On Common Fixed Point Theorem of Four Mappings

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Abstract. In this paper we shall prove common fixed point theorems for four mappings in complete metric space. Our theorems generalize results of Banach [1], Kannan [5], Fisher [4] and Chatterjee [2].

1. Definitions

Definition 1. A sequence \( \{x_n\} \) in a metric space \((X,d)\) is said to be convergent to a point \(x\) in \(X\) if

\[
\lim_{n \to \infty} d(x_n, x) = 0 \quad \text{for all } x \in X.
\]

Then \(x\) is called the limit of the sequence \(\{x_n\}\) in \(X\).

Definition 2. A sequence \(\{x_n\}\) in a metric space \((X,d)\) is said to be Cauchy sequence if

\[
\lim_{m,n \to \infty} d(x_m, x_n) = 0 \quad \text{for all } x \in X.
\]

Then \(x\) is called the limit of the sequence \(\{x_n\}\) in \(X\).

Definition 3. A metric space \((X,d)\) is said to be complete if every Cauchy sequence in \(X\) is convergent.

Definition 4. [3] Let \(A\) and \(S\) be mappings from a metric space \((X,d)\) into itself. Then \(A\) and \(S\) are said to be compatible of type (A) if

\[
\lim_{n \to \infty} d(ASx_n, SSx_n) = 0
\]
and

\[ \lim_{n \to \infty} d(SA_n, AA_n) = 0 \]

whenever \( \{x_n\} \) is a sequence in \( X \) such that

\[ \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Ax_n = z \quad \text{for some } z \in X. \]

The object of this paper is to prove following theorems:

**Theorem 1.** Let \( A, B, S \) and \( T \) be four mappings of complete metric space \( X \) into itself satisfying:

\[ d(Ax, By) \leq \alpha_1 \left[ \frac{d(Ty, By)d(Sx, Ty)}{d(Tx, Ax) + d(By, Tx)} \right] + \alpha_2 [d(Ax, Tx) + d(Sx, Bx)] \]

\[ + d(Ay, Sy) + \alpha_3 [d(Tx, Bx) + d(Sy, Tx) + d(By, Ty)] \]

\[ + \alpha_4 [d(Sx, Ty) + d(Tx, By)]. \]  

(1.1)

the pairs \( A, S \) and \( B, T \) are compatible of type \( (A) \), \n
(1.2)

one of \( A, B, S \) and \( T \), is continuous, \n
(1.3)

for all \( x, y \) in \( X \), where \( \alpha_i \geq 0 \) and \( \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 < 1. \) Then \( A, B, S \) and \( T \) have a unique common fixed point in \( X. \)

**Proof.** Let \( x_0 \) be an arbitrary point of \( X \). We define

\[ Ax_{2n+1} = y_{2n+2}, \quad Tx_{2n} = y_{2n}, \]

\[ Bx_{2n} = y_{2n+1}, \quad Sx_{2n+1} = y_{2n+1}, \quad n = 1, 2, \ldots \]

By putting \( x = x_{2n} \) and \( y = x_{2n+1} \) in (1.1), we write
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\[
d(Ax_{2n}, Bx_{2n+1}) \leq \alpha_1 \left[ \frac{d(Tx_{2n}, Bx_{2n})}{d(Tx_{2n}, Ax_{2n}) + d(Bx_{2n}, Tx_{2n})} \right] + \alpha_2 \left[ d(Ax_{2n}, Tx_{2n}) + d(Sx_{2n}, Tx_{2n}) + d(Ax_{2n}, Bx_{2n}) + d(Sx_{2n}, Bx_{2n}) \right] + \alpha_3 \left[ d(Tx_{2n}, Bx_{2n}) + d(Sx_{2n}, Bx_{2n}) + d(Ax_{2n}, Bx_{2n}) + d(Sx_{2n}, Bx_{2n}) \right] + \alpha_4 \left[ d(Sx_{2n}, Tx_{2n}) + d(Bx_{2n}, Tx_{2n}) \right]
\]

\[
= \alpha_1 \left[ \frac{d(y_{2n+1}, y_{2n+2})}{d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n})} \right] + \alpha_2 \left[ d(y_{2n+1}, y_{2n}) + d(y_{2n+2}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2}) + d(y_{2n+2}, y_{2n+1}) \right] + \alpha_3 \left[ d(y_{2n+1}, y_{2n}) + d(y_{2n+2}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2}) + d(y_{2n+2}, y_{2n+1}) \right] + \alpha_4 \left[ d(y_{2n+1}, y_{2n+2}) + d(y_{2n+2}, y_{2n+1}) \right]
\]

\[
d(y_{2n+1}, y_{2n+2}) \leq (\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4) d(y_{2n}, y_{2n+1}) + (\alpha_2 + \alpha_3 + \alpha_4) d(y_{2n+1}, y_{2n+2})
\]

Putting \( h = \frac{(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4)}{(1-\alpha_2 - \alpha_3 - \alpha_4)} \), we find \( h < 1 \), since \( \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 < 1 \). Hence

\[
d(y_{2n+1}, y_{2n+2}) \leq h d(y_{2n}, y_{2n+1}).
\]

Similarly by putting \( x = x_{2n-1} \) and \( y = x_{2n} \) in (1.1), we have

\[
d(Ax_{2n-1}, Bx_{2n}) \leq \alpha_1 \left[ \frac{d(Tx_{2n}, Bx_{2n})}{d(Tx_{2n}, Ax_{2n}) + d(Bx_{2n}, Tx_{2n})} \right] + \alpha_2 \left[ d(Ax_{2n-1}, Tx_{2n}) + d(Sx_{2n-1}, Tx_{2n}) + d(Ax_{2n-1}, Bx_{2n}) + d(Sx_{2n-1}, Bx_{2n}) \right] + \alpha_3 \left[ d(Tx_{2n}, Bx_{2n}) + d(Sx_{2n}, Bx_{2n}) + d(Ax_{2n}, Bx_{2n}) + d(Sx_{2n}, Bx_{2n}) \right] + \alpha_4 \left[ d(Sx_{2n}, Tx_{2n}) + d(Bx_{2n}, Tx_{2n}) \right]
\]

\[
= \alpha_1 \left[ \frac{d(y_{2n}, y_{2n+1})}{d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n-1})} \right] + \alpha_2 \left[ d(y_{2n-1}, y_{2n}) + d(y_{2n+1}, y_{2n}) + d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n}) \right] + \alpha_3 \left[ d(y_{2n-1}, y_{2n}) + d(y_{2n+1}, y_{2n}) + d(y_{2n}, y_{2n-1}) + d(y_{2n+1}, y_{2n}) \right] + \alpha_4 \left[ d(y_{2n-1}, y_{2n}) + d(y_{2n-1}, y_{2n+1}) \right]
\]
\[d(y_{2n}, y_{2n+1}) \leq (\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4) d(y_{2n-1}, y_{2n}) + (\alpha_2 + \alpha_3 + \alpha_4) d(y_{2n}, y_{2n+1}) \]
\[\leq \frac{(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4)}{(1-\alpha_2 - \alpha_3 - \alpha_4)} d(y_{2n-1}, y_{2n}) \]

\[d(y_{2n}, y_{2n+1}) \leq h \cdot d(y_{2n-1}, y_{2n}), \quad \text{as} \quad h = \frac{(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4)}{(1-\alpha_2 - \alpha_3 - \alpha_4)} \]

We find \( h < 1 \), since \((\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4) < 1 \). Proceeding in this way, we have

\[d(y_{2n}, y_{2n+1}) \leq h^{2n} d(y_0, y_1) \]

By routine calculations the following inequaities hold for \( k > n \)

\[d(y_n, y_{n+k}) \leq \sum_{i=1}^{k} d(y_{n+i-1}, y_{n+i}) \leq \sum_{i=1}^{k} h^{n+i-1} d(y_0, y_1) \leq \frac{h^n}{1-h} d(y_0, y_1) \to 0 \quad \text{as} \quad n \to \infty \]

Here \( h < 1 \). Hence \( \{y_n\} \) is a Cauchy sequence and by completeness of \( X \) we see that \( \{y_n\} \) is converges to a point \( z \) in \( X \). Since \( \{y_n\} \) is a Cauchy sequence and taking \( n \to \infty \), we write

\[A x_{2n} = T x_{2n+1} \to z \quad \text{and} \quad B x_{2n+1} = S x_{2n+2} \to z \]

Now, suppose \( A \) is continuous. Since \( A \) and \( S \) are compatible mappings of type \( (A) \), then

\[A A x_{2n} \quad \text{and} \quad S A x_{2n} \to A z \quad \text{as} \quad n \to \infty.\]

Now putting \( x = A x_{2n} \) and \( y = x_{2n+1} \) in (1.1), we write
\[ d(AAx_{2n}, Bx_{2n+1}) \leq \alpha_1 \left[ \frac{d(Tx_{2n+1}, Bx_{2n+1})}{d(TTx_{2n+1}, AAx_{2n}) + d(Bx_{2n+1}, TTx_{2n+1})} \right] + \alpha_2 \left[ d(AAx_{2n}, TTx_{2n+1}) + d(SA_x_{2n}, BTTx_{2n+1}) \right] + \alpha_3 \left[ d(Ax_{2n+1}, Sx_{2n+1}) + d(TTx_{2n+1}, BTTx_{2n+1}) \right] + \alpha_4 \left[ d(SAx_{2n}, Tx_{2n+1}) + d(TTTx_{2n+1}, Bx_{2n+1}) \right] \]

Taking the limit \( n \to \infty \), we write

\[ d(Az, z) \leq \alpha_2 d(Az, z) \]

giving a contradiction as \( \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 < 1 \).

Therefore \( Az = z \).

Similarly by putting \( x = Sx_{2n} \) and \( y = x_{2n+1} \) in (1.1), we write

\[ d(Ax_{2n}, Bx_{2n+1}) \leq \alpha_1 \left[ \frac{d(Tx_{2n+1}, Bx_{2n+1})}{d(Tx_{2n+1}, AAx_{2n}) + d(Bx_{2n+1}, TTx_{2n+1})} \right] + \alpha_2 \left[ d(Ax_{2n}, TTx_{2n+1}) + d(SSx_{2n}, BBx_{2n+1}) \right] + \alpha_3 \left[ d(Ax_{2n+1}, Sx_{2n+1}) + d(TT Tx_{2n+1}, BBx_{2n+1}) \right] + \alpha_4 \left[ d(SSx_{2n}, Tx_{2n+1}) + d(BTx_{2n+1}, Bx_{2n+1}) \right] \]

Taking the limit \( n \to \infty \), we write

\[ d(Sz, z) \leq (\alpha_2 + \alpha_4) d(Sz, z) \]

giving a contradiction as \( \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 < 1 \), therefore \( Sz = z \).

Similarly \( Bz = Tz = z \). Thus \( z \) is a common fixed point of \( A, B, S \) and \( T \).

For uniqueness let \( z \) and \( w = w(z) \neq w \) be two fixed points in \( X \) such that

\[ Az = Bz = Sz = Tz = z \quad \text{and} \quad Aw = Bw = Sw = Tw = w, \]
then by (1.1), we have

\[
d(Az, Bw) \leq \alpha_1 \left( \frac{d(Tw, Bw) \cdot d(Sz, Tw)}{d(Tz, Az) + d(Bw, Tz)} \right) + \alpha_2 \left[ d(Az, Tz) + d(Sz, Bz) + d(Aw, Sw) + d(Tz, Bz) + d(Tw, Sw) \right] + \alpha_4 \left[ d(Sz, Tw) + d(Tz, Bw) \right]
\]

which is a contradiction, since \( \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 < 1 \). Hence \( z = w \).

This implies the uniqueness of common fixed point of \( A, B, S \) and \( T \).

**Theorem 2.** Let \( A, B, S \) and \( T \) be four mappings of complete metric space \( X \) into itself and satisfying (1.2), (1.3), (1.4) and

\[
d(Ax, By) \leq \alpha_1 \left[ \frac{d(Tx, B^2y) \cdot d(Ty, Sx)}{d(Sx, A^2y)} \right] + \alpha_2 \left[ d(Tx, Ax) + d(Ty, By) \right] + \alpha_3 \left[ d(Tx, By) + d(Ty, Sx) + d(Ty, Bx) \right] + \alpha_4 \left[ d(Tx, Ty) + d(Ty, Bx) \right].
\]

Theorem 2 can be proved in the similar manner as Theorem 1.

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