On $\delta D$-Sets and Associated Weak Separation Axioms

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Abstract. Veličko [4] introduced the notions of $\delta$-open sets and $\delta$-closure. In this paper, we introduce some weak separation axioms by utilizing $\delta$-open sets and the $\delta$-closure operator.

1. Introduction

Throughout this paper, $(X, \tau)$ and $(Y, \sigma)$ (or $X$ and $Y$) denote topological spaces. A subset $A$ of a topological space $X$ is said to be regular open (resp. regular closed) if $A = \text{Int}(\text{Cl}(A))$ (resp. $A = \text{Cl}(\text{Int}(A))$), where $\text{Int}(A)$ and $\text{Cl}(A)$ the interior and the closure of a set $A$. A point $x \in X$ is called the $\delta$-cluster point of $A$ if $A \cap U \neq \emptyset$ for every regular open set $U$ of $X$ containing $x$. The set of all $\delta$-cluster points of $A$ is called the $\delta$-closure of $A$, denoted by $\text{Cl}_\delta(A)$. A subset $A$ is called $\delta$-closed if $A = \text{Cl}_\delta(A)$. The complement of a $\delta$-closed set is called $\delta$-open. We denote the collection of all $\delta$-open (resp. $\delta$-closed) sets by $\delta O(X, \tau)$ (resp. $\delta C(X, \tau)$). A set $U$ is a $\delta$-neighborhood of a point $x$ if $U$ is $\delta$-open such that $x \in U$.

**Lemma 1.1.** Intersection of arbitrary of $\delta$-closed sets in $(X, \tau)$ is $\delta$-closed.

In what follows, $(X, \tau)$ is a regular topological space.

**Corollary 1.2.** Let $A$ be a subset of a topological space $(X, \tau)$, $\text{Cl}_\delta(A) = \cap \{F \in \delta C(X, \tau) | A \subset F\}$.

**Corollary 1.3.** $\text{Cl}_\delta(A)$ is $\delta$-closed, that is $\text{Cl}_\delta(\text{Cl}_\delta(A)) = \text{Cl}_\delta(A)$. 
Lemma 1.4. For subsets \( A \) and \( A_i (i \in I) \) of a space \((X, \tau)\), the following hold:

1. \( A \subset \text{Cl}_\delta(A) \).
2. If \( A \subset B \), then \( \text{Cl}_\delta(A) \subset \text{Cl}_\delta(B) \).
3. \( \text{Cl}_\delta(\cap\{A_i : i \in I\}) \subset \cap\{\text{Cl}_\delta(A_i) : i \in I\} \)
4. \( \text{Cl}_\delta(\cup\{A_i : i \in I\}) = \cup\{\text{Cl}_\delta(A_i) : i \in I\} \).

2. \( \delta D \)-sets and associated separation axioms

Definition 1. A subset \( A \) of a topological space \( X \) is called a \( \delta D \)-sets if there are two \( U, V \in \delta O(X, \tau) \) such that \( U \neq X \) and \( A = U - V \).

Clearly every \( \delta \)-open set \( U \) different from \( X \) is a \( \delta D \)-set if \( A = U \) and \( V = \emptyset \).

Definition 2. A topological space \((X, \tau)\) is called \( \delta D_0 \) if for any distinct pair of points \( x \) and \( y \) of \( X \) there exists a \( \delta D \)-set of \( X \) containing \( x \) but not \( y \) or a \( \delta D \)-set of \( X \) containing \( y \) but not \( x \).

Definition 3. A topological space \((X, \tau)\) is called \( \delta D_1 \) if for any distinct pair of points \( x \) and \( y \) of \( X \) there exists a \( \delta D \)-set of \( X \) containing \( x \) but not \( y \) and a \( \delta D \)-set of \( X \) containing \( y \) but not \( x \).

Definition 4. A topological space \((X, \tau)\) is called \( \delta D_2 \) if for any distinct pair of points \( x \) and \( y \) of \( X \) there exists disjoint \( \delta D \)-sets \( G \) and \( E \) of \( X \) containing \( x \) and \( y \), respectively.

Definition 5. A topological space \((X, \tau)\) is called \( \delta T_0 \) if for any distinct pair of points in \( X \), there is a \( \delta \)-open set containing one of the points but not the other.

Definition 6. A topological space \((X, \tau)\) is called \( \delta T_1 \) if for any distinct pair of points \( x \) and \( y \) in \( X \), there is a \( \delta \)-open \( U \) in \( X \) containing \( x \) but not \( y \) and a \( \delta \)-open set \( V \) in \( X \) containing \( y \) but not \( x \).

Definition 7. A topological space \((X, \tau)\) is called \( \delta T_2 \) if for any distinct pair of points \( x \) and \( y \) in \( X \), there exist \( \delta \)-open sets \( U \) and \( V \) in \( X \) containing \( x \) and \( y \), respectively, such that \( U \cap V = \emptyset \).

Remark 2.1. (i) If \((X, \tau)\) is \( \delta T_i \), then it is \( \delta T_{i-1}, i = 1, 2 \).
(ii) Obviously, if \((X, \tau)\) is \( \delta T_i \), then \((X, \tau)\) is \( \delta D_i, i = 0, 1, 2 \).
(iii) If \((X, \tau)\) is \( \delta D_i \), then it is \( \delta D_{i-1}, i = 1, 2 \).
Theorem 2.2. For a topological space \((X, \tau)\) the following statements are true:

1. \((X, \tau)\) is \(\delta-D_0\) if and only if it is \(\delta-T_0\).
2. \((X, \tau)\) is \(\delta-D_1\) if and only if it is \(\delta-D_2\).

Proof. (1) The sufficiency is stated in Remark 2.1(ii). To prove necessity, let \((X, \tau)\) be \(\delta-D_0\). Then for each distinct pair \(x, y \in X\), at least one of \(x, y\), say \(x\), belongs to a \(\delta-D\)-set \(G\) but \(y \notin G\). Let \(G = U_1 \setminus U_2\) where \(U_1 \neq X\) and \(U_1, U_2 \in \partial O(X, \tau)\). Then \(x \in U_1\), and for \(y \notin G\) we have two cases: (a) \(y \notin U_1\); (b) \(y \in U_1\) and \(y \in U_2\).

In case (a), \(x \in U_1\) but \(y \notin U_1\); In case (b), \(y \notin U_2\) but \(x \notin U_2\). Hence \(X\) is \(\delta-T_0\).

(2) Sufficiency. Remark 2.1(iii).

Necessity. Suppose \(X\) \(\delta-D_1\). Then for each distinct pair \(x, y \in X\), we have \(\delta-D\)-sets \(G_1, G_2\) such that \(x \in G_1\), \(y \notin G_1\), \(y \in G_2\), \(x \notin G_2\). Let \(G_1 = U_1 \setminus U_2, G_2 = U_3 \setminus U_4\). From \(x \notin G_2\), it follows that either \(x \notin U_3\) or \(x \in U_3\) and \(x \in U_4\). We discuss the two cases separately.

1. \(x \notin U_3\). By \(y \notin G_1\) we have two subcases:
   - (a) \(y \notin U_1\). From \(x \in U_1 \setminus U_2\), it follows that \(x \in U_1 \setminus (U_2 \cup U_3)\) and by \(y \in U_3 \setminus U_4\) we have \(y \in U_3 \setminus (U_1 \cup U_4)\). Therefore \((U_1 \setminus (U_2 \cup U_3)) \cap (U_3 \setminus (U_1 \cup U_4)) = \emptyset\).
   - (b) \(y \in U_1\) and \(y \in U_2\). We have \(x \in U_1 \setminus U_2, y \in U_2, (U_1 \setminus U_2) \cap U_2 = \emptyset\).

2. \(x \in U_3\) and \(x \in U_4\). We have \(y \in U_3 \setminus U_4, x \in U_4, (U_3 \setminus U_4) \cap U_4 = \emptyset\). Therefore \(X\) is \(\delta-D_2\).

Corollary 2.3. If \((X, \tau)\) is \(\delta-D_1\), then it is \(\delta-T_0\).

Theorem 2.4. A topological space \((X, \tau)\) is \(\delta-T_0\) if and only if for each pair of distinct points \(x, y\) of \(X\), \(\text{Cl}_\delta\{\{x\}\} \neq \text{Cl}_\delta\{\{y\}\}\).

Proof. Sufficiency: Suppose that \(x, y \in X\), \(x \neq y\) and \(\text{Cl}_\delta\{\{x\}\} \neq \text{Cl}_\delta\{\{y\}\}\). Let \(z\) be a point of \(X\) such that \(z \in \text{Cl}_\delta\{\{x\}\}\) but \(z \notin \text{Cl}_\delta\{\{y\}\}\). We claim that \(x \notin \text{Cl}_\delta\{\{y\}\}\). For, if \(x \in \text{Cl}_\delta\{\{y\}\}\) then \(\text{Cl}_\delta\{\{x\}\} \subset \text{Cl}_\delta\{\{y\}\}\). This contradicts the fact that \(z \notin \text{Cl}_\delta\{\{y\}\}\). Consequently \(x\) belongs to the \(\delta\)-open set \([\text{Cl}_\delta\{\{y\}\}]^c\) to which \(y\) does not belong.
Necessity: Let $(X, \tau)$ be a $\delta-T_0$ space and $x, y$ be any two distinct points of $X$. There exists a $\delta$-open set $G$ containing $x$ or $y$, say $x$ but not $y$. Then $G^c$ is a $\delta$-closed set which does not contain $x$ but contains $y$. Since $\text{Cl}_\delta(\{y\})$ is the smallest $\delta$-closed set containing $y$ (Corollary 1.2), $\text{Cl}_\delta(\{y\}) \subset G^c$, and therefore $x \notin \text{Cl}_\delta(\{y\})$. Consequently $\text{Cl}_\delta(\{x\}) \neq \text{Cl}_\delta(\{y\})$.

**Theorem 2.5.** A topological space $(X, \tau)$ is $\delta-T_1$ if and only if the singletons are $\delta$-closed sets.

**Proof.** Let $(X, \tau)$ be $\delta-T_1$ and $x$ any point of $X$. Suppose $y \in \{x\}^c$. Then $x \neq y$ and so there exists a $\delta$-open set $U_y$ such that $y \in U_y$ but $x \notin U_y$. Consequently $y \in U_y \subset \{x\}^c$ i.e., $\{x\}^c = \bigcup \{U_y / y \in \{x\}\}$ which is $\delta$-open.

Conversely. Suppose $\{p\}$ is $\delta$-closed for every $p \in X$. Let $x, y \in X$ with $x \neq y$. Now $x \neq y$ implies $y \in \{x\}^c$. Hence $\{x\}^c$ is a $\delta$-open set containing $y$ but not $x$. Similarly $\{y\}^c$ is a $\delta$-open set containing $x$ but not $y$. Accordingly $X$ is a $\delta-T_1$ space.

**Definition 8.** A point $x \in X$ which has $X$ as the unique $\delta$-neighborhood is called $\delta$-neat point.

**Theorem 2.6.** For a $\delta-T_0$ topological space $(X, \tau)$ the following are equivalent:

1. $(X, \tau)$ is $\delta-D_1$,
2. $(X, \tau)$ has no $\delta$-neat point.

**Proof.** (1) $\Rightarrow$ (2). Since $(X, \tau)$ is $\delta-D_1$, then each point $x$ of $X$ is contained in a $\delta-D$-set $O = U - V$ and thus in $U$. By definition $U \neq X$. This implies that $x$ is not a $\delta$-neat point.

(2) $\Rightarrow$ (1). If $X$ is $\delta-T_0$, then for each distinct pair of points $x, y \in X$, at least one of them, $x$ (say) has a $\delta$-neighborhood $U$ containing $x$ and not $y$. Thus $U$ which is different from $X$ is a $\delta-D$-set. If $X$ has no $\delta$-neat point, then $y$ is not a $\delta$-neat point. This means that there exists a $\delta$-neighborhood $V$ of $y$ such that $V \neq X$. Thus $y \in (V - U)$ but not $x$ and $V - U$ is a $\delta-D$-set. Hence $X$ is $\delta-D_1$.

**Remark 2.7.** It is clear that a $\delta-T_0$ topological space $(X, \tau)$ is not $\delta-D_1$ if and only if there is a unique $\delta$-neat point in $X$. It is unique because if $x$ and $y$ are both $\delta$-neat point in $X$, then at least one of them say $x$ has a $\delta$-neighborhood $U$ containing $x$ but not $y$. But this is a contradiction since $U \neq X$. 
Definition 9. A topological space \((X, \tau)\) is \(\delta\)-symmetric if for \(x\) and \(y\) in \(X\), \(x \in Cl_\delta(\{y\})\) implies \(y \in Cl_\delta(\{x\})\).

Definition 10. A subset \(A\) of a topological space \((X, \tau)\) is called a \((\delta, \delta)\)-generalized-closed set \([1]\) (briefly \((\delta, \delta)\)-g-closed) if \(Cl_\delta(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is \(\delta\)-open in \((X, \tau)\).

Lemma 2.8. Every \(\delta\)-closed set is \((\delta, \delta)\)-g-closed.

Theorem 2.9. A topological space \((X, \tau)\) is \(\delta\)-symmetric if and only if \(\{x\}\) is \((\delta, \delta)\)-g-closed for each \(x \in X\).

Proof. Assume that \(x \in Cl_\delta(\{y\})\) but \(y \notin Cl_\delta(\{x\})\). This means that \([Cl_\delta(\{y\})]^c\) contains \(y\). This implies that \(Cl_\delta(\{y\})\) is a subset of \([Cl_\delta(\{x\})]^c\). Now \([Cl_\delta(\{x\})]^c\) contains \(x\) which is a contradiction.

Conversely, suppose that \(\{x\} \subseteq E \in \delta(X, \tau)\) but \(Cl_\delta(\{x\})\) is not a subset of \(E\). This means that \(Cl_\delta(\{x\})\) and \(E^c\) are not disjoint. Let \(y\) belongs to their intersection. Now we have \(x \in Cl_\delta(\{y\})\) which is a subset of \(E^c\) and \(x \notin E\). But this is a contradiction.

Corollary 2.10. If a topological space \((X, \tau)\) is a \(\delta-T_1\) space, then it is \(\delta\)-symmetric.

Proof. In a \(\delta-T_1\) space, singleton sets are \(\delta\)-closed (Theorem 2.5) and therefore \((\delta, \delta)\)-g-closed (Lemma 2.8). By Theorem 2.9, the space is \(\delta\)-symmetric.

Corollary 2.11. For a topological space \((X, \tau)\) the following are equivalent:

1. \((X, \tau)\) is \(\delta\)-symmetric and \(\delta-T_0\);
2. \((X, \tau)\) is \(\delta-T_1\).

Proof. By Corollary 2.10 and Remark 2.1 it suffices to prove only \((1) \rightarrow (2)\). Let \(x \neq y\) and by \(\delta-T_0\), we may assume that \(x \in G_1 \subseteq \{y\}^c\) for some \(G_1 \in \mathcal{O}(X, \tau)\). Then \(x \notin Cl_\delta(\{y\})\) and hence \(y \notin Cl_\delta(\{x\})\). There exists a \(G_2 \in \mathcal{O}(X, \tau)\) such that \(y \in G_2 \subseteq \{x\}^c\) and \((X, \tau)\) is a \(\delta-T_1\) space.
Theorem 2.12. For a \( \delta \)-symmetric topological space \((X, \tau)\) the following are equivalent:

1. \((X, \tau)\) is \(\delta-T_0\);
2. \((X, \tau)\) is \(\delta-D_1\);
3. \((X, \tau)\) is \(\delta-T_1\).

Proof. 
- \((1) \implies (3)\): Corollary 2.11.
- \((3) \implies (2) \implies (1)\): Remark 2.1.

Definition 11. A function \(f : (X, \tau) \to (Y, \sigma)\) is said to be \(\delta\)-continuous [3] if for each \(x \in X\) and each regular open set \(V\) containing \(f(x)\), there is a regular open set \(U\) in \(X\) containing \(x\) such that \(f(U) \subseteq V\).

Remark 2.13. In 1980, Noiri [3] proved that a function \(f : (X, \tau) \to (Y, \sigma)\) is \(\delta\)-continuous if and only if the inverse image of each \(\delta\)-open set is \(\delta\)-open.

Theorem 2.14. If \(f : (X, \tau) \to (Y, \sigma)\) is a \(\delta\)-continuous surjective function and \(E\) is a \(\delta\)-D-set in \(Y\), then the inverse image of \(E\) is a \(\delta\)-D-set in \(X\).

Proof. Let \(E\) be a \(\delta\)-D-set in \(Y\). Then there are \(\delta\)-open sets \(U_1\) and \(U_2\) in \(Y\) such that \(E = U_1 \setminus U_2\) and \(U_1 \neq Y\). By the \(\delta\)-continuity of \(f\), \(f^{-1}(U_1)\) and \(f^{-1}(U_2)\) are \(\delta\)-open in \(X\). Since \(U_1 \neq Y\), we have \(f^{-1}(U_1) \neq X\). Hence \(f^{-1}(E) = f^{-1}(U_1) \setminus f^{-1}(U_2)\) is a \(\delta\)-D-set.

Theorem 2.15. If \((Y, \sigma)\) is \(\delta-D_1\) and \(f : (X, \tau) \to (Y, \sigma)\) is \(\delta\)-continuous and bijective, then \((X, \tau)\) is \(\delta-D_1\).

Proof. Suppose that \(Y\) is a \(\delta-D_1\) space. Let \(x\) and \(y\) be any pair of distinct points in \(X\). Since \(f\) is injective and \(Y\) is \(\delta-D_1\), there exist \(\delta\)-D-sets \(G_x\) and \(G_y\) of \(Y\) containing \(f(x)\) and \(f(y)\) respectively, such that \(f(y) \notin G_x\) and \(f(x) \notin G_y\). By Theorem 2.14, \(f^{-1}(G_x)\) and \(f^{-1}(G_y)\) are \(\delta\)-D-sets in \(X\) containing \(x\) and \(y\), respectively. This implies that \(X\) is a \(\delta-D_1\) space.

Theorem 2.16. A topological space \((X, \tau)\) is \(\delta-D_1\) if and only if for each pair of distinct points \(x, y \in X\), there exists a \(\delta\)-continuous surjective function \(f : (X, \tau) \to (Y, \sigma)\), where \(Y\) is a \(\delta-D_1\) space such that \(f(x)\) and \(f(y)\) are distinct.
Proof. Necessity. For every pair of distinct points of $X$, it suffices to take the identity function on $X$.

Sufficiency. Let $x$ and $y$ be any pair of distinct points in $X$. By hypothesis, there exists a $\delta$-continuous, surjective function $f$ of a space $X$ onto a $\delta$-$D_1$ space $Y$ such that $f(x) \neq f(y)$. Therefore, there exist disjoint $\delta$-$D$-sets $G_x$ and $G_y$ in $Y$ such that $f(x) \in G_x$ and $f(y) \in G_y$. Since $f$ is $\delta$-continuous and surjective, by Theorem 2.14, $f^{-1}(G_x)$ and $f^{-1}(G_y)$ are disjoint $\delta$-$D$-sets in $X$ containing $x$ and $y$, respectively. Hence by Theorem 2.2, $X$ is $\delta$-$D_1$ space.

3. Sober $\delta$-$R_0$ spaces

Definition 12. Let $A$ be a subset of topological space $X$. The $\delta$-kernel of $A$, denoted by $\text{Ker}_\delta(A)$, is defined to be the set $\{x \in X \mid \text{Cl}_\delta(\{x\}) \cap A \neq \emptyset\}$.

Lemma 3.1. Let $(X, \tau)$ be a topological space and $x \in X$. Then $\text{Ker}_\delta(A) = \{x \in X / \text{Cl}_\delta(\{x\}) \cap A \neq \emptyset\}$.

Proof. Let $x \in \text{Ker}_\delta(A)$ and suppose $\text{Cl}_\delta(\{x\}) \cap A = \emptyset$. Hence $x \notin \text{Cl}_\delta(\{x\})^c$ which is a $\delta$-open set containing $A$. This is absurd, since $x \in \text{Ker}_\delta(A)$. Consequently, $\text{Cl}_\delta(\{x\}) \cap A \neq \emptyset$. Next, let $\text{Cl}_\delta(\{x\}) \cap A \neq \emptyset$ and suppose that $x \notin \text{Ker}_\delta(A)$. Then, there exists a $\delta$-open set $D$ containing $A$ and $x \notin D$. Let $y \in \text{Cl}_\delta(\{x\}) \cap A$. Hence, $D$ is a $\delta$-neighborhood of $y$ which $x \notin D$. By this contradiction $x \in \text{Ker}_\delta(A)$ and the claim.

Definition 13. A topological space $(X, \tau)$ is said to be sober $\delta$-$R_0$ if $\cap_{x \in X} \text{Cl}_\delta(\{x\}) = \emptyset$.

Therefore 3.2. A topological space $(X, \tau)$ is sober $\delta$-$R_0$ if and only if $\text{Ker}_\delta(\{x\}) \neq X$ for every $x \in X$.

Proof. Suppose that the space $(X, \tau)$ be sober $\delta$-$R_0$. Assume that there is a point $y$ in $X$ such that $\text{Ker}_\delta(\{y\}) = X$. Then $y \notin O$ which $O$ is some proper $\delta$-open subset of $X$. This implies that $y \in \cap_{x \in X} \text{Cl}_\delta(\{x\})$. But this is a contradiction.

Now assume that $\text{Ker}_\delta(\{x\}) \neq X$ for every $x \in X$. If there exists a point $y$ in $X$ such that $y \in \cap_{x \in X} \text{Cl}_\delta(\{x\})$, then every $\delta$-open set containing $y$ must contain every point of $X$. This implies that the space $X$ is the unique $\delta$-open set containing $y$. Hence $\text{Ker}_\delta(\{y\}) = X$ which is a contradiction. Therefore $(X, \tau)$ is sober $\delta$-$R_0$. 

Definition 14. A function \( f : X \to Y \) is called always \( \delta \)-closed if the image of every \( \delta \)-closed subset of \( X \) is \( \delta \)-closed in \( Y \).

Theorem 3.3. If \( f : X \to Y \) is an bijective always \( \delta \)-closed function and \( X \) is sober \( \delta \)-\( R_0 \), then \( Y \) is sober \( \delta \)-\( R_0 \).

Proof. Straightforward.

Theorem 3.4. If the topological space \( X \) is sober \( \delta \)-\( R_0 \) and \( Y \) is any topological space, then the product \( X \times Y \) is sober \( \delta \)-\( R_0 \).

Proof. By showing that \( \bigcap_{(x,y) \in X \times Y} Cl_\delta(\{x, y\}) = \emptyset \) we are done. We have:

\[
\bigcap_{(x,y) \in X \times Y} Cl_\delta(\{x, y\}) \subseteq \bigcap_{(x,y) \in X \times Y} (Cl_\delta(\{x\}) \times Cl_\delta(\{y\})) = \bigcap_{x \in X} Cl_\delta(\{x\}) \times \bigcap_{y \in Y} Cl_\delta(\{y\}) \subseteq \emptyset \times Y = \emptyset.
\]

4. \( \delta \)-\( R_0 \) spaces and \( \delta \)-\( R_1 \) spaces

Definition 15. A topological space \((X, \tau)\) is said to be \( \delta \)-\( R_0 \) space [1] if every \( \delta \)-open set contains the \( \delta \)-closure of each of its singletons.

Definition 16. A topological space \((X, \tau)\) is said to be \( \delta \)-\( R_1 \) if for \( x, y \) in \( X \) with \( Cl_\delta(\{x\}) \neq Cl_\delta(\{y\}) \), there exist disjoint \( \delta \)-open sets \( U \) and \( V \) such that \( Cl_\delta(\{x\}) \) is a subset of \( U \) and \( Cl_\delta(\{y\}) \) is a subset of \( V \).

Lemma 4.1. Let \((X, \tau)\) be a topological space and \( x \in X \). Then \( y \in Ker_\delta(\{x\}) \) if and only if \( x \in Cl_\delta(\{y\}) \).

Proof. Suppose that \( y \notin Ker_\delta(\{x\}) \). Then there exists a \( \delta \)-open set \( V \) containing \( x \) such that \( y \notin V \). Therefore we have \( x \notin Cl_\delta(\{y\}) \). The proof of converse case can be done similarly.

Lemma 4.2. The following statements are equivalent for any points \( x \) and \( y \) in a topological space \((X, \tau)\):

1. \( Ker_\delta(\{x\}) \neq Ker_\delta(\{y\}) \);
2. \( Cl_\delta(\{x\}) \neq Cl_\delta(\{y\}) \).
Proof. (1) $\rightarrow$ (2): Suppose that $\text{Ker}_\delta(\{x\}) \neq \text{Ker}_\delta(\{y\})$, then there exists a point $z$ in $X$ such that $z \in \text{Ker}_\delta(\{x\})$ and $z \notin \text{Ker}_\delta(\{y\})$. From $z \in \text{Ker}_\delta(\{x\})$ it follows that $\{x\} \cap \text{Cl}_\delta(\{z\}) \neq \emptyset$ which implies $x \in \text{Cl}_\delta(\{z\})$. By $z \notin \text{Ker}_\delta(\{y\})$, we have $\{y\} \cap \text{Cl}_\delta(\{z\}) = \emptyset$. Since $x \in \text{Cl}_\delta(\{z\})$, $\text{Cl}_\delta(\{x\}) \subset \text{Cl}_\delta(\{z\})$ and $\{y\} \cap \text{Cl}_\delta(\{x\}) = \emptyset$. Therefore it follows that $\text{Cl}_\delta(\{x\}) \neq \text{Cl}_\delta(\{y\})$.

(2) $\rightarrow$ (1): Suppose that $\text{Cl}_\delta(\{x\}) \neq \text{Cl}_\delta(\{y\})$. Then there exists a point $z$ in $X$ such that $z \in \text{Cl}_\delta(\{x\})$ and $z \notin \text{Cl}_\delta(\{y\})$. Then there exists a $\gamma$-open set containing $z$ and therefore $x$ but not $y$, namely, $y \notin \text{Ker}_\delta(\{x\})$. Hence $\text{Ker}_\delta(\{x\}) \neq \text{Ker}_\delta(\{y\})$.

Theorem 4.3. If $(X, \tau)$ is $\delta$-$R_1$, then $(X, \tau)$ is $\delta$-$R_0$.

Proof. Let $U$ be $\delta$-open and $x \in U$. If $y \notin U$, then since $x \notin \text{Cl}_\delta(\{y\})$, $\text{Cl}_\delta(\{x\}) \neq \text{Cl}_\delta(\{y\})$. Hence, there exists a $\delta$-open $V_y$ such that $\text{Cl}_\delta(\{y\}) \subset V_y$ and $x \notin V_y$, which implies $y \notin \text{Cl}_\delta(\{x\})$. Thus $\text{Cl}_\delta(\{x\}) \subset U$. Therefore $(X, \tau)$ is $\delta$-$R_0$.

Theorem 4.4. A topological space $(X, \tau)$ is $\delta$-$R_1$ if and only if for $x, y \in X$, $\text{Ker}_\delta(\{x\}) \neq \text{Ker}_\delta(\{y\})$, there exist disjoint $\delta$-open sets $U$ and $V$ such that $\text{Cl}_\delta(\{x\}) \subset U$ and $\text{Cl}_\delta(\{y\}) \subset V$.

Proof. It follows from Lemma 4.2.

Theorem 4.5. A topological space $(X, \tau)$ is a $\delta$-$R_0$ space if and only if for any $x$ and $y$ in $X$, $\text{Cl}_\delta(\{x\}) \neq \text{Cl}_\delta(\{y\})$ implies $\text{Cl}_\delta(\{x\}) \cap \text{Cl}_\delta(\{y\}) = \emptyset$.

Proof. Suppose that $(X, \tau)$ is $\delta$-$R_0$ and $x, y \in X$ such that $\text{Cl}_\delta(\{x\}) \neq \text{Cl}_\delta(\{y\})$.

Then, there exist $z \in \text{Cl}_\delta(\{x\})$ such that $z \notin \text{Cl}_\delta(\{y\})$ (or $z \in \text{Cl}_\delta(\{y\})$) such that $z \notin \text{Cl}_\delta(\{x\})$. There exists $V \in \delta\mathcal{O}(X, \tau)$ such that $y \notin V$ and $z \in V$; hence $x \in V$.

Therefore, we have $x \notin \text{Cl}_\delta(\{y\})$. Thus $x \in [\text{Cl}_\delta(\{y\})]^c \in \delta\mathcal{O}(X, \tau)$, which implies $\text{Cl}_\delta(\{x\}) \subset [\text{Cl}_\delta(\{y\})]^c$ and $\text{Cl}_\delta(\{x\}) \cap \text{Cl}_\delta(\{y\}) = \emptyset$. the proof for otherwise is similar.

Sufficiency: Let $V \in \delta\mathcal{O}(X, \tau)$ and let $x \in V$. We still show that $\text{Cl}_\delta(\{x\}) \subset V$.

Let $y \notin V$, i.e., $y \in [V]^c$. Then $x \neq y$ and $x \notin \text{Cl}_\delta(\{y\})$. This shows that $\text{Cl}_\delta(\{x\}) \neq \text{Cl}_\delta(\{y\})$. By assumption, $\text{Cl}_\delta(\{x\}) \cap \text{Cl}_\delta(\{y\}) = \emptyset$. Hence $y \notin \text{Cl}_\delta(\{x\})$. Therefore $\text{Cl}_\delta(\{x\}) \subset V$. 

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Theorem 4.6. A topological space \((X, \tau)\) is a \(\delta-R_0\) space if and only if for any points \(x\) and \(y\) in \(X\), \(\text{Ker}_\delta(\{x\}) \neq \text{Ker}_\delta(\{y\})\) implies \(\text{Ker}_\delta(\{x\}) \cap \text{Ker}_\delta(\{y\}) = \emptyset\).

Proof. Suppose that \((X, \tau)\) is a \(\delta-R_0\) space. Thus by Lemma 4.2, for any points \(x\) and \(y\) in \(X\) if \(\text{Ker}_\delta(\{x\}) \neq \text{Ker}_\delta(\{y\})\) then \(\text{Cl}_\delta(\{x\}) \neq \text{Cl}_\delta(\{y\})\). Now we prove that \(\text{Ker}_\delta(\{x\}) \cap \text{Ker}_\delta(\{y\}) = \emptyset\). Assume that \(z \in \text{Ker}_\delta(\{x\}) \cap \text{Ker}_\delta(\{y\})\). By \(z \in \text{Ker}_\delta(\{x\})\) and Lemma 4.1, it follows that \(x \in \text{Cl}_\delta(\{z\})\). Similarly, we have \(\text{Cl}_\delta(\{y\}) = \text{Cl}_\delta(\{z\})\). This is a contradiction. Therefore, we have \(\text{Ker}_\delta(\{x\}) \cap \text{Ker}_\delta(\{y\}) = \emptyset\).

Conversely, let \((X, \tau)\) be a topological space such that for any points \(x\) and \(y\) in \(X\), \(\text{Cl}_\delta(\{x\}) \neq \text{Cl}_\delta(\{y\})\), then by Lemma 4.2, \(\text{Ker}_\delta(\{x\}) \neq \text{Ker}_\delta(\{y\})\). Hence by hypothesis \(\text{Ker}_\delta(\{x\}) \cap \text{Ker}_\delta(\{y\}) = \emptyset\) which implies \(\text{Cl}_\delta(\{x\}) \cap \text{Cl}_\delta(\{y\}) = \emptyset\). Because \(z \in \text{Cl}_\delta(\{x\})\) implies that \(x \in \text{Ker}_\delta(\{z\})\) and therefore \(\text{Ker}_\delta(\{x\}) \cap \text{Ker}_\delta(\{z\}) = \emptyset\). Therefore by Theorem 4.5 \((X, \tau)\) is a \(\delta-R_0\) space.

Theorem 4.7. For a topological space \((X, \tau)\), the following properties are equivalent:

1. \((X, \tau)\) is a \(\delta-R_0\) space;
2. For any \(A \neq \emptyset\) and \(G \in \delta O(X, \tau)\) such that \(A \cap G \neq \emptyset\), there exists \(F \in \delta C(X, \tau)\) such that \(A \cap F \neq \emptyset\) and \(F \subseteq G\);
3. Any \(G \in \delta O(X, \tau)\), \(G = \cup\{F \in \delta C(X, \tau) \mid F \subseteq G\}\);
4. Any \(F \in \delta C(X, \tau)\), \(F = \cap\{G \in \delta O(X, \tau) \mid F \subseteq G\}\);
5. For any \(x \in X\), \(\text{Cl}_\delta(\{x\}) \subseteq \text{Ker}_\delta(\{x\})\).

Proof. (1) \(\rightarrow\) (2): Let \(A\) be a nonempty set of \(X\) and \(G \in \delta O(X, \tau)\) such that \(A \cap G \neq \emptyset\). There exists \(x \in A \cap G\). Since \(x \in G \in \delta O(X, \tau)\), \(\text{Cl}_\delta(\{x\}) \subseteq G\). Set \(F = \text{Cl}_\delta(\{x\})\), then \(F \in \delta C(X, \tau)\), \(F \subseteq G\) and \(A \cap F \neq \emptyset\).

(2) \(\rightarrow\) (3): Let \(G \in \delta O(X, \tau)\), then \(G \supseteq \cup\{F \in \delta C(X, \tau) \mid F \subseteq G\}\). Let \(x\) be any point of \(G\). There exists \(F \in \delta C(X, \tau)\) such that \(x \in F\) and \(F \subseteq G\). Therefore, we have \(x \in F \supseteq \cup\{F \in \delta C(X, \tau) \mid F \subseteq G\}\) hence \(G = \cup\{F \in \delta C(X, \tau) \mid F \subseteq G\}\).

(3) \(\rightarrow\) (4): This is obvious.

(4) \(\rightarrow\) (5): Let \(x\) be any of \(x\) and \(y \notin \text{Ker}_\delta(\{x\})\). There exists \(V \in \delta O(X, \tau)\) such that \(x \in V\) and \(y \notin V\); hence \(\text{Cl}_\delta(\{y\}) \cap V = \emptyset\). By (4) \((\cap\{G \in \delta O(X, \tau) \mid \text{Cl}_\delta(\{y\}) \subseteq G\}) \cap V = \emptyset\) and there exists \(G \in \delta O(X, \tau)\) such that
x \not\in G \text{ and } Cl_\delta(\{y\}) \subset G. \text{ Therefore, } Cl_\delta(\{x\}) \cap G = \emptyset \text{ and } y \not\in Cl_\delta(\{x\}). \text{ Consequently, we obtain } Cl_\delta(\{x\}) \subset Ker_\delta(\{x\}).

(5) \rightarrow (1): \text{ Let } G \in \partial O(X, \tau) \text{ and } x \in G. \text{ Suppose } y \in Ker_\delta(\{x\}) \text{, then } x \in Cl_\delta(\{y\}) \text{ and } y \in G. \text{ This implies that } Cl_\delta(\{x\}) \subset Ker_\delta(\{x\}) \subset G. \text{ This shows that } (X, \tau) \text{ is a } \delta{-}R_0 \text{ space.}

**Corollary 4.8.** For a topological space \((X, \tau)\), the following properties are equivalent:
(1) \((X, \tau)\) is a \(\delta{-}R_0\) space;
(2) \(Cl_\delta(\{x\}) = Ker_\delta(\{x\})\) for all \(x \in X\).

**Proof.** (1) \rightarrow (2): Suppose that \((X, \tau)\) is a \(\delta{-}R_0\) space. By Theorem 4.7, \(Cl_\delta(\{x\}) = Ker_\delta(\{x\})\) for each \(x \in X\). Let \(y \in Ker_\delta(\{x\})\), then \(x \in Cl_\delta(\{y\})\) and by Theorem 4.5 \(Cl_\delta(\{x\}) = Cl_\delta(\{y\})\). Therefore, \(y \in Cl_\delta(\{x\})\) and hence \(Ker_\delta(\{x\}) \subset Cl_\delta(\{x\})\). This shows that \(Cl_\delta(\{x\}) = Ker_\delta(\{x\})\).

(2) \rightarrow (1): This is obvious by Theorem 4.7.

**Theorem 4.9.** For a topological space \((X, \tau)\), the following properties are equivalent:
(1) \((X, \tau)\) is a \(\delta{-}R_0\) space;
(2) \(x \in Cl_\delta(\{y\})\) if and only if \(y \in Cl_\delta(\{x\})\), for any points \(x\) and \(y\) in \(X\).

**Proof.** (1) \rightarrow (2): Assume that \((X, \tau)\) is a \(\delta{-}R_0\) space. Let \(x \in Cl_\delta(\{y\})\) and \(D\) be any \(\delta\)-open set such that \(y \in D\). Now by hypothesis, \(x \in D\). Therefore, every \(\delta\)-open set which contain \(y\) contains \(x\). Hence \(y \in Cl_\delta(\{x\})\).

(2) \rightarrow (1): Let \(U\) be a \(\delta\)-open set and \(x \in U\). If \(y \not\in U\), then \(x \not\in Cl_\delta(\{y\})\) and hence \(y \not\in Cl_\delta(\{x\})\). This implies that \(Cl_\delta(\{x\}) \subset U\). Hence \((X, \tau)\) is \(\delta{-}R_0\).

We observed that by Definition 9 and Theorem 4.9 the notions of \(\delta\)-symmetric and \(\delta{-}R_0\) are equivalent.

**Theorem 4.10.** For a topological space \((X, \tau)\), the following properties are equivalent:
(1) \((X, \tau)\) is a \(\delta{-}R_0\) space;
(2) If \(F\) is \(\delta\)-closed, then \(F = Ker_\delta(F)\);
(3) If \(F\) is \(\delta\)-closed and \(x \in F\), then \(Ker_\delta(\{x\}) \subset F\);
(4) If \(x \in X\), then \(Ker_\delta(\{x\}) \subset Cl_\delta(\{x\})\).
Proof. (1) \implies (2): Let $F$ be $\delta$-closed and $x \notin F$. Thus $X - F$ is $\delta$-open and contains $x$. Since $(X, \tau)$ is $\delta$-$R_0$, $\text{Cl}_\delta(\{x\}) \subset X - F$. Thus $\text{Cl}_\delta(\{x\}) \cap F = \emptyset$ and by Lemma 3.1 $x \notin \text{Ker}_\delta(F)$. Therefore $\text{Ker}_\delta(F) = F$.

(2) \implies (3): In general, $A \subset B$ implies $\text{Ker}_\delta(A) \subset \text{Ker}_\delta(B)$. Therefore, it follows from (2) that $\text{Ker}_\delta(\{x\}) \subset \text{Ker}_\delta(F) = F$.

(3) \implies (4): Since $x \in \text{Cl}_\delta(\{x\})$ and $\text{Cl}_\delta(\{x\})$ is $\delta$-closed, by (3) $\text{Ker}_\delta(\{x\}) \subset \text{Cl}_\delta(\{x\})$.

(4) \implies (1): We show the implication by using Theorem 4.9. Let $x \in \text{Cl}_\delta(\{y\})$. Then by Lemma 4.1 $y \in \text{Ker}_\delta(\{x\})$. Since $x \in \text{Cl}_\delta(\{x\})$ and $\text{Cl}_\delta(\{x\})$ is $\delta$-closed, by (4) we obtain $y \in \text{Ker}_\delta(\{x\} \subset \text{Cl}_\delta(\{x\})$, Therefore $x \in \text{Cl}_\delta(\{y\})$ implies $y \in \text{Cl}_\delta(\{x\})$. The converse is obvious and $(X, \tau)$ is $\delta$-$R_0$.

Recall that a filterbase $F$ is called $\delta$-convergent to a point $x$ in $X$, if for any $\delta$-open set $U$ of $X$ containing $x$, there exists $B \in F$ such that $B$ is a subset of $U$.

Lemma 4.11. Let $(X, \tau)$ be a topological space and $x$ and $y$ any two points in $X$ such that every net in $X$ $\delta$-converging to $y$ $\delta$-converges to $x$. Then $x \in \text{Cl}_\delta(\{y\})$.

Proof. Suppose that $x_n = y$ for each $n \in N$. Then $\{x_n\}_{n \in N}$ is a net in $\text{Cl}_\delta(\{y\})$. Since $\{x_n\}_{n \in N}$ $\delta$-converges to $y$, then $\{x_n\}_{n \in N}$ $\delta$-converges to $x$ and this implies that $x \in \text{Cl}_\delta(\{y\})$.

Theorem 4.12. For a topological space $(X, \tau)$, the following statements are equivalent:

(1) $(X, \tau)$ is a $\delta$-$R_0$ space;

(2) If $x, y \in X$, then $y \in \text{Cl}_\delta(\{x\})$ if and only if every net in $X$ $\delta$-converging to $y$ $\delta$-converges to $x$.

Proof. (1) \implies (2): Let $x, y \in X$ such that $y \in \text{Cl}_\delta(\{x\})$. Suppose that $\{x_\alpha\}_{\alpha \in \Lambda}$ is a net in $X$ such that $\{x_\alpha\}_{\alpha \in \Lambda} \delta$-converges to $y$. Since $y \in \text{Cl}_\delta(\{x\})$, by Theorem 4.5 we have $\text{Cl}_\delta(\{x\}) = \text{Cl}_\delta(\{y\})$. Therefore $x \in \text{Cl}_\delta(\{y\})$. This means that $\{x_\alpha\}_{\alpha \in \Lambda}$ $\delta$-converges to $x$. Conversely, let $x, y \in X$ such that every net in $X$ $\delta$-converging to $y$ $\delta$-converges to $x$. Then $x \in \text{Cl}_\delta(\{y\})$ by Lemma 3.1. By Theorem 4.5, we have $\text{Cl}_\delta(\{x\}) = \text{Cl}_\delta(\{y\})$. Therefore $y \in \text{Cl}_\delta(\{x\})$.

(2) \implies (1): Assume that $x$ and $y$ are any two points of $X$ such that $\text{Cl}_\delta(\{x\}) \cap \text{Cl}_\delta(\{y\}) \neq \emptyset$. Let $z \in \text{Cl}_\delta(\{x\}) \cap \text{Cl}_\delta(\{y\})$. So there exists a net $\{x_\alpha\}_{\alpha \in \Lambda}$ in $\text{Cl}_\delta(\{x\})$ such that $\{x_\alpha\}_{\alpha \in \Lambda} \delta$-converges to $z$. Since $z \in \text{Cl}_\delta(\{y\})$, then
\{x_\alpha\}_{\alpha \in \Lambda} \text{ } \delta\text{-converges to } y. \text{ It follows that } y \in Cl_\delta(\{x\}). \text{ By the same token we obtain } x \in Cl_\delta(\{y\}). \text{ Therefore } Cl_\delta(\{x\}) = Cl_\delta(\{y\}) \text{ and by Theorem 4.5 } (X, \tau) \text{ is } \delta-R_0.

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References

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