ON A CLASS OF $\delta$-SUPPLEMENTED MODULES

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Abstract. Let $R$ be an arbitrary ring with identity and $M$ a right $R$-module. In this paper, we introduce a class of modules which is an analogous to $\delta$-supplemented modules and principally $\oplus$-supplemented modules. The module $M$ is called principally $\oplus\delta$-supplemented if for any $m \in M$ there exists a direct summand $A$ of $M$ such that $M = mR + A$ and $mR \cap A$ is $\delta$-small in $A$. We prove that some results of principally $\oplus$-supplemented modules can be extended to principally $\oplus\delta$-supplemented modules for this general settings. Several properties of these modules are given and it is shown that the class of principally $\oplus\delta$-supplemented modules lies strictly between classes of principally $\oplus$-supplemented modules and principally $\delta$-supplemented modules. We investigate conditions which ensure that any factor modules, direct summands and direct sums of principally $\oplus\delta$-supplemented modules are also principally $\oplus\delta$-supplemented. We give a characterization of principally $\oplus\delta$-supplemented modules over a semisimple ring and a new characterization of principally $\delta$-semiperfect rings is obtained by using principally $\oplus\delta$-supplemented modules.

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1. Introduction

Throughout this paper all rings have an identity and all modules are unitary right modules. $N \leq M$ will mean $N$ is a submodule of $M$. A submodule $N$ of a module $M$ is called small in $M$ if for every $K \leq M$ the equality $M = N + K$ implies $M = K$. Let $N$ and $P$ be submodules of $M$. We call $P$ a supplement of $N$ in $M$ if $M = P + N$ and $P \cap N$ is small in $P$. A module $M$ is called supplemented if every submodule of $M$ has a supplement in $M$ ([10]). In [18], Zhou introduced the concept of $\delta$-small submodules as a generalization of small submodules. A submodule $N$ of $M$ is said to be $\delta$-small in $M$ if whenever $M = N + K$ and $M/K$ is singular, we have $M = K$. Let $N$ be a submodule of $M$. A submodule $L$ of $M$ is called a $\delta$-supplement of $N$ in $M$ if $M = N + L$ and $N \cap L$ is $\delta$-small in $L$ (therefore in $M$), and $M$ is called $\delta$-supplemented in case every submodule of $M$ has a $\delta$-supplement.
in $M$ (see [8] in detail). Note that every supplemented module is $\delta$-supplemented. Following [10], the module $M$ is called $\oplus$-supplemented if for any submodule $N$ of $M$, there exists a direct summand $K$ of $M$ with $M = N + K$ and $N \cap K$ small in $K$, i.e., every submodule of $M$ has a direct summand supplement in $M$, while in [14] $M$ is called principally $\oplus$-supplemented if every cyclic submodule of $M$ has a direct summand supplement in $M$. Let $M$ be a module, $K$ and $L$ submodules of $M$. $K$ is called a $\oplus\delta$-supplement of $N$ in $M$ if $M = K + N$, $K$ is a direct summand of $M$ and $K \cap N$ is $\delta$-small in $K$. Also $M$ is called $\oplus\delta$-supplemented if every submodule of $M$ has a $\oplus\delta$-supplement in $M$. Clearly, $\oplus\delta$-supplemented modules are $\delta$-supplemented and $\oplus$-supplemented modules are $\oplus\delta$-supplemented.

In what follows, by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_n$, and $\mathbb{Z}/n\mathbb{Z}$ we denote, respectively, integers, rational numbers, the ring of integers and the $\mathbb{Z}$-module of integers modulo $n$. $M_n(R)$ stands for the ring of all $n \times n$ matrices over $R$. For unexplained concepts and notations, we refer the reader to [1] and [10].

2. $\delta$-Small Submodules and $\delta$-Supplement Submodules

We collect basic properties of $\delta$-small submodules in the following lemma which is contained in [18].

Lemma 2.1. Let $M$ be a module. Then we have the following.

1. If $N$ is $\delta$-small in $M$ and $M = X + N$, then $M = X \oplus Y$ for a projective semisimple submodule $Y$ with $Y \subseteq N$.
2. If $K$ is $\delta$-small in $M$ and $f : M \to N$ is a homomorphism, then $f(K)$ is $\delta$-small in $N$. In particular, if $K$ is $\delta$-small in $M \subseteq N$, then $K$ is $\delta$-small in $N$.
3. Let $K_1 \subseteq M_1 \subseteq M$, $K_2 \subseteq M_2 \subseteq M$ and $M = M_1 \oplus M_2$. Then $K_1 \oplus K_2$ is $\delta$-small in $M_1 \oplus M_2$ if and only if $K_1$ is $\delta$-small in $M_1$ and $K_2$ is $\delta$-small in $M_2$.
4. Let $N, K$ be submodules of $M$ with $K$ $\delta$-small in $M$ and $N \leq K$. Then $N$ is also $\delta$-small in $M$.

The next lemma is clear from definitions.

Lemma 2.2. Let $M$ be a module and $m \in M$. Then the following are equivalent.

1. $mR$ is not $\delta$-small in $M$.
2. There is a maximal submodule $N$ of $M$ such that $m \notin N$ and $M/N$ is singular.
Lemma 2.3. Let $M$ be a module and $K, L, H$ submodules of $M$. If $L$ is a $\delta$-supplement of $K$ in $M$ and $K$ is a $\delta$-supplement of $H$ in $M$, then $K$ is a $\delta$-supplement of $L$ in $M$.

Proof. By assumption $M = K + L = K + H$, $K \cap L$ is $\delta$-small in $L$ and $K \cap H$ is $\delta$-small in $K$. We prove $K \cap L$ is $\delta$-small in $K$. Let $X$ be a submodule of $M$ such that $(K \cap L) + X = K$ and $K/X$ is singular. Then $M = (K \cap L) + X + H$. Since $K \cap L$ is $\delta$-small in $M$, by Lemma 2.1(1), there exists a projective semisimple submodule $Y$ in $K \cap L$ such that $M = Y \oplus (X + H)$. Hence $K = (Y \oplus X) + (K \cap H)$. Since $K/(X+Y)$ is singular as a homomorphic image of $K/X$ and $K \cap H$ is $\delta$-small in $K$, $K = X \oplus Y$. Thus $Y = 0$ as $K/X$ is singular and $Y$ is projective semisimple. □

Lemma 2.4. Let $M$ be a module and $K, N, T$ submodules of $M$. If $K$ is a $\oplus$-$\delta$-supplement of $N$ in $M$ and $T$ is $\delta$-small in $M$, then $K$ is a $\oplus$-$\delta$-supplement of $N + T$ in $M$.

Proof. Let $K$ be a $\oplus$-$\delta$-supplement of $N$ in $M$. Then $K$ is a direct summand of $M$ such that $M = N + K$ and $N \cap K$ is $\delta$-small in $K$. We prove $(N + T) \cap K$ is $\delta$-small in $K$. For if $[(N + T) \cap K] + L = K$ and $K/L$ is singular for some $L \leq K$, then $M = L + N + T$ and $M/(L + N) = (K + N)/(L + N) \cong K/(K + (L \cap N))$ is singular as a homomorphic image of $K/L$. Since $T$ is $\delta$-small in $M$, $M = L + N$. Hence $K = L + (K \cap N)$. Since $K \cap N$ is $\delta$-small in $K$ and $K/L$ is singular, we have $K = L$. □

3. Principally $\oplus$-$\delta$-Supplemented Modules

In this section we define principally $\oplus$-$\delta$-supplemented modules. We study properties, characterizations and decompositions of principally $\oplus$-$\delta$-supplemented modules. We investigate the conditions under which any factor modules, direct summands and direct sums of a principally $\oplus$-$\delta$-supplemented module are principally $\oplus$-$\delta$-supplemented. For modules over a semisimple ring $R$ we obtain that every $R$-module is principally $\oplus$-$\delta$-supplemented if and only if every $R$-module is principally $\delta$-semiperfect. Principally $\oplus$-supplemented modules are investigated in [14] and principally $\delta$-lifting modules are studied in [6]. Recently, principally $\delta$-supplemented modules are done in [7]. In this vein we introduce principally $\oplus$-$\delta$-supplemented modules generalizing principally $\oplus$-supplemented modules, principally $\delta$-lifting modules and strengthening principally $\delta$-supplemented modules.

Now we define principally $\oplus$-$\delta$-supplemented modules with the next lemma.

Lemma 3.1. Let $M$ be a module, $m \in M$ and $L$ a direct summand of $M$. Then the following are equivalent.
(1) $M = mR + L$ and $mR \cap L$ is $\delta$-small in $L$.

(2) $M = mR + L$ and for any proper submodule $K$ of $L$ with $L/K$ singular, $M \not= mR + K$.

Proof. (1) $\Rightarrow$ (2) Let $K \leq L$ and $M = mR + K$ where $L/K$ is singular. Then $L = (L \cap mR) + K$. Since $L \cap mR$ is $\delta$-small in $L$, we have $L = K$.

(2) $\Rightarrow$ (1) Let $M = mR + L$ and $K \leq L$ and $L/K$ singular with $L = (mR \cap L) + K$. Then $M = mR + L = mR + K$. By (2), $K = L$. So $mR \cap L$ is $\delta$-small in $L$. □

Let $M$ be a module and $m \in M$. A submodule $L$ is called a \textit{principally $\oplus$-$\delta$-supplement} of $mR$ in $M$ if $mR$ and $L$ satisfy Lemma 3.1 and the module $M$ is called \textit{principally $\oplus$-$\delta$-supplemented} if every cyclic submodule of $M$ has a principally $\oplus$-$\delta$-supplement in $M$, that is, for each $m \in M$ there exists a submodule $A$ of $M$ such that $M = mR + A = B \oplus A$ for some $B \leq M$ with $mR \cap A$ $\delta$-small in $A$, therefore in $M$. In [6], a module $M$ is called \textit{principally $\delta$-lifting} if for each $m \in M$, $M$ has a decomposition $M = A \oplus B$ with $A \leq mR$ and $mR \cap B$ $\delta$-small in $B$ (equivalently, in $M$). Every principally $\delta$-lifting module is a principally $\oplus$-$\delta$-supplemented module. Principally $\oplus$-supplemented modules are introduced and investigated in [14]. The module $M$ is called \textit{principally $\oplus$-supplemented} if every cyclic submodule has a supplement which is a direct summand of $M$. Hence every principally $\oplus$-supplemented module is also principally $\oplus$-$\delta$-supplemented. In [7], $M$ is said to be a \textit{principally $\delta$-supplemented module} if for every cyclic submodule of $M$ has a $\delta$-supplement in $M$. Note that, every principally $\oplus$-$\delta$-supplemented module is principally $\delta$-supplemented. We show that the class of principally $\oplus$-$\delta$-supplemented modules lies strictly between classes of principally $\oplus$-supplemented modules (principally $\delta$-lifting modules) and principally $\delta$-supplemented modules.

In the same direction as preceding paragraph one may define principally $\delta$-$\oplus$-supplemented modules. A module $M$ is called \textit{principally $\delta$-$\oplus$-supplemented} if for every cyclic submodule $mR$ of $M$, $M$ has a direct summand which is a $\delta$-supplement of $mR$ in $M$, that is, for any $m \in M$ there exists a direct summand $A$ of $M$ such that $M = mR + A$ and $mR \cap A$ is $\delta$-small in $A$. So a principally $\delta$-$\oplus$-supplemented module is the same as a principally $\oplus$-$\delta$-supplemented module.

\textbf{Examples 3.2.} (1) Let $R$ be an incomplete rank one discrete valuation ring, with quotient field $K$. By [10, Lemma A.5], the module $M = K \oplus K$ is principally $\oplus$-$\delta$-supplemented but not lifting.

(2) Consider the $\mathbb{Z}$-module $M = \mathbb{Q} \oplus (\mathbb{Z}/2\mathbb{Z})$. We prove $M$ is a principally $\oplus$-$\delta$-supplemented module but neither supplemented nor lifting. It is routine to show that $M = (1, \overline{1}) \mathbb{Z} + (\mathbb{Q} \oplus \overline{0})$. Let $(u, \overline{v}) \in M$. Assume that $\overline{v} = \overline{1}$ and $u \not= 1$. In this case we prove $M = (u, \overline{v}) \mathbb{Z} + (\mathbb{Q} \oplus \overline{0})$. Let $(x, \overline{y}) \in M$. We have two possibilities.
(i) \( \overline{y} = \overline{1} \). Then \((x, \overline{y}) = (x, \overline{1}) = (u, \overline{1}) + (x - u, \overline{0}) \in (u, \overline{1})Z + (Q \oplus (\overline{0})) \).

(ii) \( \overline{y} = \overline{0} \). Then \((x, \overline{y}) = (x, \overline{0}) = (u, \overline{1})0 + (x, \overline{0}) \in (u, \overline{1})Z + (Q \oplus (\overline{0})) \). Hence \( M = (u, \overline{1})Z + (Q \oplus (\overline{0})) \). Since \((u, \overline{1})Z \cap (Q \oplus (\overline{0}))\), it is either zero or isomorphic to \(Z \oplus (\overline{0})\) which is small in \(Q \oplus (\overline{0})\), \(M\) is principally \(\oplus\)-\(\delta\)-supplemented \(Z\)-module. If \(M\) were supplemented \(Z\)-module, its direct summand \(Q\) would be supplemented \(Z\)-module. A contradiction. So \(M\) is neither supplemented nor lifting.

Recall that a submodule \(N\) of a module of \(M\) is called fully invariant if \(f(N) \leq N\) for all endomorphisms \(f\) of \(M\), and \(M\) is said to be a duo module (or weak-duo) if every submodule (or direct summand) of \(M\) is fully invariant (see for detail [12]). The module \(M\) is called distributive if for all submodules \(K\), \(L\) and \(N\) of \(M\), \(N \cap (K + L) = (N \cap K) + (N \cap L)\) or \(N + (K \cap L) = (N + K) \cap (N + L)\). Lemma 3.3 is well known and it is obvious from definitions.

**Lemma 3.3.** Let \(M = M_1 \oplus M_2 = K + N\) and \(K \leq M_1\). If \(M\) is distributive and \(K \cap N\) is \(\delta\)-small in \(N\), then \(K \cap N\) is \(\delta\)-small in \(M_1 \cap N\).

Recall the definitions for some of the terms to be used in the sequel. An \(R\)-module \(M\) is said to be \(\pi\)-projective if for every two submodules \(U\), \(V\) of \(M\) with \(U + V = M\) there exists \(f \in \text{End}_R(M)\) with \(\text{Im}(f) \leq U\) and \(\text{Im}(1 - f) \leq V\) and \(M\) is called refinable if for any submodules \(U\) and \(V\) of \(M\) with \(M = U + V\) there is a direct summand \(U'\) of \(M\) such that \(U' \subseteq U\) and \(M = U' + V\) (see, namely [16]). The module \(M\) has the summand intersection property if the intersection of two direct summands of \(M\) is again a direct summand of \(M\).

**Theorem 3.4.** Every principally \(\delta\)-lifting module is principally \(\oplus\)-\(\delta\)-supplemented. The converse holds if \(M\) satisfies any of the following conditions.

1. \(M\) is a distributive module.
2. \(M\) is a \(\pi\)-projective module.
3. \(M\) is a duo module.
4. \(M\) is a refinable module with the summand intersection property.
5. \(M\) is an indecomposable module.

**Proof.** Let \(M\) be a principally \(\delta\)-lifting module and \(m \in M\). Then \(M\) has a decomposition \(M = A \oplus B\) such that \(B \leq mR\) and \(mR \cap A\) is \(\delta\)-small in \(A\). Since \(M = mR + A\), \(M\) is principally \(\oplus\)-\(\delta\)-supplemented. Conversely,

1. Let \(M\) be a distributive principally \(\oplus\)-\(\delta\)-supplemented module and \(m \in M\). There exists a direct summand \(A\) of \(M\) such that \(M = mR + A\) with \(mR \cap A\) \(\delta\)-small in \(A\). Let \(M = A \oplus B\) for some submodule \(B\) of \(M\). Then by distributivity of \(M\), we have \(mR = (mR \cap A) \oplus (mR \cap B)\). Hence \(M = (mR \cap B) \oplus A\). Thus \(B = mR \cap B \leq mR\). Therefore \(M\) is principally \(\delta\)-lifting.
(2) Let $M$ be a $\pi$-projective principally $\oplus$-$\delta$-supplemented module and $m \in M$. Then we have $M = mR + A$ and $mR \cap A$ is $\delta$-small in $A$ for some direct summand $A$ of $M$. Since $M$ is $\pi$-projective, by [15, 41.14], there exists $N \leq mR$ with $M = A \oplus N$. Therefore $M$ is principally $\delta$-lifting.

(3) Similar to the case (1).

(4) Let $M$ be a refinable principally $\oplus$-$\delta$-supplemented module with the summand intersection property and $m \in M$. Then there exists a direct summand $A$ of $M$ such that $M = mR + A$ and $mR \cap A$ is $\delta$-small in $A$. Since $M$ is refinable, there exists a direct summand $U$ of $M$ such that $U$ is contained in $mR$ and $M = U + A$. By the summand intersection property of $M$, $U \cap A$ is a direct summand of $M$. Let $M = (U \cap A) \oplus K$ for some submodule $K$ of $M$. Then $A = (U \cap A) \oplus (K \cap A)$, and so $M = U \oplus (K \cap A)$. On the other hand, $mR \cap (K \cap A)$ is $\delta$-small in $A$. Since $K \cap A$ is a direct summand of $A$, $mR \cap (K \cap A)$ is also $\delta$-small in $K \cap A$. This completes the proof.

(5) Let $M$ be an indecomposable module and $m \in M$. Since $M$ is principally $\oplus$-$\delta$-supplemented, there exist submodules $A$ and $B$ of $M$ such that $mR \cap A$ is $\delta$-small in $A$ and $M = A \oplus B = mR + A$. By hypothesis, $A = M$ and $B = 0$. So that $mR \cap A = mR$ is $\delta$-small in $M$. Note that in this case, every cyclic submodule of $M$ is $\delta$-small in $M$.

Next example shows that there exists a principally $\oplus$-$\delta$-supplemented module which is not principally $\delta$-lifting.

**Example 3.5.** Consider the $\mathbb{Z}$-module $M = (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/8\mathbb{Z})$. Then $N_1 = (1, 0)\mathbb{Z}$, $N_2 = (0, 1)\mathbb{Z}$, $N_3 = (0, 0)\mathbb{Z}$, $N_4 = (0, 0)\mathbb{Z}$, $N_5 = (1, 1)\mathbb{Z}$, $N_6 = \mathbb{Z}/2\mathbb{Z}$ and $N_7 = \mathbb{Z}/8\mathbb{Z}$ are nonzero cyclic submodules of $M$. Hence $M = N_6 \oplus N_7 = N_2 \oplus N_5$ and $N_3, N_4$ are small submodules of $M$. Thus $M$ is a principally $\oplus$-supplemented module and so principally $\oplus$-$\delta$-supplemented. On the other hand, $M$ is not principally $\delta$-lifting, by [6].

Every principally $\oplus$-$\delta$-supplemented module need not be principally $\oplus$-supplemented, as Example 3.43 shows. But in some cases these modules coincide.

**Proposition 3.6.** Let $M$ be a singular module. Then $M$ is principally $\oplus$-supplemented if and only if it is principally $\oplus$-$\delta$-supplemented.

**Proof.** The necessity is clear. For the sufficiency, let $m \in M$. Then there exists a direct summand $A$ of $M$ with $M = mR + A$ and $mR \cap A$ $\delta$-small in $A$. Assume that $A = (mR \cap A) + K$ for some submodule $K$ of $A$. Since $M$ is singular, $A/K$ is also singular. Hence we have $A = K$. Thus $mR \cap A$ is small in $A$. Therefore $M$ is principally $\oplus$-supplemented. \qed
**Proposition 3.7.** Let $M$ be a principally $⊕$-$δ$-supplemented module. If every cyclic submodule of $M$ has a uniform principally $⊕$-$δ$-supplement, then $M$ is principally $⊕$-supplemented.

*Proof.* Let $m \in M$. By hypothesis, there exists a uniform direct summand $A$ of $M$ with $M = mR + A$ and $mR \cap A$ $δ$-small in $A$. Assume that $(mR \cap A) + K = A$ for some submodule $K$ of $A$. If $K = 0$, then there is nothing to do. Let $K \neq 0$. Since $K$ is essential in $A$, $A/K$ is singular. Then we have $K = A$. Hence $mR \cap A$ is small in $A$. Thus $M$ is principally $⊕$-supplemented. □

**Proposition 3.8.** Every principally $⊕$-$δ$-supplemented module is principally $δ$-supplemented. The converse is true for refinable modules.

*Proof.* The first assertion is clear. Let $M$ be a principally $δ$-supplemented module and $m \in M$. Let $A$ be a submodule of $M$ with $M = mR + A$ and $mR \cap A$ $δ$-small in $A$. Since $M$ is refinable, there is a direct summand $U$ of $M$ such that $U \subseteq A$ and $M = U + mR$. Also $U$ is a direct summand of $A$. This implies that $mR \cap U$ is $δ$-small in $A$. Hence $mR \cap U$ is $δ$-small in $U$. □

Next example shows that there exists a principally $δ$-supplemented module which is not principally $⊕$-$δ$-supplemented.

**Example 3.9.** Let $F$ be a field and $x$ and $y$ commuting indeterminates over $F$. Consider the polynomial ring $R = F[x, y]$, the ideals $I_1 = (x^2)$ and $I_2 = (y^2)$ of $R$, and the ring $S = R/(x^2, y^2)$. Let $M = πS + \pi S$. Then $M$ is an indecomposable $S$-module, principally supplemented but not principally $⊕$-supplemented. Hence $M$ is principally $δ$-supplemented. On the other hand, since $M$ is singular, it is not principally $⊕$-$δ$-supplemented by Proposition 3.6.

Because of the following example it can be said that any submodule of a principally $⊕$-$δ$-supplemented module may not be principally $⊕$-$δ$-supplemented.

**Example 3.10.** Consider $Q$ as a $Z$-module. Since every cyclic submodule of $Q$ is small and so $δ$-small in $Q$, $Q$ is principally $⊕$-$δ$-supplemented. But the submodule $Z$ of $Q$ is not principally $⊕$-$δ$-supplemented as a $Z$-module since $2Z$ does not have any principally $⊕$-$δ$-supplement in $Z$.

Now we investigate conditions which ensure that a homomorphic image and so a direct summand of a principally $⊕$-$δ$-supplemented module is principally $⊕$-$δ$-supplemented.

**Theorem 3.11.** Let $M$ be a distributive principally $⊕$-$δ$-supplemented module. Then every homomorphic image of $M$ is principally $⊕$-$δ$-supplemented.
Proof. Let $L$ be a submodule of $M$ and $(mR + L)/L$ a cyclic submodule of $M/L$. Then there exists a direct summand $A$ of $M$ such that $M = A ⊕ B = mR + A$ for some $B \leq M$ and $mR \cap A$ is $δ$-small in $A$. Now $M/L = (mR + L)/L + (A + L)/L$ and, since $M$ is distributive, $(mR + L)/(A + L) = L + (mR \cap A)/L$. So $((mR + L)/(A + L))\cap ((A + L)/L) = (L + (mR \cap A))/L$ is $δ$-small in $(A + L)/L$ as a homomorphic image of $δ$-small $mR \cap A$ in $A$ under the natural map $π$ from $A$ onto $(A + L)/L$ by Lemma 2.1(2). Again by distributivity of $M$ and $A \cap B = 0$, we have $(A + L)\cap (B + L) = L$. Hence $(A + L)/L$ is a direct summand of $M/L$. □


Proposition 3.13. Let $M$ be a module and $N$ a submodule of $M$. If every cyclic submodule of $M$ has a principally $⊕$-$δ$-supplement which contains $N$, then $M/N$ is principally $⊕$-$δ$-supplemented.

Proof. Let $m \in M$ and consider the submodule $\overline{m}R$ of $M/N$. By hypothesis, there exists a direct summand $L$ of $M$ such that $N \leq L$, $M = mR + L$ and $mR \cap L$ is $δ$-small in $L$. Let $M = K \oplus L$ for some submodule $K$ of $M$ and $π$ denote the natural epimorphism from $M$ onto $M/N$. Then we have $M/N = (K + N)/(N \oplus (L/N) = \overline{m}R + (L/N)$. On the other hand, $π(mR \cap L) = π(mR) \cap π(L) = \overline{m}R \cap (L/N)$ is $δ$-small in $π(L) = L/N$. Hence the proof is completed. □

Lemma 3.14. Let $M$ be a module and $N$ a fully invariant submodule of $M$. If $M = M_1 \oplus M_2$ for some submodules $M_1$ and $M_2$ of $M$, then $M/N = (M_1 + N)/N \oplus (M_2 + N)/N$.

Proof. Clearly, $M/N = (M_1 + N)/N + (M_2 + N)/N$. If $m_1 + N = m_2 + N$ with $m_i \in M_i$ ($i = 1, 2$), then $m_1 - m_2 \in N$. As $N$ is a fully invariant submodule of $M$, we see that $m_1, m_2 \in N$. Hence $(M_1 + N)/N \cap (M_2 + N)/N = 0$, as required. □

Proposition 3.15. Let $M$ be a principally $⊕$-$δ$-supplemented module. Then $M/N$ is principally $⊕$-$δ$-supplemented for every fully invariant submodule $N$ of $M$.

Proof. Let $N$ be a fully invariant submodule of $M$ and $\overline{m}R$ a submodule of $M/N$, where $m \in M$. Since $M$ is principally $⊕$-$δ$-supplemented, there exists a direct summand $A$ of $M$ such that $M = mR + A$ and $mR \cap A$ is $δ$-small in $A$. Let $M = A \oplus B$ for some submodule $B$ of $M$. By Lemma 3.14, we have $M/N = (A + N)/N \oplus (B + N)/N$. Also $M/N = (A + N)/N + \overline{m}R$. It is clear that $(A + N)/N \cap \overline{m}R$ is $δ$-small in $(A + N)/N$. This completes the proof. □

As an immediate consequence of Proposition 3.15, we deduce that if $M$ is principally $⊕$-$δ$-supplemented, then so are $M/Rad(M)$ and $M/Soc(M)$. 

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Corollary 3.16. Let $M$ be a weak-duo and principally $\oplus$-\(\delta\)-supplemented module. Then every direct summand of $M$ is principally $\oplus$-\(\delta\)-supplemented.

Recall that a module $M$ has $D_3$ if whenever $M_1$ and $M_2$ are direct summands of $M$ with $M = M_1 + M_2$, $M_1 \cap M_2$ is also a direct summand of $M$ ([10]).

Proposition 3.17. Let $M$ be a principally $\oplus$-\(\delta\)-supplemented module. If $M$ has $D_3$, then every direct summand of $M$ is also principally $\oplus$-\(\delta\)-supplemented.

Proof. Let $N$ be a direct summand of $M$ and $n \in N$. Since $M$ is principally $\oplus$-\(\delta\)-supplemented, there exists a direct summand $A$ of $M$ with $M = A + nR$ and $A \cap nR$ $\delta$-small in $A$. Hence $M = A + N$ and $N = (A \cap N) + nR$. Due to $D_3$, $A \cap N$ is a direct summand of $M$, $N$ and $A$. By Lemma 2.1(3), $(A \cap N) \cap nR$ is $\delta$-small in $A \cap N$ because $A \cap N$ is a direct summand of $A$. Thus $N$ is principally $\oplus$-\(\delta\)-supplemented. \(\Box\)

Due to Proposition 3.17 and [5, Lemma 2.4] we obtain the following result.

Corollary 3.18. Let $M$ be a principally $\oplus$-\(\delta\)-supplemented and UC extending module. Then every direct summand of $M$ is principally $\oplus$-\(\delta\)-supplemented.

It is obvious that every module with the summand intersection property has $D_3$. Then the following result is an immediate consequence of Proposition 3.17 and [4, Theorem 4.6].

Corollary 3.19. Let $R$ be a right semihereditary ring and $F$ a principally $\oplus$-\(\delta\)-supplemented finitely generated free $R$-module. Then $R$ is principally $\oplus$-\(\delta\)-supplemented as an $R$-module.

Next example shows that for a module $M$ and a submodule $N$, if $M/N$ is principally $\oplus$-\(\delta\)-supplemented, then $M$ need not be principally $\oplus$-\(\delta\)-supplemented.

Example 3.20. Consider the $\mathbb{Z}$-module $\mathbb{Z}/p^n\mathbb{Z}$, where $p$ is a prime number and $n$ is a positive integer. Then $\mathbb{Z}/p^n\mathbb{Z}$ is principally $\delta$-lifting and so principally $\oplus$-$\delta$-supplemented, but $\mathbb{Z}$ is not principally $\oplus$-$\delta$-supplemented.

Proposition 3.21. Let $M = M_1 \oplus M_2$ be a distributive module. Then $M$ is principally $\oplus$-$\delta$-supplemented if and only if $M_1$ and $M_2$ are principally $\oplus$-$\delta$-supplemented.

Proof. Let $M$ be a principally $\oplus$-$\delta$-supplemented module. Due to Corollary 3.12, $M_1$ and $M_2$ are principally $\oplus$-$\delta$-supplemented. Assume that $M_1$ and $M_2$ are principally $\oplus$-$\delta$-supplemented modules and $m \in M$. By distributivity of $M$, we have $mR = (mR \cap M_1) \oplus (mR \cap M_2)$. Since $mR \cap M_1$ and $mR \cap M_2$ are cyclic submodules of $M_1$ and $M_2$ respectively, there exist direct summands $A$ of $M_1$ and $B$ of $M_2$ such
that $M_1 = (mR \cap M_1) + A = A' \oplus A$ and $A \cap (mR \cap M_1) = A \cap mR$ is $\delta$-small in $A$, and $M_2 = (mR \cap M_2) + B = B' \oplus B$ and $B \cap (mR \cap M_2) = B \cap mR$ is $\delta$-small in $B$. Then $M = mR + A + B = (A' \oplus B') \oplus (A \oplus B)$. Again by distributivity, $mR \cap (A + B) = (mR \cap A) + (mR \cap B)$ is $\delta$-small in $A + B$ by Lemma 2.1(3). This completes the proof. \qed

**Proposition 3.22.** Let $M = M_1 \oplus M_2$ be a duo module. Then $M$ is principally $\oplus$-$\delta$-supplemented if and only if $M_1$ and $M_2$ are principally $\oplus$-$\delta$-supplemented.

**Proof.** Necessity is clear from Proposition 3.17 because duo modules satisfy the summand intersection property. Sufficiency is resemble to the proof of Proposition 3.21. \qed

**Corollary 3.23.** Let $M$ be a principally $\oplus$-$\delta$-supplemented module and every finite direct sum of $M$ a distributive (or duo) module. Then every finitely $M$-generated module is principally $\oplus$-$\delta$-supplemented.

Recall that a module $M$ is called regular (in the sense of Zelmanowitz) [17] if for any $m \in M$ there exists a map $\alpha \in \text{Hom}_R(M, R)$ such that $m = m\alpha(m)$ and it is known that every cyclic submodule of a regular module is a direct summand. Hence any regular module is principally $\oplus$-$\delta$-supplemented. We give an example to show that principally $\oplus$-$\delta$-supplemented modules need not be a regular module.

**Example 3.24.** Any cyclic submodule of $\mathbb{Q}$ as a $\mathbb{Z}$-module is a small submodule of $\mathbb{Q}$. Therefore $\mathbb{Q}$ is a principally $\oplus$-$\delta$-supplemented $\mathbb{Z}$-module. On the other hand, $\mathbb{Q}$ can not be a regular $\mathbb{Z}$-module since $\text{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) = 0$.

A module $M$ is said to be principally semisimple if every cyclic submodule is a direct summand of $M$. Tuganbaev calls a principally semisimple module as a regular module in [13], and lifting modules are named as semiregular modules. Every semisimple module is principally semisimple. Every principally semisimple module is principally $\delta$-lifting and so principally $\oplus$-$\delta$-supplemented. A ring $R$ is called principally semisimple if the right $R$-module $R$ is principally semisimple. It is clear that every principally semisimple ring is von Neumann regular and vice versa. For a module $M$, we write $\text{Rad}_\delta(M) = \sum \{L \mid L \text{ is a } \delta\text{-small submodule of } M\}$. Since every small submodule of $M$ is $\delta$-small, $\text{Rad}(M) \leq \text{Rad}_\delta(M)$. In the ring case, we shall denote $\text{Rad}_\delta(M)$ by $J_\delta(R)$ and usually $\text{Rad}(M)$ by $J(R)$ for a ring $R$. It is shown that $J_\delta(R)$ is an ideal of $R$, and there are cases for a ring $R$ such that $J_\delta(R)$ strictly contains $J(R)$ (see namely [18]). Also note that for any module $M$, $\text{Rad}_\delta(M)$ is a $\delta$-small submodule of $M$ provided every proper submodule of $M$ is contained in a maximal submodule of $M$, therefore $J_\delta(R)$ is a $\delta$-small right and $\delta$-small left ideal of $R$. 


Lemma 3.25. [10, Lemma 4.47] Let $M = S \oplus T = N + T$ where $S$ is $T$-projective. Then $M = S' \oplus T$ where $S' \leq N$.

Lemma 3.26. Let $M$ be a principally $\oplus$-$\delta$-supplemented module. Then $M/\text{Rad}_\delta(M)$ is a principally semisimple module if $M$ has one of the following conditions.

1. $M$ is a distributive module.
2. $M$ is a projective module.

Proof. (1) For any $m \in M$, there exists a direct summand $A$ of $M$ such that $M = mR + A$ and $mR \cap A$ is $\delta$-small in $A$. So $mR \cap A$ is $\delta$-small in $M$. By distributivity of $M$, we have $(mR + \text{Rad}_\delta(M)) \cap (A + \text{Rad}_\delta(M)) = \text{Rad}_\delta(M) + (mR \cap A) = \text{Rad}_\delta(M)$ since $mR \cap A$ is $\delta$-small in $M$. Then

$$M/\text{Rad}_\delta(M) = [(mR + \text{Rad}_\delta(M))/\text{Rad}_\delta(M)] \oplus [(A + \text{Rad}_\delta(M))/\text{Rad}_\delta(M)].$$

(2) Let $m \in M$. There exists a direct summand $A$ of $M$ such that $M = mR + A$ and $mR \cap A$ is $\delta$-small in $A$. So $mR \cap A$ is $\delta$-small in $M$. By projectivity of $M$, there exists a direct summand $N$ of $M$ such that $M = N \oplus A$ with $N \leq mR$ by Lemma 3.25. Then $(mR + \text{Rad}_\delta(M))/\text{Rad}_\delta(M) = (N + \text{Rad}_\delta(M))/\text{Rad}_\delta(M)$ and $\text{Rad}_\delta(M) = \text{Rad}_\delta(N) \oplus \text{Rad}_\delta(A)$ imply

$$M/\text{Rad}_\delta(M) = [(mR + \text{Rad}_\delta(M))/\text{Rad}_\delta(M)] \oplus [(A + \text{Rad}_\delta(M))/\text{Rad}_\delta(M)].$$

Hence every principal submodule of $M/\text{Rad}_\delta(M)$ is a direct summand in either case. Therefore $M/\text{Rad}_\delta(M)$ is principally semisimple. □

Proposition 3.27. Let $M$ be a principally $\oplus$-$\delta$-supplemented module and $N$ a submodule of $M$. If $N \cap \text{Rad}_\delta(M) = 0$, then $N$ is principally semisimple.

Proof. Let $x \in N$. By hypothesis, there exists a direct summand $A$ of $M$ with $M = A + xR$ and $A \cap xR$ $\delta$-small in $A$. Hence $N = (A \cap N) + xR$ and $A \cap xR \leq \text{Rad}_\delta(M)$. Since $(A \cap N) \cap xR \leq N \cap \text{Rad}_\delta(M) = 0$, we have $N = (A \cap N) \oplus xR$. Therefore $N$ is principally semisimple. □

Theorem 3.28 may be proved easily by making use of Lemma 3.26 for distributive modules. But we prove it in another way.

Theorem 3.28. Let $M$ be a principally $\oplus$-$\delta$-supplemented module. Then $M$ has a principally semisimple submodule $M_1$ such that $M_1$ has an essential socle and $\text{Rad}_\delta(M) \oplus M_1$ is essential in $M$.

Proof. By Zorn’s Lemma we may find a submodule $M_1$ of $M$ such that $\text{Rad}_\delta(M) \oplus M_1$ is essential in $M$. By Proposition 3.27, $M_1$ is principally semisimple. Next we show that $M_1$ has an essential socle. For this we prove for any $m \in M_1$, $mR$ has
a simple submodule. If \( mR \) is simple, we have done. Otherwise let \( m_1 \in mR \) such that \( m_1 R \neq mR \). By hypothesis there exists a direct summand \( C \) of \( M \) such that \( M = m_1 R + C \) with \( m_1 R \cap C \) \( \delta \)-small in \( C \). Then \( m_1 R \cap C \leq M_1 \cap \text{Rad}_\delta(M) = 0 \). So \( M = m_1 R \oplus C \) and then \( mR = m_1 R \oplus (mR \cap C) \). Clearly, \( mR \cap C = m'_1 R \) for some \( m'_1 \in mR \) and \( mR = m_1 R \oplus m'_1 R \). If \( m_1 R \) and \( m'_1 R \) are simple, then we stop. Otherwise let \( m_2 \in m_1 R \) such that \( m_2 R \neq m_1 R \). Similarly, there is \( m'_2 \in m_1 R \) such that \( m_1 R = m_2 R \oplus m'_2 R \). Hence \( mR = m_2 R \oplus m'_2 R \oplus m'_1 R \). If \( m_2 R \) is simple, then we stop. Otherwise we continue in this way. Since \( mR \) is cyclic, this process must terminate at a finite step, say \( n \). At this step all direct summands of \( mR \) should be simple. This completes the proof.

\[ \square \]

**Theorem 3.29.** Let \( M \) be a principally \( \oplus-\delta \)-supplemented module. Assume that \( M \) satisfies ascending chain condition on direct summands. Then \( M \) has a decomposition \( M = M_1 \oplus M_2 \), where \( M_1 \) is a semisimple module and \( M_2 \) is a module with \( \text{Rad}_\delta(M_2) \) essential in \( M_2 \).

**Proof.** Let \( M_1 \) be a submodule of \( M \) such that \( \text{Rad}_\delta(M) \oplus M_1 \) is essential in \( M \) and \( m_1 \in M_1 \). By Proposition 3.27, \( M_1 \) is principally semisimple. Since \( M \) is principally \( \oplus-\delta \)-supplemented, there exists a direct summand \( A_1 \) of \( M \) such that \( M = m_1 R + A_1 \) and \( m_1 R \cap A_1 \) is \( \delta \)-small in both \( A_1 \) and \( M \). Hence \( m_1 R \cap A_1 = 0 \) and \( M = m_1 R \oplus A_1 \). Then \( M_1 = m_1 R \oplus (M_1 \cap A_1) \). If \( M_1 \cap A_1 \neq 0 \), let \( 0 \neq m_2 \in M_1 \cap A_1 \). There exists a direct summand \( A_2 \) of \( M \) such that \( M = m_2 R \oplus A_2 \) and \( m_2 R \cap A_2 \) is \( \delta \)-small in both \( A_2 \) and \( M \). Hence \( m_2 R \cap A_2 = 0 \), \( M = m_2 R + A_2 = m_1 R \oplus m_2 R \oplus (A_1 \cap A_2) \). So \( M_1 \cap A_1 = m_2 R \oplus (M_1 \cap A_1 \cap A_2) \) and \( M_1 = m_2 R \oplus (M_1 \cap A_1) = m_1 R \oplus m_2 R \oplus (M_1 \cap A_1 \cap A_2) \). If \( M_1 \cap A_1 \cap A_2 \neq 0 \), let \( 0 \neq m_3 \in M_1 \cap A_1 \cap A_2 \). There exists a direct summand \( A_3 \) of \( M \) such that \( M = m_3 R \oplus A_3 = m_1 R \oplus m_2 R \oplus m_3 R \oplus (A_1 \cap A_2 \cap A_3) \) and \( M_1 \cap A_1 \cap A_2 = m_3 R \oplus (M_1 \cap A_1 \cap A_2 \cap A_3) \) and \( M_1 = m_1 R \oplus m_2 R \oplus m_3 R \oplus (M_1 \cap A_1 \cap A_2 \cap A_3) \).

By hypothesis this procedure stops at a finite number of steps, say \( t \). At this stage we may have \( M = m_t R \oplus A_t = m_1 R \oplus m_2 R \oplus m_3 R \oplus \cdots \oplus m_t R \oplus (A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_t) \) and \( M_1 = m_1 R \oplus m_2 R \oplus m_3 R \oplus \cdots \oplus m_t R \). Let \( M_2 = A_t \cap A_2 \cap A_3 \cap \cdots \cap A_t \). Then \( M = M_1 \oplus M_2 \) with \( \text{Rad}_\delta(M) = \text{Rad}_\delta(M_2) \). Since \( M_1 \oplus \text{Rad}_\delta(M) \) is essential in \( M \), it follows that \( \text{Rad}_\delta(M_2) \) is essential in \( M_2 \). Since \( M \) has the ascending chain condition on direct summands, without loss of generality, we may assume that all cyclic submodules \( m_1 R, m_2 R, m_3 R, \ldots, m_t R \) to be simple. This completes the proof.

\[ \square \]

**Theorem 3.30.** Let \( M \) be a module with \( \text{Rad}_\delta(M) = 0 \). Then the following conditions are equivalent.

1. \( M \) is principally \( \oplus-\delta \)-supplemented.
(2) $M$ is principally $\oplus$-supplemented
(3) $M$ is principally semisimple.

Proof. We prove only (1) $\Rightarrow$ (3) since (2) $\Leftrightarrow$ (3) is proved in [14] and (3) $\Rightarrow$ (1) is clear. Let $M$ be a principally $\oplus$-$\delta$-supplemented module and $m \in M$. There exists a direct summand $A$ of $M$ such that $M = mR + A$ and $mR \cap A$ is $\delta$-small in $A$. Since $mR \cap A$ is also $\delta$-small in $M$ and $Rad_\delta(M) = 0$, $mR$ is a direct summand of $M$. Therefore $M$ is principally semisimple.

It is known that every von Neumann regular ring has zero Jacobson radical. But there are von Neumann regular rings $R$ with $J_\delta(R) \neq 0$ as the following example shows.

Example 3.31. Let $Q = \prod_{i=1}^{\infty} F_i$, where each $F_i = \mathbb{Z}_2$. Let $R$ be the subring of $Q$ generated by $\bigoplus_{i=1}^{\infty} F_i$ and $1_Q$. Then $R$ is von Neumann regular and $\bigoplus_{i=1}^{\infty} F_i = Soc(R) = J_\delta(R)$.

Corollary 3.32. Let $R$ be a ring. If $R$ is a von Neumann regular ring, then $R$ is a principally $\oplus$-$\delta$-supplemented $R$-module. The converse holds if $J_\delta(R) = 0$.

Definition 3.33. Let $M$ be a module. $M$ is called a $\delta$-hollow module (or a principally $\delta$-hollow module) if every proper submodule (or cyclic submodule) is $\delta$-small in $M$.

Note that each hollow module is $\delta$-hollow, and each $\delta$-hollow module is principally $\delta$-hollow and so principally $\oplus$-$\delta$-supplemented. Let $M$ be a module. Clearly, if $M = xR$ for every $x \in M \setminus Rad_\delta(M)$, then $M$ is principally $\delta$-hollow.

Theorem 3.34. Let $M$ be a projective module having $Rad_\delta(M)$ finite uniform dimension. Consider the following statements.

(1) $M$ is a direct sum of principally $\oplus$-$\delta$-supplemented modules.
(2) $M$ has a decomposition $M = M_1 \oplus M_2$ where $M_1$ is a direct sum of principally semisimple modules and $M_2$ is a finite direct sum of principally $\delta$-hollow modules.

Then (2) $\Rightarrow$ (1). (1) $\Rightarrow$ (2) in case $M$ satisfies ascending chain condition on direct summands.

Proof. (2) $\Rightarrow$ (1) Assume that $M$ has a decomposition $M = M_1 \oplus M_2$ with submodules $M_1$ and $M_2$ satisfying stated conditions in (2). Both $M_1$ and $M_2$ are direct sums of principally $\oplus$-$\delta$-supplemented modules as $M_1$ is a direct sum of principally semisimple modules, and $M_2$ is a direct sum of principally $\delta$-hollow modules and
each principally $\delta$-hollow module is principally $\oplus$-$\delta$-supplemented.

(1) ⇒ (2) Assume that $M = \bigoplus_{i \in I} M_i$, where each $M_i$ is a principally $\oplus$-$\delta$-supplemented module and $\text{Rad}_\delta(M)$ has finite uniform dimension. Since $\text{Rad}_\delta(M) = \bigoplus_{i \in I} \text{Rad}_\delta(M_i)$, there is a finite subset $J$ of $I$ with $\text{Rad}_\delta(M_i) = 0$ for all $i \in I \setminus J$. Therefore, by Theorem 3.30, $M_i$ is principally semisimple for all $i \in I \setminus J$. Hence $M = M_1 \oplus (\bigoplus_{j \in J} M_j)$, where $M_1$ is a direct sum of principally semisimple modules. Due to Theorem 3.29, without loss of generality, we may assume that $\text{Rad}_\delta(M_j)$ is essential in $M_j$, where $j \in J$. Then for $j \in J$, $M_j$ has finite uniform dimension by [3, Proposition 3.20]. Now we prove each $M_j$ is principally $\delta$-hollow or a finite direct sum of principally $\delta$-hollow modules, for $j \in J$. Let $j \in J$. Since $M$ is projective, $M_j$ is also projective. Then $\text{Rad}_\delta(M_j) \neq M_j$ by [18, Lemma 1.9]. We complete the proof by induction on the uniform dimension. Suppose that $M_j$ has uniform dimension 1, and let $x \in M_j \setminus \text{Rad}_\delta(M_j)$. Since $M_j$ is principally $\oplus$-$\delta$-supplemented, there exists a direct summand $K$ of $M_j$ such that $M_j = xR + K$ and $xR \cap K$ is $\delta$-small in $K$. Let $M_j = K \oplus K_1$ for some submodule $K_1$ of $M_j$. Since $M_j$ has uniform dimension 1, we have $K = 0$ or $K_1 = 0$. If $K_1 = 0$, then $xR$ is a submodule of $\text{Rad}_\delta(M_j)$. This is a contradiction. Hence $K = 0$ and so $M_j = xR$. It follows that $M_j$ is principally $\delta$-hollow. Now suppose that $n > 1$ be a positive integer and assume each $M_j$ having uniform dimension $k(1 \leq k < n)$ is principally $\delta$-hollow or a finite direct sum of principally $\delta$-hollow submodules. Let $j \in J$ and assume $M_j$ has uniform dimension $n$. Suppose $M_j$ is not principally $\delta$-hollow. Let $x \in M_j \setminus \text{Rad}_\delta(M_j)$ such that $M_j \neq xR$. Since $M_j$ is principally $\oplus$-$\delta$-supplemented, there exist submodules $K, K_1$ of $M_j$ with $M_j = xR + K = K \oplus K_1$ and $xR \cap K$ $\delta$-small in $K$. Note that $K_1 \neq 0$ and $K \neq 0$. Since projective modules have $D_3$ and then by Proposition 3.17, $K$ and $K_1$ are principally $\oplus$-$\delta$-supplemented modules by induction, $K$ and $K_1$ are principally $\delta$-hollow or a finite direct sum of principally $\delta$-hollow submodules. So (1) ⇒ (2) holds and this completes the proof.

One may ask what happens to Theorem 3.30 in which the condition “$\text{Rad}_\delta(M) = 0$” changes to “$\text{Rad}_\delta(M)$ is $\delta$-small in $M$”.

**Theorem 3.35.** Let $M$ be a projective module with $\text{Rad}_\delta(M)$ $\delta$-small in $M$ and consider the following conditions.

1. $M$ is principally $\oplus$-$\delta$-supplemented.
2. $M/\text{Rad}_\delta(M)$ is principally semisimple.

Then (1) ⇒ (2). If $M$ is a refinable module, then (2) ⇒ (1).

**Proof.** (1) ⇒ (2) Since $M$ is a principally $\oplus$-$\delta$-supplemented module, $M/\text{Rad}_\delta(M)$ is principally semisimple by Lemma 3.26.
(2) ⇒ (1) Let $mR$ be any cyclic submodule of $M$. By (2), there exists a submodule $U$ of $M$ such that $M/\text{Rad}_\delta(M) = [(mR+\text{Rad}_\delta(M))/\text{Rad}_\delta(M)] \oplus [U/\text{Rad}_\delta(M)]$. Then $M = mR + U$ and $(mR+\text{Rad}_\delta(M)) \cap U = (mR \cap U) + \text{Rad}_\delta(M)$. Hence $mR \cap U \leq \text{Rad}_\delta(M)$ and it is $\delta$-small in $M$. Since $M = mR + U$ and being $M$ refinable, there exists a direct summand $A$ of $M$ such that $A \leq U$ and $M = mR + A$. Since $mR \cap A \leq mR \cap U$ is $\delta$-small in $M$ and $A$ is a direct summand of $M$, by Lemma 2.1(3), $mR \cap A$ is $\delta$-small in $A$. Hence $A$ is a principally $\oplus -\delta$-supplement of $mR$ in $M$. This completes the proof.

Recall that $R$ is called a right V-ring if every simple right $R$-module is injective, equivalently, by [9, Theorem 3.75], for any right $R$-module $M$, $\text{Rad}(M) = 0$. In this note we shall call the ring $R$ is a right $\delta$-V-ring if for any right $R$-module $M$, $\text{Rad}_\delta(M) = 0$. Since every small submodule is $\delta$-small, $\text{Rad}(M) \leq \text{Rad}_\delta(M)$ for any module $M$.

We adopt the definition of a small projective module in [15, 19.10(8)] and we say an $R$-module $M$ $\delta$-small projective if $\text{Hom}(M, -)$ is exact with respect to the exact sequences of right $R$-modules $0 \to K \xrightarrow{i} L \to N \to 0$ with $i(K)$ a $\delta$-small submodule of $L$. If $R$ is a $\delta$-V-ring, then every module is $\delta$-small projective. In a subsequent paper the present authors study $\delta$-small projective modules in detail. As is usual, to study $\delta$-V-rings it is convenient to deal with an injective notion. A module $M$ is called $\delta$-small injective if $\text{Hom}(-, M)$ is exact with respect to the exact sequences of right $R$-modules $0 \to K \xrightarrow{i} L \to N \to 0$ with $i(K)$ a $\delta$-small submodule of $L$. Clearly for a $R$ right $\delta$-V-ring, every right $R$-module is both $\delta$-small projective and $\delta$-small injective.

**Lemma 3.36.** Let $R$ be a ring and consider the following conditions.

1. $R$ is a right $\delta$-V-ring.
2. Every right $R$-module is $\delta$-small projective.
3. Every right $R$-module is $\delta$-small injective.

Then (1) ⇒ (2) ⇔ (3).

**Proof.** (1) ⇒ (2) Clear. (2) ⇒ (3) Let $M$ be a right $R$-module and an exact sequence of right $R$-modules with $i(K)$ a $\delta$-small submodule of $L$

$$0 \to K \xrightarrow{i} L \xrightarrow{f} N \to 0 \quad (*)$$

Applying $\text{Hom}(N, -)$ to that sequence, by (2) we have an exact sequence

$$0 \to \text{Hom}(N, K) \xrightarrow{i^*} \text{Hom}(N, L) \xrightarrow{f^*} \text{Hom}(N, N) \to 0$$
For the identity map $1 \in \text{Hom}(N,N)$ we have a map $g \in \text{Hom}(N,L)$ such that $1 = f^* g$. Hence the sequence (*) splits and so any map from $K$ to $M$ extends from $L$ to $M$. $(3) \Rightarrow (2)$ Dual to $(2) \Rightarrow (3)$.

**Theorem 3.37.** Let $R$ be a right V-ring. If every right $R$-module is $\delta$-small projective, then every principally $\oplus$-$\delta$-supplemented module is a direct sum of a projective semisimple module and a principally semisimple module.

**Proof.** Let $R$ be a right V-ring and $M$ any right $R$-module. We have $\text{Rad}(M) = 0$. By [2, Proposition 3.1] or [9, Theorem 3.75] every submodule of $M$ is contained in a maximal submodule, and [18, Lemma 1.5(4)] implies $\text{Rad}_\delta(M)$ is $\delta$-small in $M$. Since every right $R$-module is $\delta$-small projective, we apply the functor $\text{Hom}(M/\text{Rad}_\delta(M), -)$ to the sequence $0 \rightarrow \text{Rad}_\delta(M) \rightarrow M \rightarrow M/\text{Rad}_\delta(M) \rightarrow 0$ we have $M = \text{Rad}_\delta(M) \oplus K$ for some submodule $K$ of $M$. By Lemma 2.1(1), there exists a projective semisimple submodule $Y$ of $\text{Rad}_\delta(M)$ such that $M = Y \oplus K$. Hence $Y = \text{Rad}_\delta(M)$. Due to Proposition 3.27, $K$ is principally semisimple and this completes the proof. □

A ring $R$ is called $\delta$-semiregular if every cyclically presented $R$-module has a projective $\delta$-cover. By combining Lemma 3.26, Theorem 3.30 and Theorem 3.37 we obtain the next result.

**Theorem 3.38.** Let $R$ be a right $\delta$-V-ring and consider the following conditions.

1. Every right $R$-module is principally $\oplus$-$\delta$-supplemented.
2. Every right $R$-module is principally $\oplus$-supplemented.
3. Every right $R$-module is principally semisimple.
4. $R$ is von Neumann regular.
5. Every projective $R$-module is principally $\oplus$-$\delta$-supplemented.
6. $R$ is $\delta$-semiregular.

Then $(1) \iff (2) \iff (3) \Rightarrow (4) \iff (5) \iff (6)$.

**Proof.** $(4) \Rightarrow (5)$ Let $M$ be a projective right $R$-module. By [13, Proposition 1.25], $M$ is principally semisimple. This implies that $M$ is principally $\oplus$-$\delta$-supplemented.

$(5) \Rightarrow (4)$ Since $R$ is projective as a right $R$-module, $R$ is principally $\oplus$-$\delta$-supplemented. Being $J_\delta(R) = 0$, $R$ is principally semisimple by Theorem 3.30. Hence $R$ is von Neumann regular.

$(4) \iff (6)$ Clear by [18, Theorem 3.5] since $J_\delta(R) = 0$.

**Theorem 3.39.** Let $R$ be a ring with $J_\delta(R) = 0$. Then the following are equivalent.

1. Every projective $R$-module is principally $\oplus$-$\delta$-supplemented.
2. Every free $R$-module is principally $\oplus$-$\delta$-supplemented.
(3) Every projective $R$-module is principally semisimple.

(4) Every free $R$-module is principally semisimple.

Proof. (2) $\Rightarrow$ (1) Let every free $R$-module be principally $\oplus$-$\delta$-supplemented and $P$ a projective module. Then there exists a free module $F$ such that $P$ is a direct summand of $F$. By (2), $F$ is principally $\oplus$-$\delta$-supplemented with $\text{Rad}_\delta(F) = 0$ since $J_\delta(R) = 0$. Lemma 3.26 implies $F$ is principally semisimple and then $P$ is principally semisimple, therefore $P$ is principally $\oplus$-$\delta$-supplemented. The rest is clear. \hfill $\square$

At the moment we have the following conjecture.

**Conjecture.** Every right $V$-ring is right $\delta$-$V$-ring.

By [18], a projective module $P$ is called a projective $\delta$-cover of a module $M$ if there exists an epimorphism $f : P \longrightarrow M$ with $\text{Ker}f$ $\delta$-small in $P$, and a ring $R$ is called $\delta$-perfect ($\delta$-semiperfect) if every $R$-module (simple $R$-module) has a projective $\delta$-cover. Clearly, every $\delta$-perfect ring is $\delta$-semiperfect. A module $M$ is said to be principally $\delta$-semiperfect if every factor module of $M$ by a cyclic submodule has a projective $\delta$-cover. A ring $R$ is called principally $\delta$-semiperfect in case the right $R$-module $R$ is principally $\delta$-semiperfect. Every $\delta$-semiperfect module is principally $\delta$-semiperfect. Next we characterize projective principally $\oplus$-$\delta$-supplemented modules.

**Theorem 3.40.** Let $M$ be a projective module. Then the following are equivalent.

1. $M$ is principally $\delta$-semiperfect.
2. $M$ is principally $\oplus$-$\delta$-supplemented.

Proof. (1) $\Rightarrow$ (2) Let $m \in M$ and $P \xrightarrow{f} M/mR$ be a projective $\delta$-cover and $M \xrightarrow{\pi} M/mR$ the natural epimorphism.

Then there exists a map $M \xrightarrow{g} P$ such that $fg = \pi$. Hence $P = g(M) + \text{Ker}f$. Since $\text{Ker}f$ is $\delta$-small, by Lemma 2.1(1), there exists a projective semisimple submodule $Y$ of $\text{Ker}f$ such that $P = g(M) \oplus Y$. So $g(M)$ is projective. Thus $M = K \oplus \text{Ker}g$ for some submodule $K$ of $M$. Let $x \in \text{Ker}g$. Then $fg = \pi$ implies $\pi(x) = 0$. Hence $\text{Ker}g \leq mR$. Next we show $g(K) \cap \text{Ker}f = g(K \cap mR)$. Let $x \in K \cap mR$. Then $0 = \pi(x) = fg(x)$. So $x \in g^{-1}(\text{Ker}f)$ and $K \cap mR \leq g^{-1}(\text{Ker}f)$ and $K \cap mR \leq g^{-1}(\text{Ker}f) \cap K$. Then $g(K \cap mR) \leq g(g^{-1}(\text{Ker}f) \cap K) = \text{Ker}f \cap g(K)$. 
Let \( x \in \text{Ker} f \cap g(K) \). There is \( y \in K \) such that \( g(y) = x \) and \( f(x) = 0 \). Then \( \pi(y) = f(g(y)) = f(x) = 0 \). So \( y \in mR \) and \( x = g(y) \in g(K \cap mR) \). Hence \( g(K) \cap \text{Ker} f = g(K \cap mR) \) and it is \( \delta \)-small in \( P \) and therefore in \( g(K) \). Since \( g \) is an isomorphism between \( K \) and \( g(K) \cap \text{Ker} f \) is \( \delta \)-small in \( K \). Because \( K \cap mR \leq g^{-1}(g(K) \cap \text{Ker} f) \), \( K \cap mR \) is \( \delta \)-small in \( K \) by Lemma 2.1(4).

(2) \( \Rightarrow \) (1) Assume that \( M \) is a principally \( \oplus \)-\( \delta \)-supplemented module. Let \( m \in M \). There exists a direct summand \( A \) of \( M \) such that \( M = mR + A \) with \( mR \cap A \) \( \delta \)-small in \( A \). Consider the maps \( A \xrightarrow{\pi} A/(mR \cap A) \xrightarrow{h} M/mR \) where \( \pi \) is the natural epimorphism and \( h \) is the isomorphism \( A/(mR \cap A) \cong M/mR \). Since \( \text{Ker}(h \pi) = \text{Ker} \pi = mR \cap A \) is \( \delta \)-small in \( A \), \( A \) is a projective \( \delta \)-cover of \( M/mR \). So \( M \) is principally \( \delta \)-semiperfect.

Now we can give a characterization of principally \( \delta \)-semiperfect rings by using the notion of principally \( \oplus \)-\( \delta \)-supplemented.

**Corollary 3.41.** Let \( R \) be a ring. Then the following are equivalent.

1. \( R \) is principally \( \delta \)-semiperfect.
2. \( R \) is principally \( \oplus \)-\( \delta \)-supplemented.

**Proof.** Clear by Theorem 3.40. \( \square \)

It is known that a ring \( R \) is semisimple if and only if every \( R \)-module is projective. As a consequence of Theorem 3.40, we have the next result.

**Corollary 3.42.** Let \( R \) be a semisimple ring. Then every \( R \)-module is principally \( \oplus \)-\( \delta \)-supplemented if and only if every \( R \)-module is principally \( \delta \)-semiperfect.

We conclude this paper by giving the aforementioned example which shows that every principally \( \oplus \)-\( \delta \)-supplemented module need not be principally \( \oplus \)-supplemented.

**Example 3.43.** Let \( F \) be a field, \( I = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix} \), and consider the ring \( R = \{(x_1, \ldots, x_n, x, x, \ldots) : n \in \mathbb{N}, x_i \in M_2(F), x \in I\} \) with component-wise operations. By [11, Example 2.15], \( J(R) = 0 \) and \( R \) is not a von Neumann regular ring. Then \( R \) is not principally \( \oplus \)-supplemented as an \( R \)-module due to [14, Theorem 3.30]. On the other hand, it is known that, from [18, Example 4.3], \( J_\delta(R) = \{(x_1, \ldots, x_n, x, x, \ldots) : n \in \mathbb{N}, x_i \in M_2(F), x \in K\} \), where \( K = \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix} \) and \( R \) is \( \delta \)-perfect. Hence \( R \) is principally \( \delta \)-semiperfect. By Corollary 3.41, \( R \) is principally \( \oplus \)-\( \delta \)-supplemented.
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