\section{Introduction}

Let \((X,d)\) be a metric space and \(C\) be a nonempty subset of \(X\). Then a mapping \(T: C \to H\) is called \textit{nonexpansive} if \(d(Tx,Ty) \leq d(x,y)\) for all \(x,y \in C\). The class of nonexpansive mappings was studied by many authors; see, for example, \([4, 5, 7, 8, 11-13]\). We denote by \(F(T)\) the set of all fixed points of \(T\), i.e., \(F(T) := \{x \in C : Tx = x\}\). A mapping \(T\) from \(C\) into itself is called \textit{quasi-nonexpansive} if \(d(Tx,p) \leq d(x,p)\) for all \(x \in C\) and \(p \in F(T)\). It is easy to see that both nonspreading and hybrid mappings with \(F(T) \neq \emptyset\) are quasi-nonexpansive. They proved fixed point theorems for such mappings; see also Kohsaka and Takahashi \([16]\), Iemoto and Takahashi \([17]\), and Takahashi \([11]\) and Takahashi and Yao \([22]\).

Furthermore, Kocourek, Takahashi and Yao \([15]\) introduced a more broad class of nonlinear mappings than the class of nonspreading and hybrid mappings. They called such a class the class of generalized hybrid mapping. A mapping \(T: C \to C\) is called \((\alpha, \beta)\)-\textit{generalized hybrid} mapping if there exist \(\alpha, \beta \in \mathbb{R}\) such that

\begin{equation}
\alpha d^2(Tx,Ty) + (1-\alpha)d^2(x,y) \leq \beta d^2(Tx,y) + (1-\beta)d^2(x,y), \quad \text{for all } x,y \in C.
\end{equation}

Such a class contains some classes of nonlinear mappings in a Hilbert space. For example, an \((\alpha, \beta)\)-generalized hybrid mapping is nonexpansive for \(\alpha = 1\) and \(\beta = 0\), nonspreading for \(\alpha = 2\) and \(\beta = 1\), and hybrid for \(\alpha = \frac{3}{2}\) and \(\beta = \frac{1}{2}\). It is easy to see that \((\alpha, \beta)\)-generalized hybrid mapping with a fixed point is quasi-nonexpansive.

They proved a demiclosed principle for this class of mapping, i.e., \(x_n \rightharpoonup z\) and \(x_n - Tx_n \rightharpoonup 0\) imply \(Tz = z\). Also, they proved a sequence of Mann’s type for this class of mappings in Hilbert spaces:

\begin{equation}
x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \quad n \geq 1
\end{equation}

converges weakly to an element \(v\) of \(F(T)\), where \(v = \lim_{n \to \infty} P_{F(T)}x_n\), \(P_C\) is the metric projection of \(H\) onto \(C\).

On the other hand, Kirk and Panyanak \([13]\) specialized Lims concept \([18]\) of \(\Delta\)-convergence in a general metric space to \(\text{CAT}(0)\) spaces and showed that many Banach space results which involve
weak convergence have precise analogs in this setting; for instance, the Opial property, the Kadec-Klee property and the demiclosedness principle for LANE mappings.

In 2008, Dhompongsa and Panyanak [4] obtained the convergence theorems for the following Picard, Mann and Ishikawa iterations, resp., for nonexpansive mappings in a CAT(0) space:

\[ x_n = T^n x, \quad n = 0, 1, 2, \ldots \]

(P) and

\[ x_{n+1} = t_n T x_n \oplus (1 - t_n) x_n, \quad n = 0, 1, 2, \ldots \]

(M) where \( \{t_n\} \) is a sequence in \([0, 1]\), and

\[ x_{n+1} = t_n T (s_n T x_n \oplus (1 - s_n) x_n) \oplus (1 - t_n) x_n, \quad n = 0, 1, 2, \ldots \]

(I) where \( s_n \) and \( \{t_n\} \) are sequence in \([0, 1]\). By using the concept of \( \Delta \)-convergence, they prove that the sequence are generated by (P), (M), and (I) \( \Delta \)-converge to an element of \( F(T) \) in a CAT(0) space.

Motivated and inspired from above results, the aim of this paper is to study the convergence results for generalized hybrid mappings in a CAT(0) space. After we review literature, we next recall some definitions and useful lemmas in Section 2. In Section 3, we give the \( \Delta \)-convergence theorems of the Picard and Mann iterate sequence for a such mapping considered by Takahashi and Yao [21]. We give a \( \Delta \)-convergence of the Ishikawa iterate sequence for a generalized hybrid mapping. Finally, we give an example of a \((\alpha, \beta)\)-generalized hybrid mapping in CAT(0) space which is not a nonexpansive mapping.

2. Preliminaries

Let \((X, d)\) be a metric space. A \textit{geodesic path} joining \(x \in X\) to \(y \in X\) (or, more briefly, a \textit{geodesic} from \(x\) to \(y\)) is a map \(c\) from a closed interval \([0, l] \subset \mathbb{R}\) to \(X\) such that \(c(0) = x, c(l) = y\), and \(d(c(t), c(t')) = |t - t'|\) for all \(t, t' \in [0, l]\). In particular, \(c\) is an isometry and \(d(x, y) = l\). The image \(c\) of \(c\) is called a \textit{geodesic} (or metric) \textit{segment} joining \(x\) and \(y\). When it is unique this geodesic segment is denoted by \([x, y]\). The space \((X, d)\) is said to be a \textit{geodesic space} if every two points of \(X\) are joined by a geodesic, and \(X\) is said to be uniquely geodesic if there is exactly one geodesic joining \(x\) and \(y\) for each \(x, y \in X\). A subset \(C\) of a CAT(0) space is convex if \([x, y] \subseteq C\) for all \(x, y \in C\). A geodesic triangle \(\triangle(x_1, x_2, x_3)\) in a geodesic metric space \((X, d)\) consists of three points \(x_1, x_2, x_3\) in \(X\) (the vertices of \(\triangle\)) and a geodesic segment between each pair of vertices (the edges of \(\triangle\)). A comparison triangle for the geodesic triangle \(\overline{\triangle}(x_1, x_2, x_3)\) in \((X, d)\) is a triangle \(\overline{\triangle}(x_1, x_2, x_3) := \triangle(\overline{x}_1, \overline{x}_2, \overline{x}_3)\) in the Euclidean plane \(\mathbb{E}^2\) such that \(d_{\mathbb{E}^2}(\overline{x}_i, \overline{x}_j) = d(x_i, x_j)\) for all \(i, j \in 1, 2, 3\).

A geodesic space is said to be a CAT(0) space if all geodesic triangles of appropriate size satisfy the following comparison axiom.

\(\text{CAT}(0)\) : Let \(\triangle\) be a geodesic triangle in \(X\) and let \(\overline{\triangle}\) be a comparison triangle for \(\triangle\). Then \(\triangle\) is said to satisfy the \(\text{CAT}(0)\) \textit{inequality} if for all \(x, y \in \triangle\) and all comparison points \(\overline{x}, \overline{y} \in \overline{\triangle}\),

\[ d(x, y) \leq d_{\mathbb{E}^2}(\overline{x}, \overline{y}). \]

If \(x, y_1, y_2\) are points in a CAT(0) space and if \(y_0\) is the midpoint of the segment \([y_1, y_2]\), then the \(\text{CAT}(0)\) inequality implies

\[ d^2(x, y_0) \leq \frac{1}{2} d^2(x, y_1) + \frac{1}{2} d^2(x, y_2) - \frac{1}{4} d^2(y_1, y_2). \]  

(CN)

This is the (CN) inequality of Bruhat and Tits [3]. In fact (cf. [1], p. 163), a geodesic space is a CAT(0) space if and only if it satisfies the (CN) inequality.

It is well known that any complete, simply connected Riemannian manifold having nonpositive sectional curvature is a CAT(0) space. Other examples include Pre-Hilbert spaces, R-trees (see [1]), Euclidean buildings (see [2]), the complex Hilbert ball with a hyperbolic metric (see [7]), and many others. Complete CAT(0) spaces are often called Hadamard spaces.

Next, we collect some useful lemmas in CAT(0) spaces.

\textbf{Lemma 2.1.} \cite{1, Proposition 2.2} Let \(X\) be a CAT(0) space, \(p, q, r, s \in X\) and \(\lambda \in [0, 1]\). Then

\[ d(\lambda p \oplus (1 - \lambda) q, \lambda r \oplus (1 - \lambda) s) \leq \lambda d(p, r) + (1 - \lambda) d(q, s). \]

\textbf{Lemma 2.2.} \cite{4, Lemma 2.4} Let \(X\) be a CAT(0) space, \(x, y, z \in X\) and \(\lambda \in [0, 1]\). Then

\[ d(\lambda x \oplus (1 - \lambda) y, z) \leq \lambda d(x, z) + (1 - \lambda) d(y, z). \]
Lemma 2.3. [4, Lemma 2.5] Let $X$ be a CAT(0) space, $x, y, z \in X$ and $\lambda \in [0,1]$. Then
\[ d^2(\lambda x \oplus (1-\lambda)y, z) \leq \lambda d^2(x, z) + (1-\lambda)d^2(y, z) - \lambda(1-\lambda)d^2(x, y). \]

We give the concept of $\Delta$-convergence and collect some basic properties.

Let $\{x_n\}$ be a bounded sequence in a CAT(0) space $X$. For $x \in X$, we set
\[ r(x, \{x_n\}) = \limsup_{n \to \infty} d(x, x_n). \]

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by
\[ r(\{x_n\}) = \inf \{r(x, \{x_n\}) : x \in X\}, \]
and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set
\[ A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}. \]

It is known from Proposition 7 of [6] that in a CAT(0) space, $A(\{x_n\})$ consists of exactly one point.

A sequence $\{x_n\} \subset X$ is said to $\Delta$-converge to $x \in X$ if $A(\{x_{n_k}\}) = \{x\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Uniqueness of asymptotic center implies that CAT(0) space $X$ satisfies Opial's property, i.e., for given $\{x_n\} \subset X$ such that $\{x_n\}$ $\Delta$-converges to $x$ and given $y \in X$ with $y \neq x$,
\[ \limsup_{n \to \infty} d(x_n, x) < \limsup_{n \to \infty} d(x_n, y). \]

Since it is not possible to formulate the concept of demiclosedness in a CAT(0) setting, as stated in linear spaces, let us formally say that "If $T$ is demiclosed at zero" if the conditions, $\{x_n\} \subset C$ $\Delta$-converges to $x$ and $d(x_n, Tx_n) \to 0$ imply $x \in F(T)$.

Lemma 2.4. [13] Every bounded sequence in a complete CAT(0) space always has a $\Delta$-convergent subsequence.

Lemma 2.5. [5] If $C$ is a closed convex subset of a complete CAT(0) space and if $\{x_n\}$ is a bounded sequence in $C$, then the asymptotic center of $\{x_n\}$ is in $C$.

3. $\Delta$-CONVERGENCE THEOREMS FOR A $(\alpha, \beta)$-GENERALIZED HYBRID MAPPING

Let $(X, d)$ be a CAT(0) space and $C$ a nonempty, closed and convex subset of $X$. A mapping $T : C \to C$ is called $(\alpha, \beta)$-generalized hybrid mapping if there exist $\alpha, \beta \in \mathbb{R}$ such that
\[ \alpha d^2(Tx, Ty) + (1-\alpha)d^2(x, Ty) \leq \beta d^2(Tx, y) + (1-\beta)d^2(x, y). \quad (3.1) \]

In [15], Kocourek, Takahashi and Yao obtained the demiclosed principle of $(\alpha, \beta)$-generalized hybrid mapping in a Hilbert space. In a similar way, we present the demiclosed principle of $(\alpha, \beta)$-generalized hybrid mapping in a CAT(0) space.

Proposition 3.1. Let $(X, d)$ be a CAT(0) space and $C$ be a nonempty, closed and convex subset of $X$. Let $T : C \to C$ be an $(\alpha, \beta)$-generalized hybrid mapping such that $\alpha \geq 1$ and $\beta \geq 0$. Let $\{x_n\}$ be a bounded sequence in $C$ such that $\lim_{n \to \infty} d(x_n, T(x_n)) = 0$ and $\{x_n\}$ $\Delta$-converges to $w$. Then $T(w) = w$.

Proof. Notice that $T : C \to C$ is an $(\alpha, \beta)$-generalized hybrid mapping, i.e., there exist $\alpha, \beta \in \mathbb{R}$ such that
\[ \alpha d^2(Tx, Ty) + (1-\alpha)d^2(x, Ty) \leq \beta d^2(Tx, y) + (1-\beta)d^2(x, y). \quad (3.2) \]

Since $T$ is generalized hybrid mapping, we have that
\[ \alpha d^2(Tx_n, Tw) + (1-\alpha)d^2(x_n, Tw) \leq \beta d^2(Tx_n, w) + (1-\beta)d^2(x_n, w). \quad (3.3) \]

Since $\alpha \geq 1$, $\beta \geq 0$ and (3.3), we get that
\[ \alpha d^2(Tx_n, Tw) \leq \beta(d(Tx_n, x_n) + d(x_n, w)) + (1-\beta)d^2(x_n, w) + \beta(\alpha - 1)(d(x_n, Tx_n) + d(Tx_n, Tw))^2. \]

Hence
\[ (\alpha - (\alpha - 1))d^2(Tx_n, Tw) \leq (\beta + (1-\beta)d^2(x_n, w) + (\beta + (\alpha - 1))d^2(x_n, Tx_n) + 2(\beta + (\alpha - 1))(d(x_n, w) + d(Tx_n, Tw))(d(Tx_n, x_n)), \]
\[ (\alpha - (\alpha - 1))d^2(Tx_n, Tw) \leq (\beta + (1-\beta)d^2(x_n, w) + (\beta + (\alpha - 1))d^2(x_n, Tx_n) + 2(\beta + (\alpha - 1))(d(x_n, w) + d(Tx_n, Tw))(d(Tx_n, x_n)), \]
\[ (\alpha - (\alpha - 1))d^2(Tx_n, Tw) \leq (\beta + (1-\beta)d^2(x_n, w) + (\beta + (\alpha - 1))d^2(x_n, Tx_n) + 2(\beta + (\alpha - 1))(d(x_n, w) + d(Tx_n, Tw))(d(Tx_n, x_n)), \]
\[ (\alpha - (\alpha - 1))d^2(Tx_n, Tw) \leq (\beta + (1-\beta)d^2(x_n, w) + (\beta + (\alpha - 1))d^2(x_n, Tx_n) + 2(\beta + (\alpha - 1))(d(x_n, w) + d(Tx_n, Tw))(d(Tx_n, x_n)), \]
\[ (\alpha - (\alpha - 1))d^2(Tx_n, Tw) \leq (\beta + (1-\beta)d^2(x_n, w) + (\beta + (\alpha - 1))d^2(x_n, Tx_n) + 2(\beta + (\alpha - 1))(d(x_n, w) + d(Tx_n, Tw))(d(Tx_n, x_n)), \]
and so
\[
\begin{align*}
  d^2(Tx_n, Tw) &\leq d^2(x_n, w) + (\beta + \alpha - 1)d^2(x_n, Tx_n) \nonumber \\
  &\quad + 2(\beta + \alpha - 1)(d(x_n, w) + d(Tx_n, Tw))d(Tx_n, x_n).
\end{align*}
\]  

(3.4)

Since \( \{x_n\} \) is bounded and \( \lim_{n \to \infty} d(Tx_n, x_n) = 0 \), we get that \( \{Tx_n\} \) is also bounded. Thus (3.4) reduces to
\[
\begin{align*}
  d^2(Tx_n, Tw) &\leq d^2(x_n, w) + (\beta + \alpha - 1)d^2(x_n, Tx_n) \nonumber \\
  &\quad + 2(\beta + \alpha - 1)d(Tx_n, x_n)M,
\end{align*}
\]  

(3.5)

where \( M \geq \sup_{n \geq 1} d(x_n, w) + d(Tx_n, Tw) \). Since \( \{x_n\} \rightharpoonup w \), we then have that \( A_C(\{x_n\}) = \{w\} \) and also \( A(\{x_n\}) = \{w\} \). Assume that \( Tw \neq w \). Then by Opial’s conditions,
\[
\begin{align*}
  \limsup_{n \to \infty} d^2(x_n, w) &< \limsup_{n \to \infty} d^2(x_n, Tw) \\
  &\leq \limsup_{n \to \infty} (d(x_n, Tx_n) + d(Tx_n, Tw))^2 \\
  &\leq \limsup_{n \to \infty} d^2(Tx_n, Tw) \\
  &\leq \limsup_{n \to \infty} (d^2(x_n, w) + (\beta + \alpha - 1)d^2(x_n, Tx_n) + 2(\beta + \alpha - 1)d(Tx_n, x_n)M) \\
  &= \limsup_{n \to \infty} d^2(x_n, w).
\end{align*}
\]

This is a contradiction. So, we have \( Tw = w \). \qed

Next, we present the \( \Delta \)-converges of Picard iterate sequence \( (P) \) for \((\alpha, \beta)\)-generalized hybrid mapping \( T \).

**Proposition 3.2.** Let \((X, d)\) be a CAT(0) space and \( C \) be a nonempty, closed and convex subset of \( X \). Let \( T : C \to C \) be an \((\alpha, \beta)\)-generalized hybrid mapping with \( F(T) \neq \emptyset \). Let \( \gamma \) be a real number with \( 0 < \gamma < 1 \) and define the mapping \( S : C \to C \) by:
\[
  S = \gamma I \oplus (1 - \gamma)T.
\]

(3.6)

Then, for each \( x \in C \), \( d(S^{n+1}x, S^n x) \) converges to 0.

**Proof.** It is easy to prove that \( F(T) = F(S) \). Since \( F(T) \neq \emptyset \), we have \( T \) and \( S \) are a quasi nonexpansive mapping. For any \( x \in C \) and \( p \in F(T) \), we get
\[
\begin{align*}
  d(S^{n+1}x, p) &= d(SS^n x, p) \\
  &= d(\gamma S^n x \oplus (1 - \gamma)TS^n x, p) \\
  &\leq \gamma d(S^n x, p) + (1 - \gamma)d(TS^n x, p) \\
  &\leq \gamma d(S^n x, p) + (1 - \gamma)d(S^n x, p) \\
  &= d(S^n x, p).
\end{align*}
\]

Hence \( d(S^n x, p) \) is decreasing sequence and bounded below, and so \( \lim_{n \to \infty} d(S^n x, p) \) exists. Therefore \( \{S^n x\} \) is bounded and \( \{TS^n x\} \) is also. It follows from Lemma 2.3 that
\[
\begin{align*}
  d^2(S^{n+1}x, p) &= d^2(\gamma S^n x \oplus (1 - \gamma)TS^n x, p) \\
  &\leq \gamma d^2(S^n x, p) + (1 - \gamma)(d^2(TS^n x, p) - \gamma(1 - \gamma)d^2(S^n x, TS^n x)) \\
  &\leq \gamma d^2(S^n x, p) + (1 - \gamma)d^2(S^n x, p) - \gamma(1 - \gamma)d^2(S^n x, TS^n x) \\
  &\leq d^2(S^n x, p) - \gamma(1 - \gamma)d^2(S^n x, TS^n x),
\end{align*}
\]

and so
\[
\gamma(1 - \gamma)d^2(S^n x, TS^n x) \leq d^2(S^n x, p) - d^2(S^{n+1}x, p).
\]

Since \( \lim_{n \to \infty} d^2(S^n x, p) \) exists and \( 0 < \gamma < 1 \), we have that \( \lim_{n \to \infty} d(S^n x, TS^n x) = 0 \). On the other hand,
\[
  d(S^{n+1} x, TS^n x) = d(\gamma S^n x \oplus (1 - \gamma)TS^n x, TS^n x) \leq \gamma d(S^n x, TS^n x).
\]

We get that
\[
  d(S^{n+1} x, S^n x) \leq d(S^{n+1} x, TS^n x) + d(TS^n x, S^n x) \leq \gamma d(S^n x, TS^n x) + d(TS^n x, S^n x) \to 0 \text{ as } n \to \infty.
\]
This completes the proof.

**Theorem 3.3.** Let \((X, d)\) be a \(CAT(0)\) space and \(C\) be a nonempty, closed and convex subset of \(X\). Let \(T : C \to C\) be an \((\alpha, \beta)\)-generalized hybrid mapping with \(F(T) \neq \emptyset\). Let \(\gamma\) be a real number with \(0 < \gamma < 1\) and define the mapping \(S : C \to C\) by:

\[
S = \gamma I \oplus (1 - \gamma)T.
\]  

(3.7)

Then, for each \(x \in C\), \(\{S^n x\}\) \(\Delta\)-converges to an element in \(F(T)\).

**Proof.** For each \(n \geq 1\), let \(x_n = S^n x\). Since \(F(T) \neq \emptyset\), we have that \(S\) is a quasi-nonexpansive mapping. For any \(p \in F(T)\), we get that

\[
d(x_{n+1}, p) = d(S^{n+1} x, p) 
\leq d(S^n x, p) = d(x_n, p).
\]

This implies that \(\{x_n\}\) is bounded. By Proposition 3.2, it follows that

\[
\lim_{n \to \infty} d(Sx_n, x_n) = \lim_{n \to \infty} d(S^{n+1} x, S^n x) = 0.
\]

Since \(\{x_n\}\) is bounded, there exists a subsequence \(\{x_{n_j}\}\) of \(\{x_n\}\) such that \(\{x_{n_j}\}\) \(\Delta\)-converges to \(u \in C\). By Proposition 3.1, \(u \in F(T)\). Let \(\{x_{n_k}\}\) be another subsequence of \(\{x_n\}\) such that \(\{x_{n_k}\}\) \(\Delta\)-converges to \(v \in C\). Suppose \(u \neq v\). Again by Proposition 3.1, \(v \in F(T)\), and so \(\lim_{n \to \infty} d(x_n, u)\) and \(\lim_{n \to \infty} d(x_n, v)\) exist. By Opial’s condition, we have

\[
\lim_{n \to \infty} d(x_n, u) = \lim_{i \to \infty} \sup_{n \to \infty} d(x_n, u) 
< \lim_{i \to \infty} \sup_{n \to \infty} d(x_n, v) 
= \lim_{n \to \infty} d(x_n, v) 
= \lim_{j \to \infty} \sup_{n \to \infty} d(x_{n_j}, v) 
< \lim_{j \to \infty} \sup_{n \to \infty} d(x_{n_j}, u) 
= \lim_{n \to \infty} d(x_n, u).
\]

This is contraction. Thus \(u = v\). Hence \(\{x_n\}\) \(\Delta\)-converges to \(u \in F(T)\).

**Theorem 3.4.** Let \((X, d)\) be a \(CAT(0)\) space and \(C\) be a nonempty, closed and convex subset of \(X\). Let \(T : C \to C\) be an \((\alpha, \beta)\)-generalized hybrid mapping such that \(\alpha \geq 1\) and \(\beta \geq 0\) with \(F(T) \neq \emptyset\). Let \(\{\gamma_n\}\) be a sequence of real number with \(0 < a \leq \gamma_n \leq b < 1\) and defined a sequence \(\{x_n\}\) in \(C\) as follows:

\[
\begin{align*}
x_1 &= x \in C, \\
x_{n+1} &= \gamma_n x_n \oplus (1 - \gamma_n)Tx_n, \quad \forall n \in \mathbb{N}.
\end{align*}
\]  

(3.8)

Then \(\{x_n\}\) \(\Delta\)-converges to an element \(u \in F(T)\).

**Proof.** Since \(F(T) \neq \emptyset\), we have that \(T\) is a quasi-nonexpansive mapping. For any \(p \in F(T)\), we get that

\[
d(x_{n+1}, p) = d((\gamma_n x_n \oplus (1 - \gamma_n)Tx_n, p) 
\leq \gamma_n d(x_n, p) + (1 - \gamma_n)d(Tx_n, p) 
\leq \gamma_n d(x_n, p) + (1 - \gamma_n)d(x_n, p) 
= d(x_n, p).
\]

This implies that \(\{x_n\}\) is bounded, and so \(\{T(x_n)\}\) is also bounded since \(T\) is quasi-nonexpansive. Moreover, we have that the limit of \(d(x_n, p)\) exists. By Lemma 2.3,

\[
d^2(x_{n+1}, p) = d^2((\gamma_n x_n \oplus (1 - \gamma_n)Tx_n, p) 
\leq \gamma_n d^2(x_n, p) + (1 - \gamma_n)d^2(Tx_n, p) - \gamma_n(1 - \gamma_n)d^2(x_n, Tx_n) 
\leq d^2(x_n, p) - \gamma_n(1 - \gamma_n)d^2(x_n, Tx_n),
\]

\[
\lim_{n \to \infty} d(x_n, p) = d(x, p).
\]

Therefore, \(\{x_n\}\) \(\Delta\)-converges to an element \(u \in F(T)\).
and so
\[ γ_n(1 - γ_n)d^2(x_n, Tx_n) ≤ d^2(x_n, p) - d^2(x_{n+1}, p). \] (3.9)
Since the limit of \( d(x_n, p) \) exists and \( 0 < a ≤ γ_n ≤ b < 1 \), it follows from (3.9) that,
\[ \lim_{n \to \infty} d(x_n, Tx_n) = 0. \] (3.10)
It follows from the proof of Theorem 3.3 that \( \{x_n\} \) \( ∆ \)-converges to an element \( z \in F(S) = F(T) \). □

**Theorem 3.5.** Let \((X, d)\) be a CAT(0) space and \( C \) be a nonempty, closed and convex subset of \( X \). Let \( T : C \to C \) be an \((α, β)\)-generalized hybrid mapping such that \( α ≥ 1 \) and \( β ≥ 0 \) with \( F(T) \neq ∅ \). Let \( \{γ_n\} \) and \( \{σ_n\} \) be sequences of real numbers with \( 0 < a ≤ γ_n, σ_n ≤ b < 1 \) and define a sequence \( \{x_n\} \) in \( C \) as follows:
\[
\begin{align*}
x_1 &= x \in C, \\
x_{n+1} &= γ_n x_n + (1 - γ_n)T(σ_n x_n + (1 - σ_n)Tx_n), \quad \forall n ∈ \mathbb{N}.
\end{align*}
\] (3.11)
Then \( \{x_n\} \) \( ∆ \)-converges to an element \( u \in F(T) \).

**Proof.** For any \( n \), let \( y_n = σ_n x_n + (1 - σ_n)Ty_n \), then \( x_{n+1} = γ_n x_n + (1 - γ_n)Ty_n \). Since \( F(T) \neq ∅ \), we have that \( T \) is a quasi-nonexpansive mapping. For any \( p ∈ F(T) \), we get that
\[
d(y_n, p) = d(σ_n x_n + (1 - σ_n)Tx_n, p) ≤ σ_n d(x_n, p) + (1 - σ_n)d(Tx_n, p) \leq d(x_n, p),
\]
and
\[
d(x_{n+1}, p) = d(γ_n x_n + (1 - γ_n)Ty_n, p) ≤ γ_n d(x_n, p) + (1 - γ_n)d(Ty_n, p) ≤ γ_n d(x_n, p) + (1 - γ_n)d(x_n, p) = d(x_n, p).
\]
This implies that \( \{x_n\} \) is bounded, and so are \( \{y_n\}, \{Tx_n\} \) and \( \{Ty_n\} \). Moreover, we have that the limit of \( d(x_n, p) \) exists. For any \( p ∈ F(T) \), we have that
\[
d^2(x_{n+1}, p) = d^2(γ_n x_n + (1 - γ_n)Ty_n, p)
\leq γ_n d^2(x_n, p) + (1 - γ_n)d^2(Ty_n, p) - γ_n(1 - γ_n)d^2(x_n, Ty_n)
\leq γ_n d^2(x_n, p) + (1 - γ_n)d^2(y_n, p) - γ_n(1 - γ_n)d^2(x_n, Ty_n)
\leq γ_n d^2(x_n, p) + (1 - γ_n)d^2(y_n, p),
\] (3.12)
and
\[
d^2(y_n, p) = d^2(σ_n x_n + (1 - σ_n)Tx_n, p)
\leq σ_n d^2(x_n, p) + (1 - σ_n)d^2(Tx_n, p) - σ_n(1 - σ_n)d^2(x_n, Tx_n)
\leq σ_n d^2(x_n, p) + (1 - σ_n)d^2(x_n, p) - σ_n(1 - σ_n)d^2(x_n, Tx_n)
\leq d^2(x_n, p) - σ_n(1 - σ_n)d^2(x_n, Tx_n).
\] (3.13)
Hence
\[
d^2(x_{n+1}, p) ≤ γ_n d^2(x_n, p) + (1 - γ_n)(d^2(x_n, p) - σ_n(1 - σ_n)d^2(x_n, Tx_n))
= γ_n d^2(x_n, p) + (1 - γ_n)d^2(x_n, p) - (1 - γ_n)σ_n(1 - σ_n)d^2(x_n, Tx_n)
= d^2(x_n, p) - (1 - γ_n)σ_n(1 - σ_n)d^2(x_n, Tx_n),
\]
that is,
\[ (1 - γ_n)σ_n(1 - σ_n)d^2(x_n, Tx_n) ≤ d^2(x_n, p) - d^2(x_{n+1}, p). \]
Since \( 0 < a ≤ γ_n, σ_n ≤ b < 1 \), we have that
\[ a(1 - b)^2d^2(x_n, Tx_n) ≤ (1 - γ_n)σ_n(1 - σ_n)d^2(x_n, Tx_n) ≤ d^2(x_n, p) - d^2(x_{n+1}, p). \]
It follows that
\[ \lim_{n \to \infty} d^2(x_n, Tx_n) = 0. \]
By the same argument as in the proof of Theorem 3.3, \( \{x_n\} \) \( \Delta \)-converges to an element \( u \) in \( F(T) \). \( \square 

**Theorem 3.6.** Let \( (X, d) \) be a complete CAT(0) space and \( C \) be a nonempty, closed and convex subset of \( X \). Let \( T : C \to C \) be an \((\alpha, \beta)\)-generalized hybrid mapping such that \( \alpha \geq 1 \) and \( \beta \geq 0 \) with \( F(T) \neq \emptyset \). Let \( \{\gamma_n\} \) be a sequence of real numbers with \( 0 < a \leq \gamma_n \leq b < 1 \) and a \( \{x_n\} \) generated by (3.11). Then \( \{x_n\} \) converges strongly to an element \( u \in F(T) \) if and only if \( \liminf_{n \to \infty} d(x_n, F(T)) = 0 \), where
\[ d(x, F(T)) = \inf_{p \in F(T)} d(x, p). \]

**Proof.** We observe that the necessity is obvious. Next, we prove the sufficiency. As proved in Theorem 3.5, we have \( d(x_{n+1}, p) \leq d(x_n, p) \), for all \( p \in F(T) \). This implies that
\[ d(x_{n+1}, F(T)) \leq d(x_n, F(T)), \]
and so the limit of \( d(x_n, F(T)) \) exists. It follows by our assumption that \( \lim_{n \to \infty} d(x_n, F(T)) = 0 \).

We claim that \( \{x_n\} \) is a Cauchy sequence in \( C \). Since \( \lim_{n \to \infty} d(x_n, F(T)) = 0 \), for any \( \varepsilon > 0 \) there exists \( N \geq 1 \) such that
\[ d(x_n, F(T)) < \frac{\varepsilon}{4}, \forall n \geq N. \]
Therefore \( d(x_N, F(T)) < \frac{\varepsilon}{4} \). From the definition of \( d(x_N, F(T)) \), there exists \( q \in F(T) \) such that \( d(x_N, q) < \frac{\varepsilon}{4} \). For any \( m, n \geq N \geq 1 \), we have
\[ d(x_n, x_m) \leq d(x_n, q) + d(x_m, q) \leq 2d(x_N, q) < 2\frac{\varepsilon}{4} = \varepsilon. \]
Hence \( \{x_n\} \) is a Cauchy sequence in a closed subset of a complete CAT(0) space. Hence \( \{x_n\} \) converges to some \( p^* \in C \). Since \( \lim_{n \to \infty} d(x_n, F(T)) = 0 \), then \( p^* \in F(T) \). This completes the proof. \( \square 

**Remark 3.7.** Consider \( \mathbb{R}^2 \) with the usual Euclidean meter \( d(\cdot, \cdot) \) and \( \| \cdot \| \) are defined by
\[ d(x, y) = \|x - y\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}, \]
where \( x = (x_1, x_2) \) and \( y = (y_1, y_2) \). We define the radial metric \( d_r \) by
\[ d_r(x, y) = \begin{cases} d(x, y), & \text{if } y = tx \\ d(x, 0) + d(y, 0), & \text{otherwise.} \end{cases} \]
Then \( X := (\mathbb{R}^2, d_r) \) is an \( \mathbb{R} \)-tree with the radial meter \( d_r \) (see [12] and [19, page 65]). We show that \( X \) is not inner product space. We prove this by showing that the norm does not satisfy the parallelogram equality. Indeed, if we take \( x = (0, 1) \), \( y_1 = (-1, 0) \), \( y_2 = (1, 0) \) and \( y_0 = (0, 0) \), a midpoint of \( y_1 \) and \( y_2 \), then
\[ 1 = d_r^2(x, y_0) < \frac{1}{2} d_r^2(x, y_1) + \frac{1}{2} d_r^2(x, y_2) - \frac{1}{4} d_r^2(y_1, y_2) = \frac{1}{2} + \frac{1}{2} - \frac{1}{4} = 3. \]
Therefore \( X \) does not satisfy Parallellogram law, and so \( X \) is not an inner product space.

The following is an example of a \((\alpha, \beta)\)-generalized hybrid mapping in CAT(0) space which is not a nonexpansive mapping.

**Example 3.8.** Let \( X \) be an \( \mathbb{R} \)-tree with the radial meter \( d_r \). We put
\[ C = \left\{ (t, 0) : t \in [0, 2] \cup \left[ 4, 5 \frac{1}{2} \right] \right\} \cup \left\{ (0, t) : t \in [0, 2] \cup \left[ 4, 5 \frac{1}{2} \right] \right\} \subset \mathbb{R}^2 \]
and define \( T : C \to C \) by
\[ T(t, 0) = \begin{cases} (0, 0), & \text{if } t \in [0, 2]; \\ \left( \frac{t}{2}, \frac{t}{2} \right), & \text{if } t \in \left[ 4, 5 \frac{1}{2} \right], \end{cases} \]
and \( T(0, t) = \begin{cases} (0, 0), & \text{if } t \in [0, 2]; \\ \left( \frac{t}{2}, \frac{t}{2} \right), & \text{if } t \in \left[ 4, 5 \frac{1}{2} \right]. \end{cases} \]
Clearly, \( F(T) = \{(0, 0)\} \).

Claim that \( T \) is \((2,1)\)-generalized hybrid mapping, i.e.,
\[ 2d_r^2(Tx, Ty) \leq d_r^2(Tx, y) + d_r^2(x, Ty). \]
Case 1. \(x = (s, 0), y = (t, 0)\).
If \(s, t \in [0, 2]\), then we have done.

If \(s, t \in [4, 5\frac{1}{2}]\), then \(Tx = \left(0, \frac{(s-4)^2}{2}\right)\) and \(Ty = \left(0, \frac{(t-4)^2}{2}\right)\), and so

\[
2d_r^2(Tx, Ty) = 2 \left(\frac{(s-4)^2}{2} - \frac{(t-4)^2}{2}\right)^2 \leq \left(\frac{9}{8}\right)^2 + \left(\frac{9}{8}\right)^2
\]

\[
\leq 4^2 + 4^2
\]

\[
\leq \left(\frac{(s-4)^2}{2} + t\right)^2 + \left(s + \frac{(t-4)^2}{2}\right)^2 = d_r^2(Tx, y) + d_r^2(x, Ty).
\]

If \(s \in [0, 2], t \in [4, 5\frac{1}{2}]\), then \(Tx = (0, 0)\) and \(Ty = \left(0, \frac{(t-4)^2}{2}\right)\), and so

\[
2d_r^2(Tx, Ty) = 2 \left(\frac{(t-4)^2}{2}\right)^2 \leq \left(\frac{(t-4)^2}{2}\right)^2 + \left(s + \frac{(t-4)^2}{2}\right)^2
\]

\[
\leq (t)^2 + \left(s + \frac{(t-4)^2}{2}\right)^2 = d_r^2(Tx, y) + d_r^2(x, Ty).
\]

If \(s \in [4, 5\frac{1}{2}], t \in [0, 2]\), then \(Tx = (0, 0)\) and \(Ty = (0, 0)\), and so

\[
2d_r^2(Tx, Ty) = 2 \left(\frac{(s-4)^2}{2}\right)^2 = \left(\frac{(s-4)^2}{2}\right)^2 + \left(\frac{(s-4)^2}{2}\right)^2
\]

\[
\leq \left(t + \frac{(s-4)^2}{2}\right)^2 + \left(s + \frac{(t-4)^2}{2}\right)^2 = d_r^2(Tx, y) + d_r^2(x, Ty).
\]

Case 2. \(x = (s, 0), y = (0, t)\).
If \(s, t \in [0, 2]\), then we have done.

If \(s, t \in [4, 5\frac{1}{2}]\), then \(Tx = \left(0, \frac{(s-4)^2}{2}\right)\) and \(Ty = \left(\frac{(t-4)^2}{2}, 0\right)\), and so

\[
2d_r^2(Tx, Ty) = 2 \left(\frac{(s-4)^2}{2} + \frac{(t-4)^2}{2}\right)^2 \leq \left(\frac{9}{4}\right)^2 + \left(\frac{9}{4}\right)^2
\]

\[
\leq \left(4 - \frac{9}{8}\right)^2 + \left(4 - \frac{9}{8}\right)^2
\]

\[
\leq \left(4 - \frac{(s-4)^2}{2}\right)^2 + \left(4 - \frac{(t-4)^2}{2}\right)^2
\]

\[
\leq \left(t - \frac{(s-4)^2}{2}\right)^2 + \left(s - \frac{(t-4)^2}{2}\right)^2 = d_r^2(Tx, y) + d_r^2(x, Ty).
\]

If \(s \in [0, 2], t \in [4, 5\frac{1}{2}]\), then \(Tx = (0, 0)\) and \(Ty = \left(\frac{(t-4)^2}{2}, 0\right)\), and so

\[
2d_r^2(Tx, Ty) = 2 \left(\frac{(t-4)^2}{2}\right)^2 = \left(\frac{(t-4)^2}{2}\right)^2 + \left(\frac{(t-4)^2}{2}\right)^2
\]

\[
\leq \left(t + \frac{9}{8}\right)^2
\]

\[
\leq \left(t + \frac{4}{8}\right)^2
\]

\[
\leq \left(t + \frac{(t-4)^2}{2}\right)^2
\]

\[
\leq \left(t + \frac{(s-4)^2}{2}\right)^2 = d_r^2(Tx, y) + d_r^2(x, Ty).
\]
If \( s \in [4, 5\frac{1}{2}] \), \( t \in [0, 2] \), then 
\[
T x = \left( 0, \frac{(s - 4)^2}{2} \right) \quad \text{and} \quad T y = (0, 0),
\]
and so
\[
2d_r^2(T x, T y) = 2 \left( \frac{(s - 4)^2}{2} \right)^2 = \left( \frac{(s - 4)^2}{2} \right)^2 + \left( \frac{(s - 4)^2}{2} \right)^2
\leq \left( \frac{9}{8} \right)^2 + (s)^2
\leq \left( 4 - \frac{9}{8} \right)^2 + (s)^2
\leq \left( 4 - \frac{(s - 4)^2}{2} \right)^2 + (s)^2
\leq \left( t - \frac{(s - 4)^2}{2} \right)^2 + (s)^2 = d_r^2(T x, y) + d_r^2(x, T y).
\]

**Case 3.** \( x = (0, s), y = (0, t) \) the proof is similarly to case 1.

**Case 4.** \( x = (0, s), y = (t, 0) \) the proof is similarly to case 2.

Hence we have the claim. By the same proof, we can conclude that \( T \) is also \((1, 1)\)-generalized hybrid.

But \( T \) is not nonexpansive. Indeed, if \( x = (0, 5\frac{1}{2}), y = (0, 5\frac{1}{2}) \), then \( x = (\frac{25}{32}, 0), y = (\frac{5}{8}, 0) \). Thus, we have
\[
d_r(T x, T y) = \frac{25}{32} - \frac{9}{8} = \frac{11}{32} > \frac{1}{4} = \frac{1}{4} - \frac{5}{2} = d_r(x, y).
\]

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**References**


