APPROXIMATELY MIDCONVEX SET-VALUED FUNCTIONS

ALIREZA KAMEL MIRMOSTAFAEE AND MOSTAFA MAHDAVI

Abstract. We will show that if $F$ is a set-valued mapping which satisfies

$$F(x) + F(y) \subseteq 2F\left(\frac{x + y}{2}\right) + K$$

for some convex compact set $K$, then under some restrictions, there are maximal superadditive and midconvex mappings which are $K$-subclose to $F$.

1. Introduction

The notion of stability of functional equations has its origins with S. M. Ulam [25], who posed the fundamental problem in 1940 and with D. H. Hyers [6], who gave the first significant partial solution in 1941. A generalized version of Hyers theorem for approximately linear mappings was given by Th. M. Rassias [19]. Since then, the stability problems of various functional equations have been extensively investigated by a number of authors (e. g. [1, 7, 8, 9, 12, 10, 11, 18, 22, 26]).

Functional inclusion is a tool for defining many notions of set-valued analysis, e. g., linear, affine, convex, midconvex, concave, superadditive and subadditive maps.

In set-valued analysis, a functional inclusion is called stable if any function which satisfies this inclusion approximately is near to a true solution of the functional inclusion. The Hyers-Ulam stability is discussed for set-valued functional equations and inclusions by some mathematicians [3, 15, 16, 17, 24].

Let $X$ and $Y$ be semigroups and $F : X \rightarrow 2^Y$. If $F$ satisfies

\begin{equation}
F(x) + F(y) \subseteq F(x + y) \quad (x \in X),
\end{equation}

then $F$ is called superadditive. A function $F : X \rightarrow 2^Y$ is called midconvex if

\begin{equation}
F(x) + F(y) \subseteq 2F\left(\frac{x + y}{2}\right) \quad (x, y \in X).
\end{equation}

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Note that these notions are different. For example, if $F, G : [0, \infty) \to 2^\mathbb{R}$ are defined by $F(x) = [0, \sqrt{x}]$ and $G(x) = [0, x^2]$ for each $x \in [0, \infty)$, then $F$ is midconvex but it is not superadditive, while the converse holds for $G$.

Some authors studied different properties of midconvex and additive set-valued functions (e.g. [2, 5, 14, 23]). In this paper, we will show that, under certain circumstances, every approximately midconvex function $F$ from an abelian semigroup to compact convex subsets of a topological vector space can be approximated by a set-valued additive mapping. We also prove that there exists a maximal midconvex set-valued mapping which approximates $F$.

2. Results

Throughout the paper, unless otherwise stated, we will assume that $X$ is an abelian semigroup divisible by two and $Y$ is a topological vector space. If $A, B \subset Y$ and $\lambda \in \mathbb{R}$, we define

$$A + B = \{a + b : a \in A, b \in B\}, \quad \lambda A = \{\lambda a : a \in A\}.$$  

One can easily see that for each $A, B \subset Y$ and $\lambda, \mu \geq 0$,

$$\lambda (A + B) = \lambda A + \lambda B, \quad (\lambda + \mu) A \subseteq \lambda A + \mu A.$$  

Moreover, if $A$ is convex, then $(\lambda + \mu) A = \lambda A + \mu A$. We denote by $C(Y)$ and $CC(Y)$ the collection of all non-empty compact subsets and all non-empty compact convex subsets of $Y$ respectively.

**Definition 2.1.** If $K$ is a subset of $Y$ and $F : X \to 2^Y$, we say that $F$ is $K$-midconvex if

$$(2.1) \quad F(x) + F(y) \subseteq 2F\left(\frac{x + y}{2}\right) + K \quad (x, y \in X).$$

The above definition is known in the case where $K$ is a convex cone. Many properties of such set-valued functions can be found, for instance in [13].

We need some auxiliary results. The first one is due to Rådström [20].

**Lemma 2.2.** Let $A, B$ and $C$ be nonempty subsets of a topological vector space $Y$. Suppose that $B$ is closed and convex and $C$ is bounded. If $A + C \subseteq B + C$, then $A \subseteq B$. If moreover, $A$ is closed and convex and $A + C = B + C$, then $A = B$.

The following result may be found in [4, Lemma 29. 2].
Lemma 2.3. Assume that \( \{ A_n \} \) and \( \{ B_n \} \) are decreasing sequences of closed subsets of topological vector space and \( A_1 \) is compact. Then

\[
\bigcap_{n=1}^{\infty} (A_n + B_n) = \bigcap_{n=1}^{\infty} A_n + \bigcap_{n=1}^{\infty} B_n.
\]

Definition 2.4. Let \( F, G : X \to C(Y) \) be two set valued functions, for subset \( K \) of \( Y \) we say that \( F \) is \( K \)-subclose to \( G \) if \( F(x) \subseteq G(x) + K \) \( \ (x \in X) \).

Theorem 2.5. Let \( F : X \to CC(Y) \) be a \( K \)-midconvex set-valued function, \( K \in CC(Y) \) and \( 0 \in F(0) \). Then there exists a superadditive set-valued function \( A : X \to CC(Y) \) which is maximal \( K \)-subclose to \( F \) and \( A(2x) = 2A(x) \) for each \( x \in X \).

Proof. We divide the proof into three steps.

Step 1. There is a superadditive function \( A : X \to CC(Y) \) such that \( A(x) \subseteq F(x) + K \) for each \( x \in X \).

Put \( y = 0 \) in (2.1) to obtain

\[
F(x) + F(0) \subseteq 2F\left(\frac{x}{2}\right) + K \ (x \in X).
\]

Since \( 0 \in F(0) \), we have

\[
F(x) \subseteq 2F\left(\frac{x}{2}\right) + K \ (x \in X).
\]

Replacing \( x \) by \( 2^n x \) in (2.3), we see that

\[
F(2^n x) \subseteq 2F(2^{n-1} x) + K \ (x \in X, n \in \mathbb{N}).
\]

By multiplying both sides of (2.4) by \( 2^{-n} \), we get

\[
2^{-n}F(2^n x) \subseteq 2^{-(n-1)} F(2^{n-1} x) + \frac{K}{2^n} \ (x \in X, n \in \mathbb{N}).
\]

It follows from (2.5) that

\[
2^{-n}F(2^n x) + \frac{K}{2^n} \subseteq 2^{-(n-1)} F(2^{n-1} x) + \frac{K}{2^{n-1}} \ (x \in X, n \in \mathbb{N}).
\]

Let \( A_n(x) = 2^{-n}F(2^n x) + \frac{K}{2^n} \ (x \in X, n \in \mathbb{N}) \). It follows from (2.6) that \( \{ A_n(x) \} \) is a non-increasing sequence of compact sets in \( Y \) for each \( x \in X \). Hence

\[
A(x) = \bigcap_{n=0}^{\infty} A_n(x) \ (x \in X).
\]
defines a non-empty compact convex valued function on $X$. In view of (2.6), $A_n(x) = A_0(x) = F(x) + K$ for each $n \in \mathbb{N}$ and $x \in X$. Therefore $A(x) \subset F(x) + K$ for each $x \in X$. Moreover,

$$A(x) + A(y) = \bigcap_{n=0}^{\infty} A_n(x) + \bigcap_{n=0}^{\infty} A_n(y)$$

$$\subseteq \bigcap_{n=0}^{\infty} \left( A_n(x) + A_n(y) \right)$$

$$\subseteq \bigcap_{n=1}^{\infty} \left( 2^{-n}F(2^n x) + \frac{K}{2^n} + 2^{-n}F(2^n y) + \frac{K}{2^n} \right)$$

$$\subseteq \bigcap_{n=1}^{\infty} \left( 2^{-n} \left( 2F(2^{n-1} x + 2^{n-1} y) + K \right) + \frac{K}{2^{n-1}} \right)$$

$$\subseteq \bigcap_{n=1}^{\infty} \left( 2^{-(n-1)} F(2^{n-1} x + 2^{n-1} y) + \frac{K}{2^{n-1}} + \frac{K}{2^n} \right)$$

$$= \bigcap_{n=1}^{\infty} \left( 2^{-(n-1)} F(2^{n-1} x + 2^{n-1} y) + \frac{K}{2^{n-1}} \right) + \bigcap_{n=1}^{\infty} \frac{K}{2^n} \quad \text{by Lemma 2.3}$$

$$= \bigcap_{n=1}^{\infty} A_{n-1}(x + y) = A(x + y)$$

for each $x, y \in X$. Hence $a$ is superadditive. □

**Step 2.** $A(2x) = 2A(x)$.

For each $x \in X$, we have

$$A(2x) = \bigcap_{n=0}^{\infty} A_n(2x) = \bigcap_{n=0}^{\infty} \left[ 2^{-n}F(2^{n+1} x) + \frac{K}{2^n} \right] = \bigcap_{n=0}^{\infty} \left[ 2^{-n} F(2^{n+1} x) + \frac{2K}{2^{n+1}} \right]$$

$$= 2 \bigcap_{n=0}^{\infty} \left[ 2^{-(n+1)} F(2^{n+1} x) + \frac{K}{2^{n+1}} \right] = 2 \bigcap_{n=0}^{\infty} A_{n+1}(x) = 2 \bigcap_{n=0}^{\infty} A_n(x) = 2A(x). \Box$$

**Step 3.** $A$ is maximal superadditive $K$-subclose to $F$.

Let $B : X \to CC(Y)$ be a superadditive $K$-subclose to $F$. Then for each $n \in \mathbb{N}$ and $x \in X$

$$2^n B(x) \subseteq B(2^n x) \subseteq F(2^n x) + K.$$ 

It follows that

$$B(x) \subseteq A_n(x) \quad (x \in X, n \in \mathbb{N}).$$
Therefore \( B(x) \subseteq A(x) \) for each \( x \in X \).

**Definition 2.6.** By a selection \( f \) of a mapping \( F : X \to 2^Y \) we mean a single-valued mapping \( f : X \to Y \) such that \( f(x) \in F(x) \) for each \( x \in X \).

**Corollary 2.7.** Let \((X,+)\) be an additive group divisible by two and \( F : X \to C(Y) \) be a midconvex function such that \( 0 \in F(0) \). Then \( F \) admits an additive selection.

**Proof.** By Theorem 2.5, there is a superadditive function \( A : X \to C(Y) \) such that \( A(x) \subseteq F(x) \) for each \( x \in X \) and \( A(2x) = 2A(x) \) for each \( x \in X \). Therefore \( A(0) + A(0) \subseteq A(0) + \{0\} \). On account of Lemma 2.2, \( A(0) = \{0\} \). It follows that for each \( x \in X \), \( A(x) + A(-x) \subseteq A(x - x) = \{0\} \). Hence \( A \) is single-valued. Let \( A(x) = \{f(x)\} \) for each \( x \in X \). Then \( f \) is a selection of \( F \). Moreover for each \( x, y \in X \),

\[
f(x) + f(y) \in A(x) + A(y) \subseteq A(x + y) = \{f(x + y)\}.
\]

This proves additivity of \( f \). \qed

We need the following well-known result (see e.g. [21, Theorem 1.13(b)]).

**Lemma 2.8.** Let \( X \) be a topological vector space and \( A, B \subseteq X \), then \( \overline{A + B} \subseteq \overline{A} + \overline{B} \).

**Theorem 2.9.** Let \( F : X \to C(Y) \) be an \( K \)-midconvex set-valued function, \( K \in CC(Y) \) and \( 0 \in F(0) \). Then there exists a maximal midconvex set-valued function \( M : X \to C(Y) \) which is \( K \)-subclose to \( F \).

**Proof.** Let

\[
\mathcal{P} = \{G : X \to C(Y) : G \text{ is midconvex and } G(x) \subseteq F(x) + K \text{ for each } x \in X\}.
\]

The proof of Theorem 2.5 ensures that \( \mathcal{P} \neq \emptyset \). Define a binary relation "\( \preceq \)" on \( \mathcal{P} \) as follows.

\[
G_1 \preceq G_2 \text{ if and only if } G_1(x) \subseteq G_2(x) \text{ for each } x \in X.
\]

Then \( (\mathcal{P}, \preceq) \) is a partially ordered set. Let \( \mathcal{P}_0 \) be a chain in \( \mathcal{P} \), define

\[
H(x) = \bigcup_{G \in \mathcal{P}_0} G(x) \quad (x \in X).
\]

Since for each \( x \in X \) and \( G \in \mathcal{P}_0 \), \( G(x) \subseteq F(x) + K \) and \( F(x) + K \) is compact, \( H \) is compact-valued. We will show that for each \( x, y \in X \),

\[
\bigcup_{G \in \mathcal{P}_0} G(x) + \bigcup_{G \in \mathcal{P}_0} G(y) \subseteq 2H\left(\frac{x+y}{2}\right).
\]
To prove (2.7), take some $x, y \in X$, $z_1 \in \bigcup_{G \in \mathcal{P}_0} G(x)$ and $z_2 \in \bigcup_{G \in \mathcal{P}_0} G(y)$. Then for some $G_1, G_2 \in \mathcal{P}_0$, $z_1 \in G_1(x)$ and $z_2 \in G_2(y)$. Let $G_1 \preceq G_2$, then
\[
z_1 + z_2 \in G_1(x) + G_2(y) \subseteq G_2(x) + G_2(y) \subseteq 2G_2\left(\frac{x + y}{2}\right) \subseteq 2H\left(\frac{x + y}{2}\right).
\]
This proves (2.7). It follows from (2.7) and Lemma 2.8 that
\[
H(x) + H(y) = \bigcup_{G \in \mathcal{P}_0} G(x) + \bigcup_{G \in \mathcal{P}_0} G(y) \subseteq \bigcup_{G \in \mathcal{P}_0} G(x) + \bigcup_{G \in \mathcal{P}_0} G(y) \subseteq 2H\left(\frac{x + y}{2}\right).
\]
Therefore $H$ is midconvex. By Zorn’s Lemma, $\mathcal{P}$ has a maximal element $M$. This completes our proof. □

**Example 2.10.** Let $X = [0, \infty)$, $Y = \mathbb{R}$ and $F : X \to CC(Y)$ be defined by
\[
F(x) = \begin{cases}
[0, \sqrt{x}] & 0 \leq x < 1 \\
[0, 2\sqrt{x}] & x \geq 1
\end{cases}
\]
Since $g(t) = \sqrt{t}$ is concave, $F|_{[0, 1)}$ and $F|_{[1, \infty)}$ satisfy (2.1). Since $F(0) + F(1) = [0, 2]$ is not subset of $2F(\frac{0+1}{2}) = [0, \sqrt{2}]$, $F$ is not midconvex. However,
\[
F(x) + F(y) \subseteq [0, 1] + [0, 2\sqrt{y}] \subseteq 2F\left(\frac{x + y}{2}\right) + [0, 1],
\]
whenever $0 \leq x < 1$ and $y \geq 0$. Hence for $K = [0, 1]$, $F$ satisfies (2.1). According to Theorem 2.9, there is a maximal midconvex set-valued map $M : [0, \infty) \to C(Y)$ such that $M(x) \subseteq F(x) + [0, 1]$.

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**References**


Center of Excellence in Analysis on Algebraic Structures  
Department of Pure Mathematics  
School of Mathematical Sciences  
Ferdowsi University of Mashhad  
Mashhad 91775  
Iran  

E-mail address: mirmostafaei@ferdowsi.um.ac.ir