On a Class of Residually Finite Groups

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Abstract. Let $n, k$ be positive integers and $t_0, t_1, \ldots, t_k$ be non-zero integers. We denote by $W_k(n)$ the class of groups $G$ in which, for every subset $X$ of $G$ of cardinality $n+1$, there exist a subset $X_0 \subseteq X$, with $2 \leq |X_0| \leq n+1$, and a function $f : \{0, 1, 2, \ldots, k\} \to X_0$, with $f(0) \neq f(i)$ such that $[x_0^i, x_1^j, \ldots, x_k^l] = 1$ whenever $x_0^i \in H$ whenever $x_i^j \in H$. The class $W_k(n)$ is defined exactly as $W_k(n)$, with additional conditions “$x_i^j \in H$ whenever $x_i^j \in H$”, where $\{x_i^j\} \neq H \leq G$.

Let $G$ be a finitely generated residually finite group. Here we prove that

1. If $G \in W_k(n)$, then $G$ has a normal nilpotent subgroup $N$ with finite index such that the nilpotency class of $N/N_i$ is bounded by a function of $k$, where $N_i$ is the torsion subgroup of $N$.

2. If $G \in W_k^*(n)$ be $d$ generated, then $G$ has a normal nilpotent subgroup $N$ whose index and the nilpotency class are bounded by a function of $k, n, t_0, t_1, \ldots, t_k$.

1. Introduction and results

In response to a question of Paul Erdös, B.H. Neumann proved [16] that a group is center-by-finite if and only if every infinite subset contains a commuting pair of distinct elements. Other problems of this type have been the object of several articles, for example [1] – [13], [16], [20] – [23].

Our notation and terminology are standard and can be found in [17]. In particular for a group $G$ and elements $x, y, x_1, x_2, \ldots, x_k \in G$ we write

\[ [x_1, x_2] = x_1^{-1}x_2^{-1}x_1x_2 = x_1^{-1}x_1^{-1}x_2^{-1}x_2, \quad [x_1, \ldots, x_k] = [(x_1, \ldots, x_{k-1}), x_k], \]

\[ [x_0, y] = x_0, \quad [x_k, y] = [(x_{k-1}, y), y]. \]
A group is said to be a $k$-Engel group (respectively, Engel group) if for all $x, y \in G$, 
$[x_k, y] = 1$ (respectively, there exists a positive integer $t$ depending on $x$ and $y$ such that 
$[x, y] = 1$). The class of $k$-Engel (respectively, Engel) groups will be denoted by $\varepsilon_k$ (respectively, $\varepsilon$).

Let $k$ be a positive integer, $n$ be a positive integer or infinity (denoted by $\infty$). We denote by $\varepsilon_k(n)$ (respectively, $\varepsilon(n)$) the class of all groups $G$ such that for every subset $X$ of cardinality $n + 1$, there exist distinct elements $x, y \in X$ such that $[x_k, y] = 1$ (respectively, $[x, y] = 1$ for some positive integer $t$ depending on $x, y$).

Longobardi and Maj [12] (see also [8]) proved that a finitely generated soluble group $G$ has the property $\varepsilon(\infty)$ if and only if $G$ is finite-by-nilpotent. Abdollahi [2], showed that finite $\varepsilon(2)$-groups (respectively, $\varepsilon(15)$-groups) are nilpotent (respectively, soluble) and that a finitely generated residually finite $\varepsilon_k(n)$-group, $n, k$ are positive integers, is finite-by-nilpotent. In [21] the class of all groups $G$ in which for every subset $X$ of cardinality $n + 1$, there exist a positive integer $k$ and distinct elements $x, y \in X$ and non-zero integers $t_0, t_1, \ldots, t_k$ such that $[x^{t_0}, x^{t_1}, \ldots, x^{t_k}] = 1$, where $x_i \in \{x, y\}$, $x_0 \neq x_1$, is denoted by $\Omega(n)$. The author in [21] proved that a finitely generated soluble group $G$ has the property $\Omega(\infty)$ if and only if $G$ is nilpotent-by-finite.

Now, in order to generalize the classes of groups mentioned above, we define a new class of groups as follows. Let $n$ be a positive integer or infinity. We denote by $W(n)$ the class of groups $G$ such that, for every subset $X$ of cardinality $n + 1$, there exists a subset $X_0 \subseteq X$, with $2 \leq |X_0| \leq n + 1$, such that the following condition holds.

There exist a positive integer $k$ and a function $f : \{0, 1, 2, \ldots, k\} \rightarrow X_0$ and non-zero integers $t_0, t_1, \ldots, t_k$ such that $[x^{t_0}, x^{t_1}, \ldots, x^{t_k}] = 1$ where $x_i = f(i)$, $i = 0, 1, \ldots, k$ and $x_0 \neq x_1$.

If $k$ is fixed for every subset $X$ then we obtain the class $W_k(n)$. Clearly the classes $W(n)$ and $W_k(n)$ are subgroup and quotient closed. Since all torsion groups are belong to $W(n)$ we define another class of groups: the class $W^*(n)$ is defined exactly as $W(n)$ with additional conditions “$x_j \in H$ whenever $x_j \in H$ , for some subgroup $H = \left\{ x_j^i \right\}$ of $G$”. It is clear that the class $W^*(n)$ is subgroup and quotient closed.

In fact we impose the above condition to ensure that the class $W^*(n)$ be quotient closed. To see this let $K$ be a normal subgroup of a $W^*(n)$-group $G$, and let $X = \{g_0 K, g_1 K, \ldots, g_n K\}$ be a subset of $G / K$ of size $n + 1$. Since $G \in W^*(n)$, there exists a positive integer $k$, and non-zero integers $t_0, t_1, \ldots, t_k$, such that
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Let \([x_0, x_1, \ldots, x_k] = 1\), where \(x_i \in \{g_0, g_1, \ldots, g_k\}\), \(x_0 \neq x_1\), and if \(x_j' \in H\), where \(\langle x_j' \rangle \neq H \leq G\), then \(x_j \in H\). Now \([(x_0 K)^k, (x_1 K)^k, \ldots, (x_k K)^k] = K\), \(x_0 K \neq x_1 K\) and if \((x_j K)^k \in H / K \leq G / K\), where \(H / K \neq \langle x_j K \rangle\), then we have \(x_j' \in H\) and \(\langle x_j' \rangle \neq H \leq G\). So \(x_j \in H\) and \(x_j K \in H / K\). Therefore \(G / K \in W^*(n)\).

Similarly we can define \(W_k^*(n)\). The classes \(W(n), W^*(n)\) are considered in [22]. In that paper the author proved that a finite group with the property \(W^*(2)\) (respectively \(W^*(4)\)) is nilpotent (respectively soluble), and that a finitely generated soluble group \(G\) has the property \(W^*(\infty)\) if and only if \(G\) is finite-by-nilpotent.

Now we define another classes of groups, which we want to consider in this paper, namely the classes \(W_k(n)\) and \(W_k^*(n)\): if \(k\) and \(t_0, t_1, \ldots, t_k\), in the definition of \(W(n)\) (respectively \(W^*(n)\)) are the same for every subset \(X_0\), one obtains the class \(W_k(n)\) (respectively \(W_k^*(n)\)). Note that we just fix \(k\) and \(t_0, t_1, \ldots, t_k\), so that the bar notation causes no confusion here.

If the subset \(X_0\) in the definition of \(W_k(n)\) (respectively, \(W_k^*(n)\)) has always 2 elements, say \(X_0 = \{x, y\}\), and the function \(f\) is always of the form \(f(0) = x\) and \(f(i) = y\) and \(t_0 = t_1 = \cdots = t_k = 1\) one obtains the class \(\epsilon_k(n)\). Note that

\[
\epsilon_k(n) \subseteq W_k^*(n) \subseteq W_k(n) \subseteq W_k(n + 1).
\]

Throughout the paper we assume that \(n, k\) are fixed positive integers and \(t_0, t_1, \ldots, t_k\) are non-zero fixed positive integers. In this paper we consider residually finite groups. Our first result is about \(\overline{W_k(n)}\)-groups and sharpens [21, Theorem 3]:

**Theorem A.** Let \(G\) be a finitely generated residually finite \(\overline{W_k(n)}\)-group. Then \(G\) has a normal nilpotent subgroup \(N\) with finite index such that the nilpotency class of \(N / N_i\) is bounded by a function of \(k\), where \(N_i\) is the torsion subgroup of \(N\).

If we consider the stronger condition \(\overline{W_k^*(n)}\) we are able to prove the stronger result:

**Theorem B.** Let \(G\) be a \(d\)-generated residually finite \(\overline{W_k^*(n)}\)-group. Then \(G\) has a normal nilpotent subgroup \(N\) whose index and the nilpotency class are bounded by a function of \(k, n, t_0, t_1, \ldots, t_k\).
2. Proof of Theorem A

To consider finitely generated residually finite group in \( \widetilde{W}_k(n) \) we can use a result of Wilson [24], which states that if \( G \) is finitely generated residually finite group and \( N \) is positive integer such that \( G \) has no section isomorphic to the twisted wreath product \( A \text{tw}_C B \), with \( B \) is finite and cyclic, \( A \) an elementary Abelian group acted on faithfully and irreducibly by \( C \), and \( |B:C| > N \), then \( G \) is virtually a soluble minimax group. For the definition of the twisted wreath products we refer readers to Neumann [15]. Now as in the proof of the Lemma 5 in [21], we can see that

**Lemma 2.1.** Let \( A \) be a non-trivial Abelian group, \( B = \{b\} \) a finite cyclic group of order \( n \), \( C \) a subgroup of \( B \) with \( |B:C| = N \), and suppose that \( C \) acts on \( A \). Let \( W = A \text{tw}_C B \) be the twisted wreath product of \( A \) by \( B \) with respect to the action of \( C \) on \( A \). If \( G \in \widetilde{W}_k(n) \), then \( N \leq n + t_0 + t_1 + \cdots + t_k \).

Lemma 2.1 and the result of Wilson reduces the residually finite case to the soluble case. For finitely generated soluble groups we may argue exactly as in the proof of [21, Theorem 3], and prove the following.

**Proposition 2.2.** Let \( G \in \widetilde{W}_k(n) \), be a finitely generated soluble group. Then \( G \) is nilpotent-by-finite.

Now we prove that nilpotency class of torsion free nilpotent \( \widetilde{W}_k(n) \)-groups is \( k \)-bounded. The following lemma is proved in [22], but we include it for completeness.

**Lemma 2.3.** Let \( G \in \widetilde{W}_k(n) \), be torsion free nilpotent. Then the nilpotency class of \( G \) is bounded by a function of \( k \).

**Proof.** Let \( G \) be nilpotent of class \( c \). Then \( \Gamma_{[c/2]}(G) \) is Abelian, where \( [c/2] \) equals \( (c + 2)/2 \) if \( c \) is even and \( (c + 1)/2 \) if \( c \) is odd (\( \Gamma_{s+1}(G) \) is the \( s \)th term of lower central series of \( G \)). Let \( A = \{x \in G \mid x^m \in \Gamma_{[c/2]}(G)\} \), for some non-zero integer \( m \) denote the isolator of \( \Gamma_{[c/2]}(G) \). Then \( A \) is Abelian, since \( G \) is torsion free. Now let \( a \in A \) and \( g \in G \). Considering the elements \( ag, ag^2, ag^3, \ldots \) we find positive integers \( i_0, i_1, \ldots, i_k \), \( i_0 \neq i_1 \), such that

\[
1 = [(a^h g)^{i_0}, (a^h g)^{i_1}, \ldots, (a^h g)^{i_k}] = [(a^h g)^{i_0}, (a^h g)^{i_1}, g^{i_2}, \ldots, g^{i_k}].
\]
Now, since $G$ is metabelian, $[a, u, v] = [a, v, u]$, for all $a \in G'$ and $u, v \in G$. It follows that in the above equation we may replace $t_j$ by $|t_j|$, for $j \geq 2$. Now suppose that $t_0 < 0$. Then since $A$ is Abelian normal, we have

$$ [(a^h g)^{t_0}, (a^h g)^{t_0}] = [(a^h g^{t_0})(a^h g^{t_0})]^{-(a^h g)^{t_0}} $$

$$ = [(a^h g^{t_0}, (a^h g^{t_0})]^{t_0} \cdot (g^{t_0} g^{t_0} \cdots g^{t_0}) $$

$$ = [(a^h g^{t_0}, (a^h g^{t_0})]^{t_0}. $$

Therefore, since the identity $[x_1, x_2, x_3, \ldots, x_k]^{-1} = [x_2, x_1, x_3, \ldots, x_k]$ holds in a metabelian group, we have

$$ [(a^h g)^{t_0}, (a^h g)^{t_0}, g^{t_0}, \ldots, g^{t_0}] = [(a^h g)^{t_0}, (a^h g^{t_0}, g^{t_0}, \ldots, g^{t_0})]^{t_0} $$

$$ = [(a^h g)^{t_0}, (a^h g^{t_0}, g^{t_0}, \ldots, g^{t_0})]^{t_0}. $$

Thus we may replace $t_0$ by $|t_0|$. Similarly we may replace $t_i$ by $|t_i|$. Hence we may assume that $t_i > 0$, for all $i = 0, 1, \ldots, k$. We treat $A$ as a $Z(g)$-module and show that $A(g - 1)^N = 0$, where $N$ is a function of $k$. If $g \in A$, then $A(g - 1) = 0$ and we are done. So suppose that $g \notin A$. Now, since

$$ \left[ g^{t_0} a^h(g^{t_0} g^{t_0} + \cdots + g^{t_0}), g^{t_0} a^h(g^{t_0} g^{t_0} + \cdots + g^{t_0}), g^{t_0}, \cdots, g^{t_0} \right] = 1, $$

we have $af_i(g) = 0$, where

$$ f_i(x) = \left( t_i x^i + x^{i-1} + \cdots + x \right)(x^{t_i} - 1) + t_0 (x^{t_0} + x^{t_0-1} + \cdots + x)(x^{t_i} - 1). $$

Put $f(x) := (x - 1) f_i(x) = (t_i + t_0) x(x^{t_i} - 1) (x^{t_0} - 1)(1 - x^{t_i}) \cdots (1 - x^{t_i})$. Then $af(g) = 0$ and $f(x) = \sum c_i x^t$, where $c_i \neq 0$, $t \leq N$, and $N$ is a function of $k$. Let $A = A \otimes Z Q$. We consider $g$ as an operator on $A_i$, and obtain that $af(g) = 0$. Since $\{A_i, g\}$ is also nilpotent of class at most $c$, $(g - 1)^c$ annihilates $a$, as $f(g)$ does. Now if $(x - 1)^c$ divides $f(x)$, then $f(1) = f'(1) = \cdots = f^{(c-1)}(1) = 0$. Thus
\[
\begin{align*}
&c_1 + \cdots + c_i = 0 \\
m_1c_1 + \cdots + m_ic_i = 0 \\
&\vdots \quad \vdots \quad \vdots \\
m_1^{e-1}c_1 + \cdots + m_i^{e-1}c_i = 0.
\end{align*}
\]

Note that \(0 = f'(1) = m_i(m_1 - 1)c_1 + \cdots + m_i(m_i - 1)c_i\) implies that \(m_i^2c_1 + \cdots + m_i^2c_i = 0\), since \(m_1c_1 + \cdots + m_ic_i = 0\), and so on. If \(e \geq N\), then \(e \geq t\) and since the matrix
\[
\begin{bmatrix}
1 & 1 & \cdots & 1 \\
m_1 & m_2 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
m_1^{e-1} & m_2^{e-1} & \cdots & m_i^{e-1}
\end{bmatrix}
\]
is invertible, the only solution of the system is \(c_i = 0\), for all \(i = 0, 1, \ldots, t\). This means that \(f(x) = 0\), a contradiction. Therefore \(e < N\). Thus \(a(g - 1)^N = 0\), for all \(a \in A\) and \(g \in G\). In multiplicative notation of the group \(G\), we have
\[
[A, g, \ldots, g] = 1.
\]

Since \(G\) is torsion free, a result of Zel’manov (see [25 p.166]) implies that, \(A\) lies in \(Z_{\mu(N)}(G)\), the \(\mu(N)\) th center of \(G\), where \(\mu(N)\) is a function of \(N\) and independent of the number of generators of \(G\). Thus the nilpotency class of \(G\) is at most \(\lfloor e/2 \rfloor + \mu(N)\) and hence \(e \leq 2\mu(N)\).

**Proof of the Theorem A.** By Lemma 2.1 \(G\) has no section isomorphic to \(W = A \times B\), where \(A\) elementary Abelian, \(B\) finite cyclic, \(C\) a subgroup of \(B\) in which acts faithfully irreducibly on \(A\), such that \(|B : C| > n + t_0 + t_1 + \cdots + t_k\). Thus by a result of Wilson [24], \(G\) is virtually a soluble minimax group. Hence there exist a normal subgroup \(H\) of \(G\) with finite index, such that \(H\) in a soluble minimax group. By Proposition 2.2 \(H\) nilpotent-by-finite and so is \(G\). Thus there exists a normal nilpotent subgroup \(N\) of \(G\) such that \(G / N\) is finite. Now the nilpotency class of \(N / N_i\) is bounded by a function of \(k\), by Lemma 2.3, so the result follows.
3. Proof of Theorem B

To prove Theorem B we follow the arguments given in [19] and obtain the corresponding results.

**Lemma 3.1.** The variety $V_{n,k,t_0,t_1,...,t_k}$ generated by all $\overline{W}_k(n)$-groups is not the variety of all groups.

**Proof.** Let $x_0, x_1, ..., x_n$ be letters. Let $w_1, w_2, ..., w_r$ be the list of all commutators of weight $k + 1$ of the form $[y_{i_0}^{y_0}, y_{i_1}^{y_1}, ..., y_{i_k}^{y_k}] = 1$ where $y_i \in \{x_0, x_1, ..., x_n\}$, and $y_0 \neq y_1$. Let $w = [w_1, w_2, ..., w_r]$, then $w$ is a non-trivial word. It is clear that if $G \in \overline{W}_k(n)$ then $G$ satisfies the law $w = 1$. So the Lemma is proved.

Now as in the proof of [19, Lemma 2.2] we have

**Lemma 3.2.** For all positive integers $n, k$ and non-zero integers $t_0, t_1, ..., t_k$ there exists a positive integer $s(n, k, t_0, t_1, ..., t_k)$ such that if $S$ is a finite simple $\overline{W}_k^+(n)$-group, then $|S| \leq s(n, k, t_0, t_1, ..., t_k)$.

Therefore as in the proof of [19, Proposition 2.4], where we use Lemma 2.1 and Lemma 3.2 instead of Lemma 2.3 and Lemma 2.2 of [19], respectively, we have

**Proposition 3.3.** For all positive integers $n, k$ and non zero integers $t_0, t_1, ..., t_k$ there exists a positive integer $h(n, k, n, t_0, t_1, ..., t_k)$ such that if $G$ is any finite $\overline{W}_k^+(n)$-group, then $G$ has a soluble characteristic subgroup $H$ such that $\exp(G / H)$ divides $h(n, k, n, t_0, t_1, ..., t_k)$.

According to the Proposition 3.3 we must consider finite soluble $\overline{W}_k^+(n)$-group. But firstly we prove the following Lemma, which is similar to the [18, Lemma 3.3] and stated in the proof of [19, Lemma 3.1].

**Lemma 3.4.** Let $f \in F_p[x]$ be polynomial of degree $d$, and let $n_0, n_1, ..., n_d$ be $d + 1$ distinct positive integers. Let $I$ be the ideal generated by $f(x^{n_0}), f(x^{n_1}), ..., f(x^{n_d})$ in $F_p[x]$. Then $(x^m - 1)^r \in I$ for some positive integer $r, m$. 

Proof. Let $M$ be any maximal ideal of $F_p[x]$, and consider the field $F = F_p[x]/M$. By the definition of $I$ and $M$ we have $f(\pi^c) = \overline{0}$, $i = 0, 1, \ldots, d$, where $\overline{\pi}$ denotes the image of $u \in F_p[x]$ in $F$. But $f$ cannot have more than $d = \deg(f)$ distinct roots in $F$. Hence $\pi^n = \pi^c$ for some $n_i < n_j$, which implies $\pi^c = \overline{1}$, where $c = n_j - n_i$. Thus $x^c - 1 \in M$. Now $\sqrt{I}$, the radical of $I$, is the intersection of finitely many maximal ideals, $M_1, M_2, \ldots, M_s$ say, and we have obtained positive integers $c_i$, $i = 1, 2, \ldots, s$, such that $x^{c_i} - 1 \in M_i$. If $m = \prod_{i=1}^s c_i$, we get $x^m - 1 \in \bigcap_{i=1}^s M_i = \sqrt{I}$.

Therefore there exists positive integer $r$ such that $(x^m - 1)^r \in I$, as required.

Observe that if $a, b, a_0, a_1, \ldots, a_k \in A$, where $A$ is Abelian normal subgroup of a group $G$ and $u, v, x \in G$, and $i_j, t_j, t, s$ are positive integers, then

$$[(a_0^{x^b} x)^{y_0}, (a_1^{x^b} x)^{y_1}, \ldots, (a_k^{x^b} x)^{y_k}] = [(a_0^{x^b} x)^{y_0}, (a_1^{x^b} x)^{y_1}, x^{t_2}, \ldots, x^{t_s}]$$

and

$$[x^t a^u, x^s b^v] = a^{u(t-x^s)+v(x^t-1)}.$$ 

We use these facts in the following Lemma.

**Lemma 3.5.** Let $G = \langle g \rangle \cong A$, be the splitting extension of an Abelian normal subgroup $A$ by a cyclic subgroup $\langle g \rangle$. If $G \in \overline{W}_k(n)$ then, $[A, g^m] = 1$, for some positive integers $r, m$ depending only on $n, t_0, t_1, \ldots, t_k$.

*Proof.* Fix $a \in A$ and first suppose that $t_0, t_1, \ldots, t_k$ are positive. Consider the elements $a^{x^b} g, a^{x^b} g, \ldots, a^{x^b} g$. Then, since $G \in \overline{W}_k(n)$, there exist $1 \leq t_0, t_1, \ldots, t_k \leq n + 1$, such that

$$1 = [(a^{x^b} g)^{t_0}, (a^{x^b} g)^{t_1}, \ldots, (a^{x^b} g)^{t_k}] = [g^{x^b} a^{x^b (g^{t_0} + \cdots + g)} a^{x^b (g^{t_1} + \cdots + g)} a^{x^b (g^{t_2} + \cdots + g)} a^{x^b (g^{t_3} + \cdots + g)} a^{x^b (g^{t_4} + \cdots + g)} \ldots a^{x^b (g^{t_k} + \cdots + g)}].$$

Now, since $G$ is metabelian, $[a, a, u, v] = [a, v, u, a]$, for all $a \in G'$ and $u, v \in G$. It follows that in the above equation we may replace $t_j$ by $|t_j|$, for $j \geq 2$. Let $b = a^{x^b (g^{t_0} + \cdots + g)}$ and $c = a^{x^b (g^{t_1} + \cdots + g)}$, and $t_0, t_1 > 0$. Then

$$(a^{x^b} g)^{t_0} = ((a^{x^b} g)^{t_1})^{-1} = g^{x^b} b^{-1}$$ and so
Therefore 1 = [(g^h b)^{-1}, g^i c] = [g^h b, g^i c] = [g^i c, g^h b]^{g^{-h}}.

Therefore 1 = [(g^h b)^{-1}, g^i c, g^j, \ldots, g^k]^{g^{-h}}, and since in a metabelian group we have

\[ [g_1, g_2, g_3, \ldots, g_k]^{-1} = [g_2, g_1, g_3, \ldots, g_k], \]

we obtain that \([g^h b, g^i c, g^j, \ldots, g^k]^{g^{-h}} = 1\). Thus we have shown that we may replace \(t_0\) by \(|t_0|\). Similarly we may replace \(t_1\) by \(|t_1|\). Therefore if we put

\[ f(x) = \left( x^{|i_0|} + x^{|i_1|} + \cdots + x \right) (1 - x^{|i_2|}) \cdots (1 - x^{|i_n|}) \]

we have, in additive notation, that \(a \cdot f(g) = 0\). Note that since \(1 \leq i_0, i_1 \leq n + 1\),

\[ \text{deg}(f) \leq d = (n + 1) \max \{|t_0|, |t_1|\} + \sum_{i=2}^{k} |t_i| \]

Denote the polynomial \( f \) just constructed by \( f_i \in F_p[x] \). By applying the same argument for \( g^m \) instead of \( g \) we obtain a similar polynomial \( f_m(x) \in F_p[x] \) with the property that \( a \cdot f_m(g^m) = 0 \). Observe that there are not more than \((n + 1)^2\) possibilities for the polynomial \( f_m \), since \(t_0, t_1, \ldots, t_k\) are fixed and \(i_0, i_1\) may vary in \(\{1, 2, \ldots, n + 1\}\). Letting \(m\) range between 1 and \(N = 1 + (n + 1)^2\), we conclude that some polynomial \( f \) is obtained, more than \(d\) times in this way, namely

\[ f = f_{m_1} = f_{m_2} = \cdots = f_{m_{d+1}}, \]

where \(m_i \leq N\), are distinct positive integers, and \(a \cdot f(x^m) = 0\), \(i = 1, 2, \ldots, d + 1\).

Let \(I \subseteq F_p[x]\) be the ideal generated by above polynomial \( f(x^m) \). Applying Lemma 3.4 we conclude that some polynomial of the form \( (x^m - 1)^r \) lies in \(I\). Moreover since given \(n, i_0, i_1, \ldots, i_k\), there are boundedly many possibilities for \(m\) and the integers \(m_1, m_2, \ldots, m_{d+1}\), the parameters \(r, m\) are bounded in terms of \(n, i_0, i_1, \ldots, i_k\), say \(r \leq r(n, i_0, i_1, \ldots, i_k)\), \(m \leq m(n, i_0, i_1, \ldots, i_k)\). This implies that \( (x^m - 1)^r \in I \), where \(r := r(n, i_0, i_1, \ldots, i_k)\), \(m := m(n, i_0, i_1, \ldots, i_k)\).
Since $I$ is generated by polynomials which act trivially on $a$, $(g^m - 1)^r$ acts trivially on $a$. However the choice of $r, m$ is independent of $a$. Hence in $End(A)$ we have $(g^m - 1)^r = 0$, and in multiplicative notation, we have $[a, g^m] = 1$, for all $a \in A$.

**Corollary 3.6.** Let $G = \langle A, g \rangle$, where $A$ is an elementary Abelian normal $p$-subgroup of $G$, $p$ a prime. If $G \in \overline{W}_k^n(n)$, then $[A, g^m] = 1$, for some positive integers $r, m$ depending only on $n, t_0, t_1, \ldots, t_k$.

**Proof.** Obviously $G/(A \cap <g>)$ is the splitting extension of $A/(A \cap <g>)$ by $<g>/(A \cap <g>)$. Thus, by Lemma 3.5, there exist positive integers $s, m$ depending only on $n, k, t_0, t_1, \ldots, t_k$, such that $[A, g^m] \in A \cap <g>$. So $1 = [A, g^m, g^m] = [A, g^m, g^m]$.

If we use Corollary 3.6 and follow the proof of the [19, Corollary 3.3] we obtain that

**Corollary 3.7.** Let $G$ be finite soluble $G \in \overline{W}_k^n(n)$-group, then there exists a positive integer $e = e(n, k, t_0, t_1, \ldots, t_k)$ such that $G^e$ is nilpotent.

Using the above results and arguing exactly as the [19, sections 4, 5] we obtain that

**Theorem 3.8.** There exist functions $f, g$ such that every finite group $G \in \overline{W}_k^n(n)$ possesses a nilpotent normal subgroup $N$ satisfying

1. $\exp(G / N)$ divides $f(n, k, t_0, t_1, \ldots, t_k)$, and
2. every $d$-generator subgroup of $N$ has class at most $g(n, k, t_0, t_1, \ldots, t_k)$.

Now the proof of Theorem B follows from argument given in [19, page 52].

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**References**


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