Locally maximal regular subsemibands of $\mathcal{SOP}_n$

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Abstract

In this paper we describe locally maximal regular subsemibands of finite singular orientation-preserving transformation semigroups and completely obtain their classification.

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1 Introduction

Let $[n] = \{1, 2, \ldots, n\}$ ordered in the standard way. We denote by $T_n$ the semigroup of all full transformations of $[n]$, and by $\text{Sing}_n$ its subsemigroup of all singular transformations of $[n]$. We say that a transformation $\alpha$ in $\text{Sing}_n$ is order-preserving if, for all $x, y \in [n]$, $x \leq y$ implies $x\alpha \leq y\alpha$.

We denote by $O_n$ the subsemigroup of $\text{Sing}_n$ of all order-preserving singular transformations. Let

$$O^k_n = \{a^{-k}fa^k : f \in O_n\}, \ k \in [n],$$

where $a = (123\cdots n)$ is the fixed generator of the cyclic group $\mathbb{Z}_n$. Catarino and Higgins [1] proved that $O^k_n$ and $O_n$ are isomorphic subsemigroups of $T_n$.

Let $\alpha \in T_n$, we say that $\alpha$ is orientation-preserving if the sequence $(1\alpha, 2\alpha, \ldots, n\alpha)$ is cyclic, that is, there exists no more than one subscript $i$ such that $i\alpha > (i + 1)\alpha$. The notion of an orientation-preserving transformation was introduced by McAlister in [15] and, independently, by Catarino and Higgins in [1]. We denote by $\mathcal{OP}_n$ the subsemigroup of $T_n$ of all orientation-preserving full transformations of $[n]$, and by $\mathcal{SOP}_n$ the subsemigroup of $\text{Sing}_n$ of all orientation-preserving singular transformations of $[n]$.

A semigroup $S$ is called idempotent-generated or semiband if it is generated by its idempotents. The latter term was introduced by F. Pastijn [16].

Let $S$ be a semigroup. The set of all subsemigroups or subsemigroups with particular properties of $S$ is partially ordered with respect to inclusions, and the maximal elements of this set are called maximal subsemigroups or maximal subsemigroups with particular properties of $S$. The history of the problem to classify (or describe) all maximal subsemigroups or maximal subsemigroups with
particular properties of a given semigroup go back at least to [9]. Various special subsemigroups of $T_n$ have been studied by many authors (see for example [1-3, 5-8, 10, 12, 17-33]). In recent years the problem was studied for several classes of transformation semigroups in [2-5, 8-9, 11, 13-14, 17-33]. In particular, Zhao, Bo and Mei [28] classified completely locally maximal subsemibands of $SOP_n$. Recently, Zhao [29] characterized completely maximal subsemibands of $SOP_n$. Further, Zhao [30] completely described maximal regular subsemibands of $SOP_n$. In this paper, we aim to give more insight into the subsemigroup structure of the semigroup $SOP_n$ by characterizing the locally maximal regular subsemibands of $SOP_n$.

**Remark 1** In the paper it will always be clear from context when additions are modular.

For convenience, we introduce the following notation. From Catarino and Higgins [1], Green’s equivalences in $SOP_n$ can be characterized as:

$$\alpha \mathcal{L} \beta \Leftrightarrow im(\alpha) = im(\beta),$$
$$\alpha \mathcal{R} \beta \Leftrightarrow ker(\alpha) = ker(\beta),$$
$$\alpha \mathcal{J} \beta \Leftrightarrow |im(\alpha)| = |im(\beta)|.$$

Thus $SOP_n$ has $n-1$ $\mathcal{J}$-classes: $J_1, J_2, \ldots, J_{n-1}$, where

$$J_r = \{ \alpha \in SOP_n : |im(\alpha)| = r \}.$$ 

Obviously, we have $SOP_n = \bigcup_{r=1}^{n-1} J_r$. Let

$$K(n, r) = \{ \alpha \in SOP_n : |im(\alpha)| \leq r \} = J_0 \cup J_1 \cdots \cup J_r,$$

where $1 \leq r \leq n-1$. The sets $K(n, r)$ are the two-sided ideals of $SOP_n$. We want to focus on the class $J_{n-1}$ at the top of the semigroup $SOP_n$. As in [29], we use the notation

$$L_k = \{ \alpha \in SOP_n : im(\alpha) = [n] \setminus \{k\} \},$$

$$R_{(i,i+1)} = \{ \alpha \in SOP_n : the\ unique\ non-singleton\ class\ of\ ker(\alpha)\ is\ \{i, i+1\} \}$$

for $\mathcal{L}$-classes and $\mathcal{R}$-classes in $J_{n-1}$. Hence $J_{n-1}$ has $n$ $\mathcal{L}$-classes $L_1, L_2, \ldots, L_n$ and $n$ $\mathcal{R}$-classes $R(1, 2), R(2, 3), \ldots, R(n-1, n), R(n, 1)$.

Gomes and Howie [10] used the notation $[i \to i - 1]$ for the decreasing idempotent $e$ defined by $ie = i - 1$, $xe = x$ $(x \neq i)$. They also used the notation $[i \to i + 1]$ for the increasing idempotent $f$ defined by $if = i + 1$, $xf = x$ $(x \neq i)$.

As usual, we denote by $E(S)$ the set of all idempotents of a subset $S$ of $SOP_n$. Employing above notation the set $E(J_{n-1})$ consists of $n$ decreasing idempotents $[i \to i - 1]$ $(i \in [n])$ and $n$ increasing idempotents $[i \to i + 1]$ $(i \in [n])$. Let $E_{n-1}^+ = \{ [i \to i + 1] : i \in [n] \}$ and $E_{n-1}^- = \{ [i \to i - 1] : i \in [n] \}$ be the increasing and decreasing idempotent sets, respectively. Then $E(J_{n-1}) = E_{n-1}^+ \cup E_{n-1}^-$. Note that $[0 \to 1] = [n \to 1]$, $[1 \to 0] = [1 \to n]$, $[n \to n + 1] = [n \to 1]$, $[n + 1 \to n] = [1 \to n]$, etc., by Remark 1.

With above notation, we have the following simple observations:
Lemma 1.1  Let \( n \geq 3 \). Then
\[
E(R_{(k,k+1)}) = \{ [k \to k+1], [k+1 \to k] \}, \ k \in [n],
\]
\[
E(L_k) = \{ [k \to k-1], [k \to k+1] \}, \ k \in [n].
\]

Lemma 1.2  Let \( n \geq 3 \). Then
\[
E(O_k^{\infty} \cap J_{n-1}) = E(J_{n-1}) \setminus \{ [k \to k+1], [k+1 \to k] \}, \ k \in [n].
\]

2  Locally maximal regular subsemibands of \( SOP_n \)

Let \( I \) be a subset of \( E(J_{n-1}) \). A subsemiband \( \langle I \rangle \) of \( SOP_n \) is called **locally maximal regular subsemiband** if \( \langle I \rangle \) is a regular, and any regular subsemiband \( \langle J \rangle \) \((J \subseteq E(J_{n-1})) \ of \ SOP_n \) properly containing \( \langle I \rangle \) must be \( SOP_n \). In this section, we obtain a classification of locally maximal regular subsemibands of \( SOP_n \).

Our main result is

**Theorem 2.1** Let \( n \geq 3 \). Then each one of the following types \((A), (B)\) is a locally maximal regular subsemiband of \( SOP_n \):

\((A)\) \( S_i = O_i, \ i \in [n] \).
\((B)\) \( T_i = \{ \alpha \in SOP_n : i\alpha = i \}, \ i \in [n] \).

Conversely, every locally maximal regular subsemiband of \( SOP_n \) is of one of types \((A), (B)\).

\( SOP_n \) has \( 2n \) locally maximal regular subsemibands.

To prove Theorem 2.1 we need the following series of lemmas. First, we need the following notation.

As in [1], let \( k \in \{ 0, 1, 2, \ldots, n-1 \} \), define a total order \( \leq_k \) on \([n]\) by
\[
k + 1 \leq_k k + 2 \leq_k \cdots \leq_k n \leq_k 1 \leq_k \cdots \leq_k k.
\]
We write \( i \leq_k j \) if \( i < k \) \( j \) and \( i \neq j \). Note that \( i \leq_k j \) if and only if \( i - k \leq j - k \). We say that \( A = (a_1, a_2, \ldots, a_t) \) is cyclic with respect to \( \leq_k \) if there exist no more than one subscript \( i \) such that \( a_{i+1} <_k a_i \). Clearly \( A = (a_1, a_2, \ldots, a_t) \) is cyclic with respect to \( \leq_k \) if and only if there exists \( j \in \{ 0, \ldots, t-1 \} \) such that
\[
a_{j+1} \leq_k \cdots \leq_k a_t \leq_k a_1 \leq_k \cdots \leq_k a_j.
\]

The following lemma was proved by Catarino and Higgins [1, Lemma 1.4].

**Lemma 2.2** Let \( A = (a_1, a_2, \ldots, a_t) \) be any sequence of elements from \([n]\). Then the following are equivalent:
\((a)\) \( A \) is cyclic with respect to \( \leq_0 = \leq \).
\((b)\) \( A \) is cyclic with respect to \( \leq_k \) for some \( k \).
\((c)\) \( A \) is cyclic with respect to \( \leq_k \) for all \( k \).
Lemma 2.4
Let
\[ M_{i,j} = \langle E(J_{n-1}) \setminus \{[i \rightarrow i + 1], [j \rightarrow j - 1]\} \rangle, i, j \in [n], \]  
\[ S_j^k = \{ \alpha \in \mathcal{O}_n^k : (\forall x \in [n]) \ j \leq_k x \implies j \leq_k x \alpha \}, j \neq k + 1 \ (mod(n)), \]
\[ T_j^k = \{ \alpha \in \mathcal{O}_n^k : (\forall x \in [n]) x \leq_k j \implies x \alpha \leq_k j \}, j \neq k \ (mod(n)). \]

Proof. See [29, Lemma 3.5]. □

The following lemma is the main result (Theorem 3.5) of [28].

Lemma 2.5
Let \( M_{i,j}, S_j^i \) and \( T_j^{i-1} \) be defined as (2.1), (2.2) and (2.3), respectively. Then
\[ M_{i,j} \cap J_{n-1} = (S_j^i \cup T_j^{i-1}) \cap J_{n-1}, \ j \neq i + 1 \ (mod(n)). \]

Proof. See [28, Theorem 3.5]. □

We can use Lemmas 1.2, 2.2 and 2.4 to obtain the following.

Lemma 2.3
Let \( M_{i,j}, S_j^i \) and \( T_j^{i-1} \) be defined as (2.1), (2.2) and (2.3), respectively. Then
\[ M_{i,j} \cap J_{n-1} = (S_j^i \cup T_j^{i-1}) \cap J_{n-1}, \ j \neq i + 1 \ (mod(n)). \]

Proof. See [29, Lemma 3.5]. □

Let \( M_{i,i} \) be defined as (2.1). Then
\[ M_{i,i} = \{ \alpha \in \mathcal{SOP}_n : i \alpha = i \}. \]

Proof. Let \( D_i = \{ \alpha \in \mathcal{SOP}_n : i \alpha = i \} \) and \( F = E(J_{n-1}) \setminus \{[i \rightarrow i + 1], [i \rightarrow i - 1]\} \). Then
\[ M_{i,i} = \langle F \rangle. \] It is easy to verify that \( D_i \) is a subsemigroup of \( \mathcal{SOP}_n \). Note that \( F \subseteq D_i \). Thus
\[ M_{i,i} = \langle F \rangle \subseteq D_i. \]

For the reverse containment, let \( T_i^{i-1} \) be defined as (2.3). By Lemmas 1.2 and 2.4, we have
\[ T_i^{i-1} = \langle E(J_{n-1}) \setminus \{[i - 1 \rightarrow i], [i \rightarrow i - 1], [i \rightarrow i + 1]\} \rangle. \]

It follows easily that
\[ M_{i,i} = \langle F \rangle = \langle T_i^{i-1} \cup \{[i - 1 \rightarrow i]\} \rangle. \]

By the definition of \( T_i^{i-1} \), we easily deduce that
\[ T_i^{i-1} = \{ \alpha \in \mathcal{O}_n^{i-1} : i \alpha = i \}. \]

We now prove that \( D_i \subseteq M_{i,i} \). Let \( \alpha \in D_i \). We partition into two cases.

Case 1. \(|im(\alpha)| = 1\). Clearly, \( \alpha = \binom{n}{i} \) (since \( \alpha \in D_i \), we have \( i \alpha = i \)). Then, by (2.4) and (2.5),
\[ \alpha = \binom{n}{i} \in \{ \alpha \in \mathcal{O}_n^{i-1} : i \alpha = i \} = T_i^{i-1} \subseteq M_{i,i}. \]
Case 2. $|\text{im}(\alpha)| = r \geq 2$. From Theorem 3.3 in [1], we know that $\alpha$ can be expressed as

$$
\alpha = \begin{pmatrix}
A_1 & A_2 & \cdots & A_r \\
b_1 & b_2 & \cdots & b_r
\end{pmatrix},
$$

where $A_s = \{a_s, a_s + 1, \ldots, a_s + 1 \} = s = 1, 2, \ldots, r$, the subset $\{a_1, a_2, \ldots, a_r\}$ is an initial points set of kernel classes $A_1, A_2, \ldots, A_r$, $A_1 < a_2 < \cdots < a_r$ and $(b_1, b_2, \ldots, b_r)$ is cyclic. Since $\alpha \in D_i$, we have $\alpha = i$. Then there exist $k \in \{1, 2, \ldots, r\}$ such that $b_k = i$ and $i \in A_k$. Note that $A_k = \{a_k, a_k + 1, \ldots, a_k + 1 \}$. We may now partition into two cases according to $a_k = i$ or $a_k \neq i$.

Case 2.1. $a_k = i$. Note that $b_k = i$ and $(b_1, b_2, \ldots, b_r)$ is a cyclic. From Lemma 2.2, we easily deduce that

$$
i = b_k \leq i - 1 \leq i - 1 b_r \leq i - 1 \leq \frac{a - b_k - 1}{b_k - 1}.
$$

(2.6)

Note that $A_k = \{i, i + 1, \ldots, a_k + 1 \}$ and $A_k = \{a_k + 1, a_k + 1, \ldots, i - 1 \}$ (since $a_k = i - 1$).
Then, by (2.5), (2.6) and the definition of $O_{i-1}^n$, $\alpha = \begin{pmatrix} A_k \ A_k+1 \ \cdots \ A_r \ A_1 \ \cdots \ A_k-1 \\
b_k \ b_{k+1} \ \cdots \ b_r \ b_1 \ \cdots \ b_{k-1}\end{pmatrix} \in T_i^{i-1}$.

Thus, by (2.4), $\alpha \in T^{i-1}_i \subseteq M_{i,j}$.

Case 2.2. $a_k \neq i$. Since $i \in A_k$ and $A_k = \{a_k, a_k + 1, \ldots, a_k + 1 \}$, we have $i - 1 \in A_k$. Let $A_k = \{a_k, a_k + 1, \ldots, i - 1 \}$ and $A_k^* = \{i, i + 1, \ldots, a_k + 1 \}$. Then $A_k = A_k^* \cup A_k^*$. Note that $b_k = i$ and $A_s = \{a_s, a_s + 1, \ldots, a_s + 1 \}$, $s = 1, 2, \ldots, r$. Let

$$
\beta = \begin{pmatrix}
A_k^{*} & A_k & \cdots & A_r & A_1 & \cdots & A_k \\
i & a_k+1 & \cdots & a_r & a_1 & \cdots & a_k-1 & i - 1
\end{pmatrix},
$$

$$
\gamma = \begin{pmatrix}
A_k^{*} & A_k & \cdots & A_r & A_1 & \cdots & A_k & A_k^* \\
b_k & b_{k+1} & \cdots & b_r & b_1 & \cdots & b_{k-1} & b_{k-1}
\end{pmatrix}.
$$

Then clearly $\alpha = \beta[i - 1 \to i] \gamma$. Note that $[i - 1 \to i] \in M_{i,j}$ (by (2.4)). To prove that $\alpha \in M_{i,j}$, it suffices to prove that $\beta, \gamma \in M_{i,j}$. By (2.5), (2.6) and the definition of $O_{i-1}^n$, we have $\gamma \in T_i^{i-1}$.

Thus, by (2.4), $\gamma \in T_i^{i-1} \subseteq M_{i,j}$. Since $i - 1, i \in A_k$, we have $a_k \leq i - 1 < i \leq a_k + 1 < a_k + 1$.

Note that $a_1 < a_2 < \cdots < a_r$. It easily follows from Lemma 2.2 that

$$
i \leq a_k \leq a_k + 1 \leq \cdots \leq a_k + 1 \leq a_{k-1} \leq \cdots \leq a_k < a_i \leq a_{i-1} \leq i - 1.
$$

Then, by (2.5) and the definition of $O_{i-1}^n$, $\beta \in T_i^{i-1}$. Thus, by (2.4), $\beta \in T_i^{i-1} \subseteq M_{i,j}$. \boxdot

Recall that Catarino and Higgins [1] had already proved that $\text{SOP}_n$ is a subsemiband of $\mathcal{T}_n$.
We have proved in [28, Theorem 2.1] the following result.

**Lemma 2.6** For $n \geq 3$, let $A \subseteq E(\text{SOP}_n)$. Then

$$
\langle A \rangle = \text{SOP}_n \text{ if and only if } E_n^+ \subseteq A \text{ or } E_n^- \subseteq A.
$$

The following lemma was proved by the author in [28, Lemma 2.3].
Lemma 2.7 Let \( i \in [n] \), \( e = [i+1 \to i] \), \( f \in E(J_{n-1}) \) and \( ef \notin \{e, f\} \). Then

\[
ef \in J_{n-1} \text{ if and only if } f = [i+2 \to i+1].
\]

Let \( I \) be a subset of \( E(J_{n-1}) \). It is obvious that \( I \subseteq E((I) \cap J_{n-1}) \). In general, \( E((I) \cap J_{n-1}) \subseteq I \) is false. For example, let \( I = E_{n-1}^+ \), then, by Lemma 2.6, \( (I) = SOP_n \) and so \( E((I) \cap J_{n-1}) = E(J_{n-1}) = E_{n-1}^+ \cup E_{n-1}^- \). Then clearly \( E((I) \cap J_{n-1}) \nsubseteq I \). However, we have the following.

Lemma 2.8 Let \( I \) be a subset of \( E(J_{n-1}) \). If \( \langle I \rangle \subset SOP_n \). Then

\[
E((I) \cap J_{n-1}) = I.
\]

Proof. Clearly, \( I \subseteq E((I) \cap J_{n-1}) \). Now, we need to prove that \( E((I) \cap J_{n-1}) \subseteq I \). Note that \( I \subseteq E(J_{n-1}) \) and \( E(J_{n-1}) = E_{n-1}^+ \cup E_{n-1}^- \). Let \( I_1 = \langle I \rangle \cap E_{n-1}^+ \) and \( I_2 = \langle I \rangle \cap E_{n-1}^- \), then \( E((I) \cap J_{n-1}) = I_1 \cup I_2 \). Let \( G_i = I_i \setminus I, i = 1, 2 \). To prove that \( E((I) \cap J_{n-1}) \subseteq I \), we only need to prove that \( G_i \subseteq \emptyset \), \( i = 1, 2 \). Note that \( G_1 \subseteq I_1 \subseteq E_{n-1}^+ \) and \( G_2 \subseteq I_2 \subseteq E_{n-1}^- \). Now, we assume that \( G_1 \neq \emptyset \) and so there is some idempotent element \( e = [k \to k+1] \in I_i \setminus I \). Note that \( I \subseteq I \) and \( I \subseteq E(J_{n-1}) \). Obviously, we may suppose that

\[
e = e_1 e_2 \cdots e_r, \text{ where } e_i \in I, i = 1, 2, \ldots, r, \ r > 1,
\]

and

\[
e_i e_{i+1} \cdots e_j \neq e_i, e_j, \ 1 \leq i < j \leq r.
\]

Since \( e \in J_{n-1} \) and \( e \notin I \) it follows that \( eRe_1, eLe_r \). By Lemma 1.1, we have \( e_i = [k+1 \to k] \) and \( e_r = [k \to k-1] \). By repeated use of Lemma 2.7, we have \( e_i = [k+i \to k+i-1] (i = 1, 2, \ldots, r) \). Then \( e_r = [k+r \to k+r-1] = [k \to k-1] \) and so \( r \equiv 0 \ (mod \ n) \). It follows immediately that \( E_{n-1}^- = \{[k+i \to k+i-1] : i \in [n] \} = \{e_1, e_2, \ldots, e_r \} \subseteq I \). Thus, by Lemma 2.6, \( (I) = SOP_n \), contradicting the assumption that \( \langle I \rangle \subset SOP_n \). Similarly, we can prove that \( G_2 = \emptyset \). \( \Box \)

Let \( I \) and \( J \) be nonempty subsets of \( E(J_{n-1}) \). It is obvious that \( I \subseteq J \Rightarrow \langle I \rangle \subseteq \langle J \rangle \Rightarrow K(n, n-2) \cup \langle I \rangle \subseteq K(n, n-2) \cup \langle J \rangle \). In general, \( I \subseteq J \Rightarrow \langle I \rangle \subseteq \langle J \rangle \subseteq K(n, n-2) \cup \langle J \rangle \) are false. For example, let \( I = E_{n-1}^+ \) and \( J = E_{n-1}^- \), then, by Lemma 2.6, \( (I) = SOP_n \) and so \( K(n, n-2) \cup (I) = K(n, n-2) \cup (J) = SOP_n \). Clearly \( I \cap J = \emptyset \). However, we can use Lemma 2.8 to obtain the following.

Lemma 2.9 Let \( I \) and \( J \) be nonempty subsets of \( E(J_{n-1}) \). If \( \langle J \rangle \subset SOP_n \). Then

(i) \( I \subseteq J \Leftrightarrow \langle I \rangle \subseteq \langle J \rangle \Leftrightarrow K(n, n-2) \cup (I) \subseteq K(n, n-2) \cup (J) \).

(ii) \( I \subseteq J \Leftrightarrow \langle I \rangle \subseteq \langle J \rangle \Leftrightarrow K(n, n-2) \cup (I) \subseteq K(n, n-2) \cup (J) \).

Proof. (i) Clearly,

\[
I \subseteq J \Rightarrow \langle I \rangle \subseteq \langle J \rangle \Rightarrow K(n, n-2) \cup (I) \subseteq K(n, n-2) \cup (J).
\]

To prove that

\[
I \subseteq J \Leftrightarrow \langle I \rangle \subseteq \langle J \rangle \Leftrightarrow K(n, n-2) \cup (I) \subseteq K(n, n-2) \cup (J).
\]
Lemma 2.10

Let \( K(n, n - 2) \cup \langle I \rangle \subseteq K(n, n - 2) \cup \langle J \rangle \Rightarrow I \subseteq J. \)

Suppose that \( K(n, n - 2) \cup \langle I \rangle \subseteq K(n, n - 2) \cup \langle J \rangle. \) Then \( \langle I \rangle \cap J_{n-1} = (K(n, n - 2) \cup \langle I \rangle) \cap J_{n-1} \subseteq (K(n, n - 2) \cup \langle J \rangle) \cap J_{n-1} = (J) \cap J_{n-1}. \) Thus, by Lemma 2.8,

\[
I \subseteq E(\langle I \rangle \cap J_{n-1}) \subseteq E((J) \cap J_{n-1}) = J.
\]

(ii) By (i), we easily deduce that

\[
I = J \iff \langle I \rangle = \langle J \rangle \iff K(n, n - 2) \cup \langle I \rangle = K(n, n - 2) \cup \langle J \rangle.
\]

It follows immediately that

\[
I \subset J \iff \langle I \rangle \subset \langle J \rangle \iff K(n, n - 2) \cup \langle I \rangle \subset K(n, n - 2) \cup \langle J \rangle.
\]

\( \square \)

We can use Lemmas 2.5 and 2.9 to obtain the following.

Lemma 2.10 Let \( i \in \{n\}, \) \( T_i = \{ \alpha \in \text{SOP}_n : i\alpha = i \}. \) Then \( T_i \) is a locally maximal regular subsemiband of \( \text{SOP}_n. \)

Proof. Let \( M_{i,i} \) be as defined in (2.1). By Lemma 2.5, we have

\[
T_i = \{ \alpha \in \text{SOP}_n : i\alpha = i \} = M_{i,i} = \langle E(J_{n-1}) \setminus \{[i \to i + 1], [i \to i - 1]\} \rangle. \tag{2.7}
\]

Then \( T_i \) is a subsemiband of \( \text{SOP}_n. \) Let \( \alpha \in T_i. \) If \( |\text{im}(\alpha)| = 1, \) then clearly \( \alpha \) is an idempotent, and so \( \alpha \) is regular. If \( |\text{im}(\alpha)| \geq 2, \) from Theorem 3.3 in [1], we know that \( \alpha \) can be expressed as

\[
\alpha = \begin{pmatrix}
A_1 & A_2 & \cdots & A_r \\
b_1 & b_2 & \cdots & b_r
\end{pmatrix},
\]

where \( A_s = \{a_s, a_s + 1, \ldots, a_s + 1\}, s = 1, 2, \ldots, r, \) the subset \( \{a_1, a_2, \ldots, a_r\} \) is an initial points set of kernel classes \( A_1, A_2, \ldots, A_r, a_1 < a_2 < \cdots < a_r \) and \( (b_1, b_2, \ldots, b_r) \) is cyclic. Since \( \alpha \in T_i, \) we have \( i\alpha = i. \) Then there exist \( k \in \{1, 2, \ldots, r\} \) such that \( b_k = i \) and \( i \in A_k. \) Let \( C_j = \{b_j, b_j + 1, \ldots, b_j + 1\}, j = 1, 2, \ldots, r, \) and let

\[
\beta = \begin{pmatrix}
C_k & C_{k+1} & \cdots & C_r \\
i & a_{k+1} & \cdots & a_r \\
& a_1 & \cdots & a_{k-1}
\end{pmatrix},
\]

then \( \alpha = \alpha\beta\alpha \) (since \( i \in A_k \)) and \( i\beta = i \) (since \( i = b_k \in C_k \)). Since \( i \in A_k, \) we have \( a_k \leq i \leq a_{k+1} - 1 < a_{k+1}. \) Note that \( a_1 < a_2 < \cdots < a_r. \) It easily follows from Lemma 2.2 that

\[
i \leq i - 1 \leq a_{k+1} \leq i - 1 \leq \cdots \leq i - 1 \leq a_r \leq \cdots \leq i - 1 \leq a_{k-1} \leq i - 1 \leq i - 1.
\]

Then, by (2.5) and the definition of \( \text{SOP}_n, \) \( \beta \in T_i^{-1}. \) Thus, by (2.4) and (2.7), \( \beta \in T_i^{-1} \subseteq M_{i,i} = T_i \) and so \( \alpha \) is regular (note that \( \alpha = \alpha\beta\alpha \)). Hence \( T_i \) is a locally regular subsemiband of \( \text{SOP}_n. \)
Let \( \langle J \rangle \) \((J \subseteq E(J_{n-1}))\) be a locally regular subsemiband of \(SOP_n\) properly containing \(T^8\). Then, by Lemma 2.9 \((ii)\) and \((2.7)\),

\[
E(J_{n-1}) \setminus \{[i \rightarrow i + 1], [i \rightarrow i - 1]\} \subseteq J,
\]

and so

\[
E(J_{n-1}) \setminus \{[i \rightarrow i + 1]\} \subseteq J \text{ or } E(J_{n-1}) \setminus \{[i \rightarrow i - 1]\} \subseteq J.
\]

Note \(E(J_{n-1}) = E_{n-1}^+ \cup E_{n-1}^-\). It follows that

\[
E_{n-1}^- \subseteq J \text{ or } E_{n-1}^+ \subseteq J.
\]

Thus, by Lemma 2.6, \(\langle J \rangle = SOP_n\). \(\Box\)

A proper subsemigroup \(S\) of \(SOP_n\) is called \textit{maximal regular subsemiband} of \(SOP_n\) if \(S\) is a regular subsemiband, and any regular subsemiband of \(SOP_n\) properly containing \(S\) must be \(SOP_n\). The following lemma is the main result of [30].

\textbf{Lemma 2.11} Let \(n \geq 3\). Then each maximal regular subsemiband of \(SOP_n\) must be of one of the following forms:

\((C)\) \(K(n, n - 2) \cup O_i^n, i \in [n]\).

\((D)\) \(K(n, n - 2) \cup \{\alpha \in O_i^{n-1} \cup O_i^n : i \alpha = i\}, i \in [n]\).

\textbf{Proof.} See [30, Theorem 4]. \(\Box\)

The following lemma gives a necessary condition for a locally regular subsemiband of \(SOP_n\) to be maximal.

\textbf{Lemma 2.12} Let \(I\) be a nonempty subset of \(J_{n-1}\). If \(\langle I \rangle\) is a locally maximal regular subsemiband of \(SOP_n\) then \(T = K(n, n - 2) \cup \langle I \rangle\) is a maximal regular subsemiband of \(SOP_n\).

\textbf{Proof.} From Lemma 13 of [30] we know that if \(S\) is a regular semigroup and \(I\) is an ideal of \(S\), then \(I\) is also a regular semigroup. Note that \(SOP_n\) is a regular semigroup (see [1, Theorem 3.1]) and \(K(n, n - 2)\) is an ideal of \(SOP_n\). Then \(K(n, n - 2)\) is regular and so \(T = K(n, n - 2) \cup \langle I \rangle\) is regular (since \(\langle I \rangle\) is regular). From Theorem 2.1 of [29] we know that \(K(n, n - 2) = \langle E(J_{n-2}) \rangle\). Then

\[
T = K(n, n - 2) \cup \langle I \rangle = \langle E(J_{n-2}) \rangle \cup I.
\]

Thus \(T = K(n, n - 2) \cup \langle I \rangle\) is a regular subsemiband of \(SOP_n\).

Let \(S\) be a regular subsemiband of \(SOP_n\) properly containing \(T\). Clearly \(S = \langle E(S) \rangle\) and \(K(n, n - 2) \subseteq T \subset S\). We easily deduce that \(S = K(n, n - 2) \cup S = K(n, n - 2) \cup \langle E(S \cap J_{n-1}) \rangle\) and so

\[
K(n, n - 2) \cup \langle I \rangle = T \subset S = K(n, n - 2) \cup \langle E(S \cap J_{n-1}) \rangle.
\]

Note that \(E(S \cap J_{n-1}) \subseteq E(J_{n-1})\). Then, by Lemma 2.9 \((ii)\), \(\langle I \rangle \subseteq \langle E(S \cap J_{n-1}) \rangle\) and so, by the locally maximality of \(\langle I \rangle\), \(\langle E(S \cap J_{n-1}) \rangle = SOP_n\). Thus \(S = SOP_n\) and so \(T = K(n, n - 2) \cup \langle I \rangle\) is a maximal regular subsemiband of \(SOP_n\). \(\Box\)
Lemma 2.13 Let $n \geq 3$. Then
\[ O^k_n = (E(O^k_n \cap J_{n-1})). \]

Proof. See [28, Lemma 2.2]. □

Now, we can prove Theorem 2.1.

Proof Theorem 2.1 From Lemma 2.10 we know that $T_i$ is a locally maximal regular subsemiband of $SOP_n$. By Lemmas 1.2 and 2.13, we have
\[ O^i_n = (E(J_{n-1}) \setminus \{[i \to i + 1], [i + 1 \to i]\}). \]

It is well known that $O_n$ is regular. From Lemma 4.1 in [1], we know that the mapping $\varphi_i : f \to a^{-i}fa^i$ is an isomorphism between $O_n$ and $O^i_n$, where $a = (123 \cdots n)$ is the fixed generator of the cyclic group $Z_n$. Then $O^i_n$ is regular and so, by (2.8), $S_i = O^i_n$ is a locally regular subsemiband of $SOP_n$. Let $\langle J \rangle (J \subseteq E(J_{n-1}))$ be a locally regular subsemiband of $SOP_n$ properly containing $S_i = O^i_n$. Then, by Lemma 2.9 (ii) and (2.8),
\[ E(J_{n-1}) \setminus \{[i \to i + 1], [i + 1 \to i]\} \subseteq J, \]
and so
\[ E(J_{n-1}) \setminus \{[i \to i + 1]\} \subseteq J \text{ or } E(J_{n-1}) \setminus \{[i + 1 \to i]\} \subseteq J. \]

Note that $E(J_{n-1}) = E^+_{n-1} \cup E^-_{n-1}$. It follows that
\[ E^-_{n-1} \subseteq J \text{ or } E^+_{n-1} \subseteq J. \]
Thus, by Lemma 2.6, $\langle J \rangle = SOP_n$. Hence $S_i$ is a locally maximal regular subsemiband of $SOP_n$.

Conversely, we shall prove that each locally maximal regular subsemiband of $SOP_n$ must be of the form $S_i$ or $T_i$. Let $C_i = K(n, n - 2) \cup O^i_n$ and $D_i = K(n, n - 2) \cup \{\alpha \in O^i_{n-1} \cup O^i_n : \alpha \alpha = i\}$. By (2.8), we have
\[ C_i = K(n, n - 2) \cup O^i_n = K(n, n - 2) \cup (E(J_{n-1}) \setminus \{[i \to i + 1], [i + 1 \to i]\}). \]

Let $M_i$, $S_i^t$ and $T_i^{t-1}$ be defined as (2.1), (2.2) and (2.3), respectively. By the definition of $S_i^t$, $T_i^{t-1}$, we easily deduce that $S_i^t = \{\alpha \in O^i_n : \alpha \alpha = i\}$ and $T_i^{t-1} = \{\alpha \in O^i_{n-1} : \alpha \alpha = i\}$. Then, by Lemma 2.3,
\[ D_i = K(n, n - 2) \cup \{\alpha \in O^i_{n-1} \cup O^i_n : \alpha \alpha = i\} = K(n, n - 2) \cup S_i^t \cup T_i^{t-1} \]
\[ = K(n, n - 2) \cup ((S_i^t \cup T_i^{t-1}) \cap J_{n-1}) = K(n, n - 2) \cup (M_i \cap J_{n-1}) \]
\[ = K(n, n - 2) \cup M_{i,i} = K(n, n - 2) \cup (E(J_{n-1}) \setminus \{[i \to i + 1], [i \to i - 1]\}). \]

Suppose that $\langle I \rangle (I \subseteq E(J_{n-1}))$ is a locally maximal regular subsemiband of $SOP_n$. Then, by Lemma 2.12, $T = K(n, n - 2) \cup I$ is a maximal regular subsemiband of $SOP_n$. Thus, by Lemma 2.11, (2.9) and (2.10), there exist $s \in [n]$ such that $T = C_s = K(n, n - 2) \cup (E(J_{n-1}) \setminus \{[s \to s + 1], [s +
1 \to s\}) or there exist \(t \in [n]\) such that \(T = D_t = K(n,n-2) \cup \langle E(J_{n-1}) \setminus \{[t \to t+1], [t \to t-1]\}\rangle\).

It follows easily from Lemma 2.9 (i) that

\[
\langle I \rangle = \langle E(J_{n-1}) \setminus \{[s \to s+1], [s+1 \to s]\} \rangle \text{ or } \langle I \rangle = \langle E(J_{n-1}) \setminus \{[t \to t+1], [t \to t-1]\} \rangle.
\]

Thus, by (2.8) and Lemma 2.5,

\[
\langle I \rangle = \langle E(J_{n-1}) \setminus \{[s \to s+1], [s+1 \to s]\} \rangle = O_n^s = S_s
\]
or

\[
\langle I \rangle = \langle E(J_{n-1}) \setminus \{[t \to t+1], [t \to t-1]\} \rangle = M_{t,t} = \{\alpha \in SOP_n : t\alpha = t\} = T_t.
\]

It now is obvious that \(SOP_n\) has \(n\) locally maximal regular subsemibands of type (A), and \(n\) locally maximal regular subsemibands of type (B). Hence \(SOP_n\) has \(2n\) locally maximal regular subsemibands. This completes the proof of Theorem 2.1.

**Remark 2** By Lemma 2.5 and (2.10), we have

\[
K(n,n-2) \cup \{\alpha \in O_n^{i-1} \cup O_n^i : i\alpha = i\} = K(n,n-2) \cup M_{t,i} = K(n,n-2) \cup \{\alpha \in SOP_n : i\alpha = i\}.
\]

From this fact and Lemma 2.11 (the main result of [30]), we immediately obtain the following result, which is a clearer than the main result of [30] (see [30, Theorem 4]).

**Theorem 2.14** Let \(n \geq 3\). Then each maximal regular subsemiband of \(SOP_n\) must be of one of the following forms:

(C) \(K(n,n-2) \cup O_n^i, i \in [n]\).

(D) \(K(n,n-2) \cup \{\alpha \in SOP_n : i\alpha = i\}, i \in [n]\).

### 3 Some related problems

In [21], You described the maximal regular subsemigroup of the ideals of \(T_n\). In turn, the maximal subsemigroup of the ideals of \(T_n\) was given by Yang and Yang [20]. In [22], You and Yang classified the maximal subsemibands of \(Sing_n\). Yang and Yang [27] obtained the classification of maximal regular subsemibands of \(Sing_n\). For the semigroup \(T_n\), it is then natural to ask for the problem concerning the description of (locally) maximal subsemibands or (locally) maximal regular subsemibands of the ideals of \(T_n\) which are open questions.

On the other hand, as the notions of order-preserving transformation and orientation-preserving transformation have been widely considered for several classes of transformation semigroups, it is also natural to consider the semigroups \(O_n\) and \(OP_n\). We also may ask for the problem concerning the description of (locally) maximal subsemigroups or (locally) maximal regular subsemigroups of the ideals of the two semigroup. Dimitrova and Koppitz [2] determined all the maximal subsemigroups of the ideals of \(O_n\). The same authors [3] classified completely maximal regular subsemigroups of the ideals of \(O_n\). Recently, Zhao [31] classified completely maximal regular subsemibands of the ideals of \(O_n\). Further, Dimitrova, Fernandes and Koppitz [5] described all the maximal subsemibands of the ideals of \(OP_n\). All the other cases remain as open problems.
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References


