The axiom of hemi-slant 3-spheres in almost Hermitian geometry

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Abstract

The axiom of hemi-slant 3-spheres is introduced. It is proved that if an almost Hermitian manifold \( M \) with dimension \( 2m \geq 6 \) satisfies this axiom for some slant angle \( \theta \in (0, \frac{\pi}{2}) \), then \( M \) has pointwise constant type \( \alpha \) if and only if \( M \) has pointwise constant anti-holomorphic sectional curvature \( \alpha \), and using this result some conditions for constancy of sectional curvature of a considered almost Hermitian manifold are given.

1 Introduction

In [2], E. Cartan defined the axiom of \( n \)-planes. A Riemannian manifold \( M \) of dimension \( m \geq 3 \) is said to satisfy the axiom of \( n \)-planes, where \( n \) is a fixed integer \( 2 \leq n \leq m - 1 \), if for each point \( p \in M \) and any \( n \)-dimensional subspace \( \sigma \) of the tangent space \( T_pM \) there exists an \( n \)-dimensional totally geodesic submanifold \( N \) such that \( p \in N \) and \( T_pN = \sigma \). He gave a criterion for constancy of sectional curvature in the following theorem.

**Theorem 1.1.** Let \( M \) be a Riemannian manifold of dimension \( m \geq 3 \). If \( M \) satisfies the axiom of \( n \)-planes for some \( n, 2 \leq n \leq m - 1 \), then \( M \) has constant sectional curvature.

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In [21], K. Yano and I. Mogi applied Cartan’s idea to Kaehlerian manifolds. A Kaehlerian manifold $M$ is said to satisfy the axiom of holomorphic planes if for each point $p \in M$ and each holomorphic plane $\sigma \subset T_p M$, there exists a totally geodesic submanifold $N$ such that $p \in N$ and $T_p N = \sigma$. They proved the following theorem.

**Theorem 1.2.** A Kaehlerian manifold satisfying the axiom of holomorphic planes is a complex space form.

In [12], D.S. Leung and K. Nomizu defined the axiom of $n$-spheres by taking totally umbilical submanifold $N$ with parallel mean curvature vector field instead of totally geodesic submanifold $N$ in the axiom of $n$-planes. They proved the following theorem.

**Theorem 1.3.** If a Riemannian manifold $M$ of dimension $m \geq 3$ satisfies the axiom of $n$-spheres for some $n$, $2 \leq n \leq m - 1$, then $M$ has constant sectional curvature.

Afterwards, many studies have been made in this direction. Kaehlerian manifolds were studied in [4, 6, 8, 11, 19], the papers [17] and [19] discussed nearly Kaehlerian (almost Tachibana) manifolds, and results concerning larger classes of almost Hermitian manifolds can be found in [9, 10, 16, 17]. In this paper, we shall introduce the axiom of hemi-slant 3-spheres and as an application, we shall give an interesting relation between the notion of constant type and anti-holomorphic sectional curvature for a $2m(m \geq 3)$-dimensional almost Hermitian manifold satisfying this axiom for some slant angle $\theta \in (0, \frac{\pi}{2})$. Using this fact, we shall prove some theorems related to sectional curvature for a considered even dimensional almost Hermitian manifold. We shall also give some results related to the Weyl conformal curvature tensor and the Bochner curvature tensor of a certain almost Hermitian manifold satisfying the axiom of hemi-slant 3-spheres. Our work is motivated by the above-cited papers.

## 2 Preliminaries

A $C^\infty$-manifold $M$ is called almost Hermitian if its tangent bundle has an almost complex structure $J$ and a Riemannian metric $g$ such that $g(JX, JY) = g(X, Y)$ for all $X, Y \in \chi(M)$, where $\chi(M)$ is the Lie algebra of $C^\infty$ vector fields on $M$. Let $\nabla$ be the covariant derivative on $M$, the Riemannian curvature tensor $R$ associated with $\nabla$ defined by $R(X, Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$.
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We denote $g(R(X,Y)Z,U)$ by $R(X,Y,Z,U)$. The sectional curvature $K$ of $M$ determined by orthonormal vector fields $X$ and $Y$ is given by $K(X,Y) = R(X,Y,X,Y)$. The Weyl conformal curvature tensor $W$ is defined by

$$W(X,Y,Z,U) = R(X,Y,Z,U) - \frac{1}{2m-1} \left\{ g(X,U)Ric(Y,Z) - g(X,Z)Ric(Y,U) 
+ g(Y,Z)Ric(X,U) - g(Y,U)Ric(X,Z) \right\} + \frac{S}{(2m-1)(2m-2)} \left\{ g(X,U)g(Y,Z) - g(X,Z)g(Y,U) \right\}$$

for all $X,Y,Z,U \in T_pM$, where $Ric$ and $S$ are the Ricci tensor and the scalar curvature of $M$, respectively. A $2m$-dimensional almost Hermitian manifold with $m \geq 2$ is conformally flat if and only if $W = 0$ identically [10, 20].

By an $r$-plane we mean an $r$-dimensional linear subspace of a tangent space $T_pM, p \in M$. Motivated from [3], we have the following definition.

**Definition 2.1.** Let $\sigma$ be a 2-plane. The angle $\theta \in [0, \frac{\pi}{2}]$ between $\sigma$ and $J\sigma$ is defined by

$$\cos \theta = |g(X, JY)|,$$

where $\{X, Y\}$ is an orthonormal basis of $\sigma$. If $\theta = constant$, then $\sigma$ is called a slant-plane and $\theta$ is called slant angle of $\sigma$.

This is a generalization of holomorphic and anti-holomorphic planes. In fact, holomorphic and anti-holomorphic planes are slant planes with slant angle $\theta$ equal to 0 and $\frac{\pi}{2}$, respectively, see ([4, 6, 8]). Now, motivated from [1] and [14] we have the following definition.

**Definition 2.2.** A 3-plane $\sigma$ in $T_pM$ is called hemi-slant if it contains a slant 2-plane with slant angle $\theta \in [0, \frac{\pi}{2})$ and a nonzero vector $Z \in T_pM$ such that $JZ$ is perpendicular to $\sigma$, in which case $\sigma = D^\theta \oplus \{Z\}$ with $JZ \perp \sigma$, where $D^\theta$ is the corresponding slant 2-plane.

The sectional curvature of $M$ restricted to a holomorphic (resp. an anti-holomorphic) plane $\sigma$ is called holomorphic (resp. anti-holomorphic) sectional curvature. If the holomorphic (resp. anti-holomorphic) sectional curvature at each point $p \in M$, does not depend on $\sigma$, then $M$ is said to be pointwise constant holomorphic (resp. pointwise constant anti-holomorphic) sectional curvature. A connected Riemannian (resp. Kaehlerian) manifold of (global) constant sectional curvature (resp. of constant holomorphic sectional curvature) is called a real space form (resp. a complex space form) ([9, 20]). The following useful notion was defined by A. Gray in [7].
Definition 2.3. Let $M$ be an almost Hermitian manifold. Then $M$ is said to be of constant type at $p \in M$ provided that for all $X \in T_p M$, we have $\lambda(X, Y) = \lambda(X, Z)$ whenever the planes $\text{span}\{X, Y\}$ and $\text{span}\{X, Z\}$ are anti-holomorphic and $g(Y, Y) = g(Z, Z)$, where the function $\lambda$ is defined by $\lambda(X, Y) = R(X, Y, X, Y) − R(JX, JY, X, Y)$. If this holds for all $p \in M$, then we say that $M$ has (pointwise) constant type. Finally, if for $X, Y \in \chi(M)$ with $g(X, Y) = g(JX, Y) = 0$, the value $\lambda(X, Y)$ is constant whenever $g(X, X) = g(Y, Y) = 1$, then we say that $M$ has global constant type.

L. Vanhecke introduced to the notion of RK-manifold in [16]. An almost Hermitian manifold $M$ is called an RK-manifold if

\[ R(X, Y, Z, U) = R(JX, JY, JZ, JU) \]

for all $X, Y, Z, U \in \chi(M)$. He proved many theorems. Recall some of them.

**Theorem 2.4.** ([16]) Let $M$ be an RK-manifold. Then $M$ has (pointwise) constant type if and only if there exists $\alpha \in \mathcal{F}(M)$ such that

\[ \lambda(X, Y) = \alpha\{g(X, X)g(Y, Y) − g^2(X, Y) − g^2(X, JY)\}, \]

for all $X, Y \in \chi(M)$. Furthermore, $M$ has global constant type if and only if $\alpha$ is a constant function.

**Theorem 2.5.** ([16]) Let $M$ be an RK-manifold. Suppose that $M$ has constant holomorphic sectional curvature $\mu$ at a point $p \in M$, let $X, Y \in T_p M$ be any orthonormal vectors. Then we have

\[ K(X, Y) = \mu\left\{1 + 3g^2(X, JY)\right\} + \frac{5}{8}\lambda(X, Y) + \frac{1}{8}\lambda(X, JY). \]

where $K(X, Y)$ is sectional curvature determined by $X$ and $Y$.

**Theorem 2.6.** ([16]) Let $M$ be an RK-manifold with pointwise constant anti-holomorphic (resp. holomorphic) sectional curvature $\nu$ (resp. $\mu$). Then $M$ has pointwise constant holomorphic (resp. anti-holomorphic) sectional curvature $\mu$ (resp. $\nu$) if and only if $M$ has pointwise constant type $\alpha$, in which case

\[ 4\nu = \mu + 3\alpha. \]

The dimension of $M$ is supposed to be $\geq 6$.

We shall call an almost Hermitian manifold $M$ as Kaehlerian if $\nabla_X J = 0$ for all $X \in \chi(M)$, nearly Kaehlerian (almost Tachibana or K-space) if
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\[(\nabla_X \mathcal{J})X = 0\] for all \(X \in \chi(M)\), and \textit{para-Kaehlerian} if \(R(X, Y, Z, U) = R(X, Y, JZ, JU)\) for all \(X, Y, Z, U \in \chi(M)\). These manifolds satisfy (2.1), so they are \(RK\)-manifolds. It is easy to see that a para-Kaehlerian manifold has global constant type ([16, 17]).

For a \(2m\)-dimensional Kaehlerian manifold, the Bochner curvature tensor \(B\) is defined by

\[
B(X, Y, Z, U) = R(X, Y, Z, U) - \frac{1}{2(m+2)} \left\{ g(X, U) \text{Ric}(Y, Z) - g(X, Z) \text{Ric}(Y, U) + g(Y, Z) \text{Ric}(X, U) - g(Y, U) \text{Ric}(X, Z) + g(X, JU) \text{Ric}(Y, JZ) - g(X, JZ) \text{Ric}(Y, JU) + g(Y, JZ) \text{Ric}(X, JU) - g(Y, JU) \text{Ric}(X, JZ) - 2g(X, JY) \text{Ric}(Z, JU) - 2g(Z, JU) \text{Ric}(X, JY) \right\} 
+ \frac{S}{4(m+1)(m+2)} \left\{ g(X, U)g(Y, Z) - g(X, Z)g(Y, U) \right\} 
+ g(X, JU)g(Y, JZ) - g(X, JZ)g(Y, JU) - 2g(X, JY)g(Z, JU)
\]

for all \(X, Y, Z, U \in T_pM\) and \(p \in M\), where \(\text{Ric}\) and \(S\) are the \textit{Ricci tensor} and the \textit{scalar curvature} of \(M\), respectively ([11]). The following lemma gives a criterion for vanishing of the Bochner curvature tensor of a Kaehlerian manifold.

**Theorem 2.7.** ([11]) A Kaehlerian manifold \(M\) of dimension \(2m \geq 6\) has a vanishing Bochner curvature tensor, if and only if for each point \(p \in M\) and for all unit vectors \(X, Y, Z \in T_pM\), which span an anti-holomorphic 3-plane

\[R(X, JX, Y, Z) = 2R(X, Y, JX, Z)\]

holds.

Now, we give some definitions related to submanifolds.

Let \(M\) be a \(C^\infty\)-Riemannian manifold with metric tensor \(g\) and \(N\) be a submanifold of \(M\). We denote by \(\nabla\) and \(\hat{\nabla}\) the covariant derivatives on \(M\) and \(N\) respectively. For any vector fields \(X\) and \(Y\) tangent to \(N\), the second fundamental form \(T\) is defined by

\[
T(X, Y) = \nabla_X Y - \hat{\nabla}_X Y
\]

where \(\hat{\nabla}_X Y\) is tangent to \(N\) and \(T(X, Y)\) is normal to \(N\). The normal bundle-valued form \(T\) is a symmetric tensor field of type \((0,2)\). We say that \(N\) is \textit{totally umbilical} submanifold in \(M\) if for all \(X, Y\) tangent to \(N\), we have

\begin{equation}
T(X, Y) = g(X, Y)\eta , \tag{2.2}
\end{equation}

where \(\eta\) is the normal element of \(N\).
where $\eta$ is the mean curvature vector field of $N$ in $M$. The Codazzi equation is given by

\begin{equation}
(R(X,Y)Z)^\perp = (\nabla_X T)(Y,Z) - (\nabla_Y T)(X,Z)
\end{equation}

for all $X, Y, Z$ tangent to $N$. Where $^\perp$ denotes the normal component and the covariant derivative of $T$, denoted by $\nabla_X T$, is defined by

\begin{equation}
(\nabla_X T)(Y,Z) = D_X(T(Y,Z)) - T(\nabla_X Y, Z) - T(Y, \nabla_X Z),
\end{equation}

for all $X, Y, Z$ tangent to $N$, where $D$ denotes the operator of covariant derivative in the normal bundle of $N$ [4, 6, 8, 19].

### 3 Main Results

We now introduce the following axiom.

**Definition 3.1. (Axiom of hemi-slant 3-spheres).** An almost Hermitian manifold $M$ is said to satisfy the axiom of hemi-slant 3-spheres if for each point $p \in M$ and each hemi-slant 3-plane $\sigma$ in $T_pM$, there exists a 3-dimensional totally umbilical submanifold $N$ such that $p \in N$ and $T_pN = \sigma$.

Before studying the axiom of hemi-slant 3-spheres, let us note the following.

**Remark 3.2.** Let $M$ be any $2m$-dimensional almost Hermitian manifold with $m \geq 3$ and let $\{X_1, ..., X_m, JX_1, ..., JX_m\}$ be an orthonormal $J$-basis of $T_pM$. Then we always have a hemi-slant 3-plane with the slant angle $\theta$. For example, $\sigma = D^\theta \oplus \{X_3\}$ is a hemi-slant 3-plane with the slant angle $\theta$, where $D^\theta = \text{span}\{X_1, \cos \theta JX_1 + \sin \theta X_2\}$.

**Lemma 3.3.** Let $M$ be an almost Hermitian manifold with dimension $2m \geq 6$. If $M$ satisfies the axiom of hemi-slant 3-spheres for some $\theta \in (0, \pi/2)$, then we have

\begin{equation}
\lambda(X,Y) = K(X,Y),
\end{equation}

for all orthonormal vectors $X, Y \in T_pM$ with $g(X, JY) = 0$, where $\lambda(X,Y) = R(X,Y,X,Y) - R(X,Y,JX,JY)$ and $K$ denotes anti-holomorphic sectional curvature.


**Proof.** Let $p$ be an arbitrary point of $M$ and let $X, Y$ and $Z$ be any orthonormal vectors in $T_pM$ with $g(X, JY) = g(X, JZ) = g(Y, JZ) = 0$. Consider the hemi-slant 3-plane $\sigma = D^\theta \oplus \{Y\}$ with slant angle $\theta \in (0, \frac{\pi}{2})$, where $D^\theta = \text{span}\{X, \cos \theta JX + \sin \theta Z\}$. By the axiom of hemi-slant 3-spheres, there exists a 3-dimensional totally umbilical submanifold $N$ such that $p \in N$ and $T_pN = \sigma$. Then, with the help of (2.2) and (2.4) from (2.3), we have

\begin{equation}
(R(X, Y)(\cos \theta JX + \sin \theta Z))^\perp = 0.
\end{equation}

Since $JY$ is normal to $N$, we get

\begin{equation}
R(X, Y, \cos \theta JX + \sin \theta Z, JY) = 0.
\end{equation}

Now, consider the hemi-slant 3-plane $\sigma_2 = D^\theta_2 \oplus \{Y\}$ with slant angle $\theta \in (0, \frac{\pi}{2})$, where $D^\theta_2 = \text{span}\{X, \cos \theta JX - \sin \theta Z\}$. By a similar method, we can obtain

\begin{equation}
R(X, Y, \cos \theta JX - \sin \theta Z, JY) = 0.
\end{equation}

From (3.3) and (3.4) we get

\begin{equation}
R(X, Y, JX, JY) = 0.
\end{equation}

From Definition 2.3, and the equation (3.5), we obtain (3.1). \hfill \Box

**Theorem 3.4.** Let $M$ be an almost Hermitian manifold with dimension $2m \geq 6$. If $M$ satisfies the axiom of hemi-slant 3-spheres for some $\theta \in (0, \frac{\pi}{2})$, then $M$ has pointwise constant type if and only if $M$ has pointwise constant anti-holomorphic sectional curvature.

**Proof.** Let $M$ be an almost Hermitian manifold with dimension $2m \geq 6$ satisfying the axiom of hemi-slant 3-spheres for some $\theta \in (0, \frac{\pi}{2})$. If $M$ has pointwise constant type; that is, for all $p \in M$, $M$ has constant type at $p$, then for all $X, Y, Z \in T_pM$, we have

\begin{equation}
\lambda(X, Y) = \lambda(X, Z),
\end{equation}

whenever the planes $\text{span}\{X, Y\}$ and $\text{span}\{X, Z\}$ are anti-holomorphic and $g(Y, Y) = g(Z, Z)$. Here, we can assume that $g(Y, Y) = g(Z, Z) = 1$. Thus, from Lemma 3.3, we get

\begin{equation}
K(X, Y) = K(X, Z),
\end{equation}
for all orthonormal vectors $X, Y, Z \in T_p M$ with $g(X, JY) = g(X, JZ) = 0$. On the other hand, since the dimension of $M$ is greater than 6 we can choose a unit vector $U$ in $(\text{span}\{X, JX\})^\perp \cap (\text{span}\{Z, JZ\})^\perp$. Then, from (3.7), we have

$$(3.8) \quad K(X, U) = K(X, Z) .$$

This implies that the sectional curvature is the same for all anti-holomorphic sections which contain any given vector $X$. Hence we write

$$(3.9) \quad K(X, Y) = K(Y, Z) = K(Z, U) .$$

Therefore, we find

$$(3.10) \quad K(X, Y) = K(Z, U) ,$$

for all $X, Y, Z, U \in T_p M$ whenever the planes $\text{span}\{X, Y\}$ and $\text{span}\{Z, U\}$ are anti-holomorphic. It follows that the sectional curvature is the same for all anti-holomorphic sections at $p \in M$; that is, $M$ has pointwise constant anti-holomorphic sectional curvature.

Conversely, let $M$ be of pointwise constant anti-holomorphic sectional curvature and let $p$ be any point of $M$. Then for all orthonormal vectors $X, Y, Z \in T_p M$ with $g(X, JY) = g(X, JZ) = 0$, $(\text{span}\{X, Y\})$ and $(\text{span}\{Z, U\})$ are anti-holomorphic planes and $g(X, X) = g(Y, Y) = g(Z, Z) = 1$, we have

$$(3.11) \quad K(X, Y) = K(X, Z) .$$

From Lemma 3.3, we get

$$(3.12) \quad \lambda(X, Y) = \lambda(X, Z)$$

for all orthonormal vectors $X, Y, Z \in T_p M$ whenever the planes $\text{span}\{X, Y\}$ and $\text{span}\{X, Z\}$ are anti-holomorphic. It is not difficult to see that (3.12) also holds in the case $g(Y, Y) = g(Z, Z) \neq 1$. It follows that $M$ has constant type at $p$.

With the help of Lemma 3.3, from Theorem 3.4, we have the following result.

**Corollary 3.5.** Let $M$ be a $2m$-dimensional almost Hermitian manifold with $m \geq 3$. If $M$ satisfies the axiom of hemi-slant $3$-spheres for some $\theta \in (0, \frac{\pi}{2})$, then $M$ has pointwise constant type $\alpha$ if and only if $M$ has pointwise constant anti-holomorphic sectional curvature $\alpha$. 
We now state the main result of the present work.

**Theorem 3.6.** Let $M$ be a $2m$-dimensional (connected) $\text{RK}$-manifold with pointwise constant type $\alpha$ and $m \geq 3$. If $M$ satisfies the axiom of hemi-slant 3-spheres for some $\theta \in (0, \frac{\pi}{2})$, then $M$ is a real space form with constant sectional curvature $\alpha$ and $M$ has global constant type.

**Proof.** Let $p$ be any point of $M$ and $M$ has constant type $\alpha$ at $p$. Then it follows from Corollary 3.5 that $M$ has constant anti-holomorphic sectional curvature $\alpha$ at $p$. On the other hand, from Theorem 2.6, we see that $M$ has constant holomorphic sectional curvature $\alpha$ at $p$. With the help of Theorem 2.4, from Theorem 2.5, we obtain

\[(3.13) \quad K(X,Y) = \alpha ,\]

for all orthonormal vectors $X, Y \in T_p(M)$, where $K(X,Y) = R(X,Y,X,Y)$ is sectional curvature. It is not difficult to see that (3.13) is also true for all $X, Y \in T_p(M)$. By the well-known Schur’s theorem ([20]) it follows that $M$ has constant sectional curvature $\alpha$ and $M$ has global constant type. \qed

**Corollary 3.7.** Let $M$ be a $2m$-dimensional (connected) $\text{RK}$-manifold with vanishing constant type and $m \geq 3$. If $M$ satisfies the axiom of hemi-slant 3-spheres for some $\theta \in (0, \frac{\pi}{2})$, then $M$ is flat.

Now, suppose that $M$ is a Kaehlerian manifold. Then, for all $X, Y, Z \in \chi(M)$, as a result of the Kaehler identity $R(X,Y)JZ = JR(X,Y)Z$, we get $R(JX,JY)Z = R(X,Y)Z$ ([20]). In this case, we have

\[(3.14) \quad R(JX,JY,JX,JY) = R(X,Y,JX,JY) .\]

Using (2.1) and (3.14), we can see that $\lambda(X,Y) = 0$. Thus, any Kaehlerian manifold has (global) vanishing constant type. Thus, it follows from Corollary 3.7 that:

**Corollary 3.8.** Let $M$ be a $2m$-dimensional (connected) Kaehlerian or para-Kaehlerian manifold with $m \geq 3$. If $M$ satisfies the axiom of hemi-slant 3-spheres for some $\theta \in (0, \frac{\pi}{2})$, then $M$ is flat.

**Theorem 3.9.** Let $M$ be a $2m$-dimensional (connected) non-Kaehlerian nearly Kaehlerian manifold with constant type $\alpha$ and $m \geq 3$. If $M$ satisfies the axiom of hemi-slant 3-spheres for some $\theta \in (0, \frac{\pi}{2})$, then $M$ has constant sectional curvature $\alpha > 0$ and $m = 3$. 
Proof. In [7], for a nearly Kaehlerian manifold \( M \), A. Gray proved the following.

\[
\lambda(X, Y) = R(X, Y, X, Y) - R(X, Y, JX, JY) = \|\nabla_X J\|^2 ,
\]

where \( X, Y \in \chi(M) \). Now, let \( M \) be a 2m-dimensional non-Kaehlerian nearly Kaehlerian manifold with constant type \( \alpha \) and \( m \geq 3 \). Then, it follows from (3.15) that \( \alpha = \lambda(X, Y) = \|\nabla_X J\|^2 > 0 \) due to \( M \) is non-Kaehlerian. On the other hand, since \( M \) satisfies the axiom of hemi-slant 3-spheres for some \( \theta \in (0, \frac{\pi}{2}) \), by Theorem 3.6, we have \( M \) has constant sectional curvature \( \alpha \). Hence, we see that \( M \) has constant holomorphic sectional curvature \( \alpha \). Thus, the assertion \( m = 3 \) follows from the following theorem.

**Theorem.** ([15]) Except for the 6-dimensional one, there does not exist a non-Kaehlerian nearly Kaehlerian manifold of constant holomorphic sectional curvature.

Now, we give a result related to the Weyl conformal curvature tensor.

**Theorem 3.10.** Let \( M \) be an almost Hermitian manifold with dimension \( 2m \geq 8 \). If \( M \) satisfies the axiom of hemi-slant 3-spheres for some \( \theta \in (0, \frac{\pi}{2}) \), then \( M \) is conformally flat.

**Proof.** Let \( p \) be an arbitrary point of \( M \) and let \( X, Y \) and \( Z \) be any orthonormal vectors of \( T_pM \) with \( g(X, JY) = g(X, JZ) = g(Y, JZ) = 0 \). Consider the hemi-slant 3-plane \( \sigma_1 = D_1^\theta \oplus \{Z\} \) with slant angle \( \theta \in (0, \frac{\pi}{2}) \), where \( D_1^\theta = \text{span}\{X, \cos \theta JX + \sin \theta Y\} \) and the hemi-slant 3-plane \( \sigma_2 = D_2^\theta \oplus \{Z\} \) with slant angle \( \theta \in (0, \frac{\pi}{2}) \), where \( D_2^\theta = \text{span}\{X, \cos \theta JX - \sin \theta Y\} \). As in the proof of Lemma 3.3, by the axiom of hemi-slant 3-spheres and by the equation (2.3), we have

\[
(R(X, \cos \theta JX + \sin \theta Y)Z)^\perp = 0
\]

and

\[
(R(X, \cos \theta JX - \sin \theta Y)Z)^\perp = 0 .
\]

On the other hand, since the dimension of \( M \) is greater than 8 we can choose a unit vector \( U \) in \( (\text{span}\{X, JX\})^\perp \cap (\text{span}\{Y, JY\})^\perp \cap (\text{span}\{Z, JZ\})^\perp \). Thus, we write

\[
R(X, \cos \theta JX + \sin \theta Y, Z, U) = 0
\]
and

\[(3.19) \quad R(X, \cos \theta JX - \sin \theta Y, Z, U) = 0 .\]

From (3.18) and (3.19), we have

\[(3.20) \quad R(X, JX, Z, U) = 0 .\]

On the other hand, (3.18) and (3.20) give

\[(3.21) \quad R(X, Y, Z, U) = 0 ,\]

where \(X, Y, Z, U \in T_p M\) span an anti-holomorphic 4-plane. According to a well-known theorem of Schouten [13], the Weyl conformal curvature tensor \(W\) of \(M\) vanishes. This completes the proof.

Next, we will give a result related to the Bochner curvature tensor.

**Theorem 3.11.** Let \(M\) be a Kaehlerian manifold with dimension \(2m \geq 6\). If \(M\) satisfies the axiom of hemi-slant 3-spheres for some \(\theta \in (0, \frac{\pi}{2})\), then \(M\) has a vanishing Bochner curvature tensor.

**Proof.** Let \(p\) be any point of \(M\) and let \(X, Y\) and \(Z\) be any orthonormal vectors of \(T_p M\) with \(g(X, JY) = g(X, JZ) = g(Y, JZ) = 0\); that is, they span an anti-holomorphic 3-plane. Then the 3-plane \(\sigma_1 = D_1^\theta \oplus \{Y\}\) is a hemi-slant 3-plane with slant angle \(\theta \in (0, \frac{\pi}{2})\), where \(D_1^\theta = \text{span}\{X, \cos \theta JX + \sin \theta JZ\}\). By the axiom of hemi-slant 3-spheres, there exists a 3-dimensional totally umbilical submanifold \(N_1\) such that \(p \in N_1\) and \(T_p N_1 = \sigma_1\). Then, with the help of (2.2) and (2.4) from (2.3), we have

\[(3.22) \quad (R(X, \cos \theta JX + \sin \theta JZ)Y)^\perp = 0\]

and

\[(3.23) \quad (R(X, Y)(\cos \theta JX + \sin \theta JZ))^\perp = 0 .\]

Since \(Z\) is normal to \(N_1\), from (3.22) and (3.23) we get

\[(3.24) \quad R(X, \cos \theta JX + \sin \theta JZ, Y, Z) = 0\]

and

\[(3.25) \quad R(X, Y, \cos \theta JX + \sin \theta JZ, Z) = 0 .\]
Now, consider the hemi-slant 3-plane \( \sigma_2 = D_2^\theta \oplus \{ Y \} \) with slant angle \( \theta \in (0, \frac{\pi}{2}) \), where \( D_2^\theta = \text{span}\{ X, \cos \theta JX - \sin \theta JZ \} \). Again by the axiom of hemi-slant 3-spheres, there exists a 3-dimensional totally umbilical submanifold \( N_2 \) such that \( p \in N_2 \) and \( T_pN_2 = \sigma_2 \). Then, with the help of (2.2) and (2.4) from (2.3), we have

\[
(3.26) \quad (R(X, \cos \theta JX - \sin \theta JZ)Y)_{\perp} = 0
\]

and

\[
(3.27) \quad (R(X, Y)(\cos \theta JX - \sin \theta JZ))_{\perp} = 0.
\]

Since \( Z \) is normal to \( N_2 \), from (3.26) and (3.27) we get

\[
(3.28) \quad R(X, \cos \theta JX - \sin \theta JZ, Y, Z) = 0
\]

and

\[
(3.29) \quad R(X, Y, \cos \theta JX - \sin \theta JZ, Z) = 0.
\]

From (3.24) and (3.28) we obtain

\[
(3.30) \quad R(X, JX, Y, Z) = 0.
\]

On the other hand, from (3.25) and (3.29) we obtain

\[
(3.31) \quad R(X, Y, JX, Z) = 0.
\]

Thus, our assertion follows from (3.30), (3.31) and Lemma 2.7.

\[\Box\]

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References


The axiom of hemi-slant 3-spheres


