \varphi \) is onto. Conversely, let \( ab = 1 \) in \( R \). Therefore the homomorphism \( \alpha : R \to R, \alpha(r) = br, \forall r \in R \) is monomorphism. By hypothesis \( \alpha \) is an epimorphism. There exists \( c \in R \) such that \( 1 = \alpha(c) = bc \). It follows that \( a = c \) and \( ba = 1 \). □

The series of higher left socles \( \{S^l_\alpha\} \) of the ring \( R \) are defined inductively as follows: \( S^l_1 = \text{Soc}(R_R) \), and \( S^l_{\alpha+1}/S^l_\alpha = \text{Soc}(R/S^l_\alpha) \) for each ordinal \( \alpha \geq 1 \).

Motivated by [3, Lemma 9 (ii)], we have the following proposition.

**Proposition 3.18** If \( R \) is a right pseudo GP-injective ring and satisfies ACC on essential left ideals, then

1. \( r(J) \leq^e R_R \),
2. \( J \) is nilpotent,
3. \( J = lr(J) \).

**Proof.** (1) Since \( R \) has ACC on essential left ideals, \( R/S_l \) is a left Noetherian ring. Then, there exists \( k > 0 \) such that \( S^l_k = S^l_{k+1} = \cdots \) and \( R/S^l_k \) is a right Noetherian ring. Now we will claim that \( S^l_k \leq^e R_R \). In fact, assume that \( xR \cap S^l_k = 0 \) for some \( 0 \neq x \in R \). Let \( \bar{R} = R/S^l_k \) and \( l_{\bar{R}}(\bar{a}) \) be maximal in the set \( \{ l_{\bar{R}}(\bar{y}) \mid 0 \neq y \in xR \} \). Since \( S^l_k = S^l_{k+1} \), we get \( \text{Soc}(l_{\bar{R}}) = 0 \), and so \( \bar{R} \) is not simple as left \( \bar{R} \)-module. Thus there exists \( t \in R \) such that \( 0 \neq \bar{R}t \bar{a} < \bar{R} \).
If $\bar{a}\bar{a} = 0$, then $ata \in aR \cap S_k^1 = 0$, and so $ata = 0$. From this fact and pseudo GP-injectivity of $R$, we see that if $r(ta) = r(b)$, $b \in R$ then $Rta = Rb$ by Corollary 2.2. If $r(a) = r(ta)$, then $R = Rta$, a contradiction. Thus $r(a) < r(ta)$. Then there exists $b \in R$ such that $ab \neq 0$ and $tab = 0$. That means $0 \neq ab \in xR$ and $l_R(\bar{a}) < l_R(ab)$. This contradicts to the maximality of $l_R(\bar{a}_0)$.

If $\bar{a}\bar{a} \neq 0$, then $0 \neq \bar{R}a\bar{a} < \bar{R}a$. Since $R$ is right pseudo GP-injective, there exists $m \in \mathbb{N}$ such that $(ata)^m \neq 0$ and if $r((ata)^m) = r(b)$, $b \in R$ then $b \in R((ata)^m)$. It follows that $r(a) < r((ata)^m)$. Let $c \in r((ata)^m) \setminus r(a)$. Then $0 \neq ac \in xR$, $(\bar{a}\bar{a})^{m-1}a\bar{c} \in l_R(\bar{a})$, a contradiction.

Thus $S_k^1 \leq ^e R_R$ and hence $r(J) \leq ^e R_R$ (since $S_k^1 \leq r(J)$).

(2). By [3, Lemma 9 (ii)],
(3). Since $r(J) \leq ^e R_R$, $r(J) \leq Z_v = J$. □

A module $M_R$ is called **extending (or CS)** if every submodule of $M$ is essential in a direct summand of $M$. A ring $R$ is called right CS if $R_R$ is CS (see [7]). Following [10], a module $M$ is called NCS if there are no nonzero complement submodules which is small in $M$. A ring $R$ is right NCS if $R_R$ is NCS. Clearly every CS module is NCS, but the converse is not true, as we can see that the $Z$-module $Z_2 \oplus Z_3$ is NCS but not CS. On the other hand, let $K$ be a division ring and $V$ be a left $K$-vector space of infinite dimension. Let $S = \text{End}_K(V)$. Take $R = \begin{pmatrix} S & S \\ S & S \end{pmatrix}$, then $R$ is right NCS but not right CS.

**Proposition 3.19** If $R$ is a left Noetherian, right pseudo $P$-injective and right NCS ring, then $R$ is left Artinian.

**Proof.** First, we prove that $R = R/J$ is a regular ring. Assume that $a \notin J$. Since $J = lr(J) = Z_v$, there exists a nonzero complement right ideal $I$ of $R$ such that $r(a) \cap I = 0$ by Lemma 3.18. We claim that there exists $b \in I$ such that $ab \notin J$. Suppose on the contrary that $aI \leq J$. Then $aIr(J) = 0$. Since $r(a) \cap I = 0$, $Ir(J) \leq I \cap r(a) = 0$. Thus $I \leq lr(J) = J$. It follows that $I$ is small in $R_R$, a contradiction. Hence we have $b \in I$ such that $r(a) \cap bR = 0$ and $ab \notin J$. It follows that $r(b) = r(ab)$. Hence $Rb = Rab$ and so $b = cab$ for some $c \in R$. This implies that $\bar{b} \in r_R(\bar{a} - \bar{a}c\bar{a})$, where $\bar{r} = r + J \in R/J$ for any $r \in R$. Since $\bar{ab} \neq 0$, we see that $r_R(\bar{a}) < r_R(\bar{a} - \bar{a}c\bar{a})$. If $a - ac \in J$, then $a$ is a regular element of $R$. If $a - ac \notin J$, let $a_1 = a - ac$. Then $r(a_1) = 0$ is not essential in $R_R$. By the same way, we get $a_2 = a_1 - a_1c_1a_1$ for some $c_1 \in R$ and $r_R(\bar{a}_1) < r_R(\bar{a}_2)$. If $a_2 \notin J$, then $a_1$ is a regular element of $R$. It follows that $a$ is a regular element of $R$. If $a_2 \notin J$, we have $a_3 = a_2 - a_2c_2a_2$ for some $c_2 \in R$ and $r_R(\bar{a}_2) < r_R(\bar{a}_3)$. Continuing this process, we get $a_k \in R, k = 1, 2, \ldots$. Since $R$ is left noetherian and $Jac(R) = 0$, $R$ is a semiprime and left Goldie ring. By [9, Lemma 5.8], $R$ satisfies ACC on right
annihilators. Hence there exists some positive integer \( m \) such that \( a_m \in J \), and thus \( a \) is also a regular element of \( R \). Since \( \bar{a} \) is an arbitrary nonzero element of \( \bar{R} \), we see that \( \bar{R} \) is a regular ring. Then \( \bar{R} \) is semisimple because \( R \) is left noetherian. Moreover, by Lemma 3.18, \( J \) is nilpotent and so \( R \) is semiprimary. Thus \( R \) is left artinian. \( \square \)

### 4 On maximal ideals

In this section, we study the endomorphism ring of quasi-pseudo-gp-injective modules.

Let \( S = \text{End}_R(M) \) be the endomorphism ring of a right \( R \)-module \( M \). Following [19], an element \( u \in S \) is called a right uniform element of \( S \) if \( u \neq 0 \) and \( u(M) \) is a uniform submodule of \( M \). An element \( u \in R \) is called right uniform if \( uR \) is a uniform right ideal (see [14]). In this section, we generalize some results of Sanh and Shum for quasi-p-injective modules; Nicholson and Yousif for p-injective rings to quasi-pseudo-gp-injective modules.

First, we need the following lemma:

**Lemma 4.1** Let \( M \) be a quasi-pseudo-gp-injective module and \( S = \text{End}(M) \). Then for any right uniform element \( u \) of \( S \), the set

\[
A_u = \{ s \in S | \text{Ker} s \cap \text{Im} u \neq 0 \}
\]

is the unique maximal left ideal of \( S \) containing \( \text{ls} \text{(Im} u) \).

**Proof.** Clearly, \( A_u \) is a left ideal of \( S \). It is easy to see that \( \text{ls} \text{(Im} u) \leq A_u \) and \( A_u \neq S \) (because \( 1 \notin A_u \)). We now claim that \( A_u \) is maximal. In fact, for any \( s \in S \setminus A_u \), we have \( \text{Im} u \cap \text{Ker} s = 0 \), whence \( su \neq 0 \). There exists \( m \in \mathbb{N} \) such that \( (su)^m \neq 0 \) and if \( \text{Ker} (su)^m = \text{Ker} g \), \( g \in S \) then \( g \in S(su)^m \). Since \( \text{Ker} ((su)^m) = \text{Ker} u \), we get \( S(su)^m = Su \). Then there exists \( t \in S \) such that \( (1 - t(su)^{m-1})u = 0 \). It follows from \( S = \text{ls}(u) + Ss \), that \( A_u \) is maximal in \( S \). It remains to show that \( A_u \) is unique. In fact, assume that there is another maximal left ideal \( L \) of \( S \) containing \( \text{ls}(\text{Im} u) \) and \( L \neq A_u \). Repeating above process we also have \( S = L \), a contradiction. \( \square \)

**Corollary 4.2 ([19, Lemma 1])** Let \( M \) be a quasi-p-injective module and \( S = \text{End}(M) \). Then for any right uniform element \( u \) of \( S \), the set

\[
A_u = \{ s \in S | \text{Ker} s \cap \text{Im} u \neq 0 \}
\]

is the unique maximal left ideal of \( S \) containing \( \text{ls}(\text{Im} u) \).
Lemma 4.4 Let $M$ be a quasi-pseudo-$p$-injective, self-generator module with finite Goldie dimension and $S = \text{End}(M_R)$. Then $M_u$ is the unique maximal left ideal which contains $l(u)$.

The following lemma is a generalization of Lemma 3 in [19].

Lemma 4.4 Let $M$ be a quasi-pseudo-$p$-injective module, $S = \text{End}(M_R)$ and $W = \bigoplus_{i=1}^n u_i(M)$ a direct sum of uniform submodule $u_i(M)$ of $M$. If $A \leq S$ is a maximal left ideal which is not of the form $A_u$ for some right uniform element $u$ of $S$, then there is $\psi \in A$ such that $\text{Ker}(1 - \psi) \cap W$ is essential in $W$.

Proof. Since $A \neq A_{u_1}$, we can take $k \in A \setminus A_{u_1}$. Then $\text{Im} u_1 \cap \text{Ker} k = 0$, whence $ku_1 \neq 0$. There exists $m \in \mathbb{N}$ such that $(ku_1)^m \neq 0$ and if $\text{Ker}(ku_1)^m = \text{Ker}(g)$, $g \in S$ then $g \in S(ku_1)^m$. It is easy to see that $\text{Ker}(ku_1)^m = \text{Ker}(u_1)$ and hence $S(ku_1)^m = Su_1$. Consequently we have $u_1 = \alpha_1(ku_1)^m$ for some $\alpha_1 \in S$. Let $\varphi_1 = \alpha_1(ku_1)^m - 1 k \in SA \subset A$. Then $(1 - \varphi_1)u_1 = 0$. This shows that $\text{Ker}(1 - \varphi_1) \cap u_1(M) = u_1(M) \neq 0$. If $\text{Ker}(1 - \varphi_1) \cap u_2(M) \neq 0$ for all $i \geq 2$, then we are done and in this case $\bigoplus_{i=1}^n (\text{Ker}(1 - \varphi_1) \cap u_i(M)) \leq W$. Without loss of generality, we now assume that $\text{Ker}(1 - \varphi_1) \cap u_2(M) = 0$. It follows that $(1 - \varphi_1)(u_2(M)) \cong u_2(M)$ is uniform. Since $A \neq A_{(1 - \varphi_1)u_2}$, we can take any $h \in A \setminus A_{(1 - \varphi_1)u_2}$. By using the above argument, there exists $\alpha_2 \in S$ such that $(1 - \varphi_1)u_2 = \alpha_2 h(1 - \varphi_1)u_2$. It follows that

$$(1 - (\alpha_2 h + \varphi_1 - \alpha_2 h \varphi_1))u_2 = 0.$$ 

Let $\varphi_2 = \alpha_2 h + \varphi_1 - \alpha_2 h \varphi_1$. Then $(1 - \varphi_2)u_i = 0$ for $i = 1, 2$. Continuing this way, we eventually obtain a $\psi \in A$ such that $\text{Ker}(1 - \psi) \cap u_i(M) \neq 0$ for all $i = 1, \ldots, n$. In other words, we have shown that $\text{Ker}(1 - \psi) \cap W$ is essential in $W$ as required.

The following theorem describes the properties of the endomorphism ring $S = \text{End}(M_R)$ of a quasi pseudo $p$-injective module $M_R$.

Theorem 4.5 Let $M$ be a quasi-pseudo-gp-injective, self-generator module with finite Goldie dimension and $S = \text{End}(M_R)$.

1. If $I \subset S$ is a maximal left ideal, then $I = A_u$ for some right uniform element $u \in S$.

2. $S$ is semilocal.
Proof. Since $M$ is a self-generator which has finite Goldie dimension, there exist elements $u_1, u_2, \ldots, u_n$ of $S$ such that $W = u_1(M) \oplus u_2(M) \oplus \cdots \oplus u_n(M)$ is essential in $M$, where each $u_i(M)$ is uniform. Moreover, $M$ is a quasi-p-injective module, we have $J(S) = W(S) = \{s \in S \mid \text{Ker}(s) \text{ is essential in } M\}$ by Lemma 3.7.

(1). Suppose on the contrary that $I$ is not of the form $A_u$ for some right uniform element of $u \in S$. Then by Lemma 4.4, there exists a $\varphi \in I$ such that $\text{Ker}(1-\varphi) \cap W$ is essential in $W$. It follows that $1-\varphi \in J(S) \subset I$, a contradiction. Hence $I = A_u$ for some right uniform element $u \in S$.

(2). If $\varphi \in A_{u_1} \cap A_{u_2} \cap \cdots \cap A_{u_n}$, then $\text{Ker}(\varphi) \cap u_i(M) \neq 0$ for each $i$. Hence $\text{Ker}(\varphi)$ is essential in $M$. Therefore $\varphi \in J(S)$, i.e., $A_{u_1} \cap \cdots \cap A_{u_n} = J(S)$. This shows that $S/J(S)$ is semisimple. \hfill \Box

As a consequence, we immediately get the following result for the right pseudo GP-injective rings.

Corollary 4.6 Let $R$ be a right pseudo GP-injective ring which has right finite Goldie dimension. Then

(1) If $I \subset R$ is a maximal left ideal, then $I = A_u$ for some right uniform element $u \in R$.

(2) $R$ is semilocal.

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