Convolutions with Hypergeometric Functions

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Abstract. In this paper we study the behaviour of \( I_{a,b,c}(f) (z) = z^F(a,b,c,z) \ast f(z) \) where \( F(a,b,c,z) \) is the Gaussian Hypergeometric function and the * is usual Hadamard product. In the main result, we find conditions on \( a,b,c,A,B \) and \( \beta \) so that \( I_{a,b,c}(f)(z) \) belong to \( S^*[A,B] \) whenever \( f(z) \in R(\beta), \beta < 1 \).

1. Introduction

Let \( A \) denote the family of functions \( f(z)=z+\sum_{n=2}^{\infty} a_n z^n \) that are analytic in the interior of unit disk \( \Delta = \{ z \in C : |z| < 1 \} \). Let \( g \) be analytic and univalent in \( \Delta \) and \( f \) be analytic in \( \Delta \), then \( f(z) \) is said to be subordinate to \( g(z) \), written \( f \prec g \) if \( f(0)=g(0) \) and \( f(\Delta) \subset g(\Delta) \).

For \( -1 \leq B < A \leq 1 \), let
\[
S^*[A,B] = \left\{ f \in A \mid \frac{zf'(z)}{f(z)} < \frac{1 + Az}{1 + Bz}, \ z \in \Delta \right\}
\]

For \( A=1, B=-1 \) we get the well known family \( S^* \) of starlike functions. We further get \( S[1-2\gamma,-1] = S^*(\gamma) \) and \( S^*(\gamma,0) = S^*_\gamma \). For \( \beta < 1 \), define
\[
R(\beta) = \{ f \in A \mid \exists \ \theta \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) / \text{Re} \left[ e^{i\theta} (f'(z) - \beta) \right] > 0, \ z \in \Delta \}.
\]

Note that when \( \beta \geq 0 \), we have \( R(\beta) \subset S \), the class of univalent functions in \( A \). For each \( \beta < 0 \), \( R(\beta) \) contains also nonunivalent functions.
For any complex number \( a \) we define the ascending factorial notation \((a, n) = a(a+1)\cdots(a+n-1)\) for \( n \geq 1 \) and \((a, 0) = 1\) for \( a \neq 0 \). The triangle inequality for \((a, n)\) is \(|(a, n)| \leq (|a|, n)\). When \( 'a' \) is neither zero nor a negative integer, we can write \((a, n) = \Gamma(n+a)/\Gamma(a)\).

The Gaussian hypergeometric function is defined as

\[
F(a, b, c, z) = \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{c, (n)(1, n)} z^n, \quad a, b, c \in C
\]

where \( c \) is neither zero nor a negative integer. The following well known formula

\[
F(a, b, c, 1) = \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}, \quad \text{Re}(c-a-b) > 0
\]

will be used frequently. Univalence, starlikeness and convexity properties of \( zF(a, b, c, z) \) have been studied in [6] and [8].

For \( f \in A \), we consider the Hohlov convolution operator [2] \( I_{a, b, c}(f) \) given by

\[
[I_{a, b, c}(f)](z) = zF(a, b, c, z) \ast f(z)
\]

where \( \ast \) stands for the usual Hadamard product of power series. For \( \text{Re} c > \text{Re} b > 0 \), it is known that

\[
F(a, b, c, z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \frac{dt}{(1-tz)^a}.
\]

We can write

\[
[I_{a, b, c}(f)](z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \frac{f(tz)}{t} \frac{dt}{(1-tz)^a} \ast \frac{z}{(1-z)^a}.
\]

This operator reduces to Bernardi operator

\[
B_f(z) = (1+\gamma) \int_0^1 t^{\gamma-1} f(tz) dt
\]

for \( a = 1, b = 1 + \gamma \) and \( c = 2 + \gamma \) with \( \text{Re} \gamma > -1 \). For \( \gamma = 1 \) and 2, respectively we get Alexander transform and Libera transform. These three operators are all examples of the situation where \( c = a + b \) in \( I_{a, b, c}(f) \). Also we have
\[
\frac{z}{(1-z)^{n+1}} \ast f(z) = [I_{n+1, f}(f)](z), \quad n > -1
\]
which is known as Ruscheweyh differential, studied in [7]. It represents the case \( c < a + b \) with \( a = 1, \ b = n+1 \) and \( c = 1 \). Some more special cases of the operator \( I_{a,b,c} (f) \) can be found in [10].

P.T. Mocanu [3] obtained the range for \( \gamma \) so that the Bernardi operator \( B_f \in S^* \) whenever \( f \in R(0) \). As a natural extension, here we determine conditions on \( A, B, a, b, c \) and \( \beta \), the transform by the hypergeometric function \( F(a, b, c, z) \) on the class \( R(\beta) \) so that \( I_{a,b,c}(f) \in S^*[A,B] \).

2. Auxiliary lemmas

We shall state the following Lemmas [4] which may be used in proving the main theorems.

**Lemma 2.1.** Let \( a, b, c > 0 \). Then

(i) for \( c > a + b + 1 \),

\[
\sum_{n=0}^{\infty} \frac{(n+1) (a,n) (b,n)}{(c,n) (1,n)} = \frac{\Gamma(c-a-b) \Gamma(c)}{\Gamma(c-a) \Gamma(c-b)} \left[ \frac{ab}{c-1-a-b} + 1 \right]
\]

(ii) for \( c > a + b + 2 \),

\[
\sum_{n=0}^{\infty} \frac{(n+1)^2 (a,n) (b,n)}{(c,n) (1,n)} = \frac{\Gamma(c-a-b) \Gamma(c)}{\Gamma(c-a) \Gamma(c-b)} \left[ 1 + \frac{(a,2) (b,2)}{(c-2-a-b,2)} + \frac{3ab}{c-1-a-b} \right].
\]

**Lemma 2.2.** Let \( a, b, c > 0 \) and for \( a \neq 1, b \neq 1, c \neq 1 \) with \( c > \max \{0, a+b-1\} \),

\[
\sum_{n=0}^{\infty} \frac{(a,n) (b,n)}{(c,n) (1,n+1)} = \frac{1}{(a-1) (b-1)} \left[ \frac{\Gamma(c+1-a-b) \Gamma(c)}{\Gamma(c-a) \Gamma(c-b)} - (c-1) \right].
\]

**Lemma 2.3.** Let \( a, b, c > 0 \). For \( b \neq 1 \) and \( c > 1 + b \),

\[
\sum_{n=0}^{\infty} \frac{(b,n) 1}{(c,n) (n+1)} = \frac{c-1}{(b-1)} \left( \psi(c-1) - \psi(c-b) \right)
\]

where \( \psi(x) = \Gamma'(x) / \Gamma(x) \).
3. Main theorems

Now let us study the action of the hypergeometric function on the classes $R(\beta)$ and $S$.

**Theorem 3.1.** Let $a, b \in C \setminus \{0\}$, $|a| \neq 1$, $|b| \neq 1$, $c \neq 1$ and $c > |a| + |b|$. For $-1 \leq B < A \leq 1$, assume that

$$
\frac{\Gamma(c - |a| - |b|) \Gamma(c)}{\Gamma(c - |a|) \Gamma(c - |b|)} \left\{ (1 - B) - \frac{(1 - A) (c - |a| - |b|)}{(|a| - 1) (|b| - 1)} \right\} 
\leq (A - B) \left\{ 1 + \frac{1}{2(1 - \beta)} \right\} - \frac{(1 - A) (c - 1)}{(|a| - 1) (|b| - 1)} 
$$

Then the operator $I_{a,b,c}(f)$ maps $R(\beta)$ into $S^*[A,B]$.

**Proof.** Let $a, b \in C \setminus \{0\}$ and $c > |a| + |b|$, $|a| \neq 1$, $|b| \neq 1$ and $c \neq 1$. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be a function in $R(\beta)$. Then, it is well-known that $|a_n| \leq \frac{2(1 - \beta)}{n}$. Consider $zF(a,b,c,z)_* f(z) = z + \sum_{n=2}^{\infty} B_n z^n$ where $B_1 = 1$ and for $n \geq 1$,

$$
B_n = \frac{(a,n-1)(b,n-1)}{(c,n-1)(1,n-1)} a_n
$$

A special case of Theorem 3 in [1] gives a sufficient condition for $f \in S^*[A,B]$ is that

$$
\sum_{n=2}^{\infty} \{ n(1 - B) - (1 - A) \} |a_n| \leq A - B.
$$

Then we have to show that

$$
T = \sum_{n=2}^{\infty} \{ n(1 - B) - (1 - A) \} |B_n| \leq A - B.
$$

We have

$$
T \leq \sum_{n=2}^{\infty} \{ n(1 - B) - (1 - A) \} \frac{(|a|, n-1)(|b|, n-1)}{(c,n-1)(1,n-1)} \frac{2(1 - \beta)}{n} 
= 2(1 - \beta) \left\{ (1 - B) \sum_{n=1}^{\infty} \frac{(|a|, n)(|b|, n)}{(c,n)(1,n)} - (1 - A) \sum_{n=1}^{\infty} \frac{(|a|, n)(|b|, n)}{(c,n)(1,n+1)} \right\} = T_1
$$
Using the formula (1.1) and Lemma 2.2, we observe that

\[
T_1 = 2(1 - \beta) \left\{ (1 - B) \frac{\Gamma(c - |a - b|)}{\Gamma(c - |a|)} \frac{\Gamma(c)}{\Gamma(c - |b|)} - (1 - B) \right. \\
- (1 - A) \frac{\Gamma(c - |a - b|)}{\Gamma(c - |a|)} \frac{\Gamma(c) (c - |a - b|)}{\Gamma(c - |b|)(|a| - 1)(|b| - 1)} + \frac{(1 - A)(c - 1)}{(|a| - 1)(|b| - 1)} + (1 - A) \left. \right\}
\]

\[
= 2(1 - \beta) \left\{ \frac{\Gamma(c - |a - b|)}{\Gamma(c - |a|)} \frac{\Gamma(c) (c - |a - b|)}{\Gamma(c - |b|)(|a| - 1)(|b| - 1)} \left[ (1 - B) - \frac{(1 - A)(c - |a - b|)}{(|a| - 1)(|b| - 1)} \right] \\
+ \frac{(1 - A)(c - 1)}{(|a| - 1)(|b| - 1)} - (A - B) \right\}.
\]

Then under the hypothesis (3.1) of the theorem we get

\[
T \leq T_1 \leq 2 (1 - \beta) \frac{(A - B)}{2(1 - \beta)} = A - B,
\]

thereby showing that \( f \in S^* [A, B] \).

**Note.** For \( A = \lambda, B = 0 \) we get, as a special case, Theorem 2.1 of [4].

**Theorem 3.2.** Let \( b \in C \setminus \{0\}, c > 0, |b| \neq 1 \) and \( c > 1 + |b| \). For \( -1 \leq B < A \leq 1 \), assume that

\[
\frac{(1 - B)(c - 1)}{(c - |b| - 1)} - (1 - A) \left( \frac{c - 1}{b - 1} \right) (\psi(c - 1) - \psi(c - |b|)) \leq \frac{A - B}{2(1 - \beta)} + (A - B) \quad (3.3)
\]

where \( \psi(x) = \Gamma'(x)/\Gamma(x) \). Then the operator \( I_{1,b,c}(f) \) maps \( R(\beta) \) into \( S^* [A, B] \).

**Proof.** Putting \( a = 1 \) in (3.2) we get

\[
T_1 = 2(1 - \beta) \left\{ (1 - B) \sum_{n=1}^\infty \frac{|b|, n}{(c, n)} - (1 - A) \sum_{n=1}^\infty \frac{|b|, n}{(c, n)(n + 1)} \right\}.
\]

Using (1.1) and Lemma 2.3, we get

\[
T_1 = 2(1 - \beta) \left\{ \frac{(1 - B)(c - 1)}{(c - |b| - 1)} - (1 - A) \left( \frac{c - 1}{b - 1} \right) (\psi(c - 1) - \psi(c - |b|)) \right\} - (A - B) \right\}.
\]
Thus under the hypothesis (3.3) of the theorem we get \( T \leq T_1 \leq (A - B) \), there by showing that the operator \( I_{1,b,c}(f) \) maps \( R(\beta) \) into \( S^*[A,B] \).

**Note.** For \( A = \lambda, \ B = 0 \), we get as a special case, Theorem 2.2. of [4].

From the proof of Theorems 3.1 and 3.2, we observe that for \( A = 1, \ B = 0 \). We need not treat the case \( a = 1 \) separately neither we need the aestrictions \( b \neq 1 \) and \( c \neq 1 \). In this case, we have the following result.

**Corollary 3.3.** Let \( a,b \in \mathbb{C} \setminus \{0\} \) and \( c > |a| + |b| \). Assume that

\[
\frac{\Gamma(c-a) \left| -b \right| \Gamma(c)}{\Gamma(c-a) \left| -b \right|} \leq 1 + \frac{1}{2(1-\beta)}.
\]

Then the operator \( I_{a,b,c}(f) \) maps \( R(\beta) \) into \( S^*[1, 0] \).

Let \( \pi : [0, 1] \to \mathbb{R} \) be a nonnegative function normalized so that \( \int_0^1 \pi(t)dt = 1 \) and define

\[
[V_{\pi}(f)](z) = \int_0^1 \pi(t) \frac{f(tz)}{t} dt, \quad f \in A.
\]

Let \( \Pi(t) = \int_0^1 \pi(s) \frac{ds}{s} \) and assume that \( \Pi(t) \to 0 \) when \( t \to 0^+ \). It is shown in [9] that the class \( S^*[A,B] \), \(-1 \leq B < A \leq 1 \) can be characterized in terms of convolutions that

\[
f \in S^*[A,B] \Leftrightarrow \frac{f(z)}{z} * \frac{h_{(A,B)}(z)}{z} \neq 0
\]

where

\[
h_{(A,B)}(z) = \frac{z - \frac{A - x}{A - B}}{(1 - z)^2}; \quad |x| = 1.
\]

Choose \( G(t) = \frac{(A-B) - (1-A)t}{(A-B)(1+t)^2} \).

From \( tg'(t) + g(t) + 1 = 2G(t) \), we get

\[
g(t) = \frac{2(1-B) - (A-B)(1+t)}{(A-B)(1+t)} - \frac{2(1-A)}{(A-B)} \log(1+t) - \frac{t}{t}
\]

An application of Theorem 2.1 in [5] gives the following result.
Theorem 3.4. Let $\beta$ be given by

$$\frac{\beta}{1-\beta} = -i \int_0^1 \pi(t) \left[ \frac{2(1-B) - (A-B)(1+t)}{(A-B)(1+t)} - \frac{2(1-A)}{(A-B)} \log(1+t) \right] dt .$$

Then,

$$V_\pi(R(\beta)) \subset \mathcal{S}^*[A,B] \Leftrightarrow L_{\Pi}(e^{-i\theta} h_{(A,B)}(e^{i\theta} z)) \geq 0, \quad z \in \Delta$$

Where

$$L_{\Pi}(h) = \inf_{z \in \Delta} \int_0^1 \Pi(t) \left[ \text{Re} \left( \frac{h(tz)}{iz} \right) - \frac{(A-B) + (A-1)t}{(A-B)(1+t)^2} \right] dt .$$

Note. The operator $I_{1,b,c}(f)$ corresponds to $V_\pi(f)$ with $\pi(t) = \pi_{b,c}(t)$

$$= \frac{\Gamma(c) \Gamma(b) \Gamma(c-b)}{\Gamma(c-b)} t^{b-1} (1-t)^{c-b-1} \text{ where } \int_0^1 \pi_{b,c}(t) dt = 1. \text{ The cases } A=\lambda, B=0 \text{ and } A=1-2\gamma, B=-1 \text{ were treated in [4] and [5] respectively.}$$

Next we determine the condition on $a, b, c$ and $A, B$ when $f(z)$ is in $S$ instead of $f(z) \in R(\beta)$.

Theorem 3.5. Let $a, b, c \in C \setminus \{0\}, \ c > 2 + |a| + |b|$. Suppose that $a, b$ and $-1 \leq B < A \leq 1$ satisfy the condition that

$$\frac{\Gamma(c-a-b) \Gamma(c)}{\Gamma(c-a) \Gamma(c-b)} \left[ (1-B) \frac{a(a+1) b(b+1)}{(c-2-a-b)(c-1-a-b)} + (A+2-3B) \frac{|ab|}{c-1-a-b} + (A-B) \right] \leq 2 (A-B) \quad (3.4)$$

Then the operator $I_{a,b,c}(f)$ maps $S$ into $S^*[A,B]$.

Proof. Let $a \in C \setminus \{0\}, \ c > 2 + |a| + |b| \text{ and } -1 \leq B < A \leq 1$. Let $f(z) = z + \sum_{n=2}^\infty a_n z^n \in S$. Then we have that $|a_n| \leq n$. Consider $zF(a,b,c,z) \ast f(z) = z + \sum_{n=2}^\infty B_n z^n$ where

$$B_n = \frac{(a,n-1)(b,n-1)}{(c,n-1)(1,n-1)} a_n .$$
It is enough to show that

$$T = \sum_{n=2}^{\infty} \{ n(1-B) - (1-A) \} \left| B_n \right| \leq A - B.$$ 

We have

$$T = \sum_{n=2}^{\infty} \{ n(1-B) - (1-A) \} \frac{(a,n-1)(b,n-1)}{(c,n-1)(1,n-1)} \left| a_n \right|$$

$$\leq \sum_{n=2}^{\infty} \{ (n+1)^2 (1-B) - (n+1)(1-A) \} \frac{(a,n)(b,n)}{(c,n)(1,n)}$$

$$= (1-B) \sum_{n=1}^{\infty} \frac{(n+1)^2 (a,n)(b,n)}{(c,n)(1,n)} - (1-A) \sum_{n=1}^{\infty} \frac{(n+1)(a,n)(b,n)}{(c,n)(1,n)} := T_2$$

From Lemma 2.1. we get

$$T_2 = \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)} \left[ (1-B) + \frac{(1-B)(a,2)(b,2)}{(c-2-a-b,2)} + \frac{3(1-B)}{c-1-a-b} \left| ab \right| \right.$$ 

$$- \frac{(1-A)}{c-1-a-b} - (1-A) \right] - (1-B) + (1-A)$$

$$= \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)} \left[ (1-B) \frac{a(a+1)b(b+1)}{(c-2-a-b)(c-1-a-b)} + \frac{A}{c-1-a-b} \right.$$ 

$$- \frac{ab}{c-1-a-b} \left| ab \right| + \frac{3(1-B)}{c-1-a-b} \left| ab \right| + (A-B) \right] - (A-B)$$

$$= \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)} \left[ (1-B) \frac{a(a+1)b(b+1)}{(c-2-a-b)(c-1-a-b)} + \frac{A}{c-1-a-b} \right.$$ 

$$+ \frac{(A+2-3B)}{c-1-a-b} \left| ab \right| + (A-B) \right] - (A-B) .$$

Then, under the hypothesis (3.4) of the theorem we get $T \leq T_2 \leq A - B$. Therefore the operator $I_{a,b,c}(f)$ maps $S$ into $S^*[a,b]$.

**Note.** When $A = \lambda$, $B = 0$, this reduces to Theorem 2.6. in [4].
References


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