Some Results on Anti-Invariant Submanifolds of a Trans-Sasakian Manifold

MOHAMMED HASAN SHAHID
Department of Mathematics, Faculty of Science, King Abdul Aziz University,
P.O. Box. 80203, Jeddah 21589, Kingdom of Saudi Arabia
e-mail: hasan_jmi@yahoo.com

Abstract. In [7] Oubina introduced a new class of almost contact metric structure known as trans-Sasakian structure which is a generalization of both $\alpha$-Sasakian and $\beta$-Kenmotsu structures [6]. The geometry of anti-invariant submanifolds of Sasakian manifolds have been investigated by Yano and Kon and many others [9,10] etc.

On the other hand, anti-invariant submanifolds of Kenmotsu manifold have been studied by the present author [5]. The purpose of this paper is to study anti-invariant submanifolds of trans-Sasakian manifold generalizing some results on the above mentioned topics.

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1. Introduction

Let $\bar{M}^{2n+1}$ be a $(2n + 1)$-dimensional almost contact metric manifold with structure tensors $(\phi, \xi, \eta, g)$ where $\phi$ is a tensor field of typo (1.1), $\xi$ a vector field, $\eta$ a 1-form and $g$ is the Riemannian metric on $\bar{M}$. Then these tensors satisfy [1]

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\xi) = 1, \quad \eta(\phi X) = 0 \quad (1.1)$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi) \quad (1.2)$$

for any vector fields $X, Y$ tangent to $\bar{M}$.

An almost contact structure $(\phi, \xi, \eta)$ is called normal if the almost complex structure $J$ on $\bar{M} \times R$ given by

$$J \left[ X, f \frac{d}{dt} \right] = \left[ \phi X - f\xi, \quad \eta(X) \frac{d}{dt} \right]$$
being a \( f \) function on \( \overline{M} \times R \), is integrable, or equivalently
\[
[\phi, \phi] + 2d\eta \otimes \xi = 0 \text{ where } [\phi, \phi] \text{ is the Nijenhuis tensor of } \phi.
\]

According to Gray and Hervella [4], in the classification of almost Hermitian manifolds, there appears, a class of Hermitian manifolds namely \( o_4 \) which contains locally conformal Kahler manifold. An almost contact metric structure \((\phi, \xi, \eta, g)\) on \( \overline{M} \) is called trans-Sasakian if \((\overline{M} \times R, J, g)\) belongs to the class \( o_4 \) where \( g \) is a Riemannian metric on \( \overline{M} \times R \). This may be expressed by the condition [2]
\[
\left( \nabla_X \phi \right)(Y) = \alpha \{g(X, Y)\xi - \eta(Y)X\} + \beta [g(\phi X, Y)\xi - \eta(Y)\phi(X)]
\]
\( (1.3) \)

for functions \( \alpha \) and \( \beta \) on \( \overline{M} \) and the Levi-Civita connection \( \nabla \) on \( \overline{M} \), and in this case we say that the trans-Sasakian structure is of type \((\alpha, \beta)\).

If \( \alpha = 0 \) then \( \overline{M} \) is a \( \beta \)-Kenmotsu manifold and if \( \beta = 0 \) then \( \overline{M} \) is \( \alpha \)-Sasakian manifold [6]. Moreover if \( \alpha = 1 \) and \( \beta = 0 \) then \( \overline{M} \) is a Sasakian manifold and if \( \alpha = 0 \) and \( \beta = 1 \) then \( \overline{M} \) is a Kenmotsu manifold. From (1.3), it follows that
\[
\nabla_X \xi = -\alpha \phi X + \beta [X - \eta(X)\xi]. \quad (1.4)
\]

Let \( M \) be an \( m \)-dimensional Riemannian manifold isometrically immersed in a trans-Sasakian manifold \( \overline{M} \). We denote by \( g \) the metric tensor on \( \overline{M} \) as well as that induced on \( M \). Let \( T_x M \) and \( T^\perp_x M \) denote the tangent and normal bundles of \( M \) at \( x \in M \). Let \( \nabla \) and \( \nabla^\perp \) denote the covariant differentiation with respect to the metrics on \( M \) and \( \overline{M} \), respectively. The Gauss and Weingarten formulae for \( M \) are given by
\[
\nabla_X Y = \nabla_X Y + h(X, Y) \text{ and } \nabla_X N = -A_N X + \nabla^\perp_X N \quad (1.5)
\]
respectively, where \( h \) is the second fundamental form of \( M \) in \( \overline{M} \), and \( \nabla^\perp \) is the operator of covariant differentiation with respect to the linear connection induced in the normal bundle \( T^\perp_x M \).

Moreover,
\[
g(h(X, Y), N) = g(A_N X, Y).
\]

A submanifold \( M \) of a trans-Sasakian manifold \( \overline{M} \) is called invariant if \( \phi T_x M \subset T_x M \). On the other hand, if \( \phi T_x M \subset T^\perp_x M \) for all \( x \in M \), then \( M \) is said to be anti-invariant in \( \overline{M} \).
Now suppose $M^m$ is an $m$-dimensional anti-invariant submanifold of a trans-Sasakian manifold $\tilde{M}^{2n+1}$. Then for every vector $Z$ of $\tilde{M}^{2n+1}$ at a point of $M^m$, we put
\[
Z = Z_t + Z_n
\]
where $Z_t$ and $Z_n$ are tangential and normal vectors to $M^m$, respectively. Define homomorphisms $P$ and $Q$ of the normal bundle into the tangent and normal bundles of $M^m$ respectively, by
\[
PN = (\phi N)\quad \text{and} \quad QN = (\phi N)_n
\]
for every normal vector field $N$ of $M^m$.

If $X$ is a vector field on an anti-invariant submanifold $M^m$ of a trans-Sasakian manifold $\tilde{M}^{2n+1}$, then $\phi X$ is a vector field in the normal bundle of $M^m$, where $m > 1$, as any 1-dimensional submanifold is anti-invariant.

Now operating $\phi$ on $\phi X$, $\phi N$ and $\xi$ and comparing tangential and normal components, we get the following:
\[
-X + \eta(X)\xi = P\phi X, \quad \eta(X)\xi_n = Q\phi X,
\]
\[
\eta(N)\xi_t = PQN, \quad -N + \eta(N)\xi_n = \phi PN + Q^2N,
\]
\[
\phi\xi_t + P\xi_n = 0, \quad Q\xi_n = 0
\]
for any $X \in TM$ and $N \in T^1M$.

2. Anti-invariant submainfold of trans-Sasakian manifold when $\xi$ is tangent to $M$

In what follows we assume that $\xi$ is tangent to $M^m$. Then $\xi_n = 0$ and (1.8) becomes
\[
-X + \eta(X)\xi = P\phi X, \quad Q\phi X = 0,
\]
\[
PQN = 0, \quad -N = \phi PN + Q^2N.
\]
From (2.1), we find that $Q^2 + Q = 0$, and hence, $Q$ defines an $f$-structure in the normal bundle [8].
We now assume that $M^m$ is an anti-invariant submanifold of a trans-Sasakian manifold $\bar{M}^{2n+1}$. Then differentiating $\phi X$, $N\phi$, and $\xi$ in the direction of a tangent vector field on $M^m$ and using (1.3), (1.6) and Gauss and Weingarten formulae, we have the following lemmas.

**Lemma 2.1.** Let $M$ be an anti-invariant submanifold of a trans-Sasakian manifold $\bar{M}$ such that $\xi$ is tangent to $M$. Then

\[ A_{\xi X} Y + Ph(X, Y) = \alpha \left[ \eta(X)Y - g(X, Y)\xi \right], \quad (2.2) \]

\[ \nabla^\perp Y X - Qh(X, Y) - \phi \nabla^\perp Y X = -\beta \eta(X) \phi Y \quad (2.3) \]

for any $X, Y \in T$.

**Lemma 2.2.** Let $M$ be an anti-invariant submanifold of trans-Sasakian manifold $\bar{M}$ such that $\xi$ is tangent to $M$. Then

\[ P \nabla^\perp X N = \nabla X PN + A_{\xi X} N, \quad (2.4) \]

\[ -QA_{\xi X} N + Q \nabla^\perp X N = h(X, PN) + \nabla^\perp QN \quad (2.5) \]

for any $X \in TM$ and $N \in T^\perp M$.

**Lemma 2.3.** Let $M$ be an anti-invariant submanifold of a trans-Sasakian manifold $\bar{M}^{2n+1}$ such that $\xi$ is tangent to $M$. Then

\[ \nabla X \xi = \beta (X - \eta(X)\xi), \quad (2.6) \]

\[ h(X, \xi) = \alpha \phi X \quad (2.7) \]

for all $X \in TM$.

From the above lemmas, as particular cases, we have

**Lemma 2.4.** Let $M$ be an anti-invariant submanifold of a $\alpha$-Sasakian manifold $\bar{M}$ such that $\xi$ is tangent to $M$. Then

\[ A_{\xi X} Y + Ph(X, Y) = \alpha \left[ \eta(X)Y - g(X, Y)\xi \right], \]

\[ \nabla^\perp Y X - Qh(X, Y) = \phi \nabla^\perp Y X \]

for any $X, Y \in TM$. 

Lemma 2.5. Let $M$ be an anti-invariant submanifold of a $\beta$-Kenmotsu manifold $\overline{M}$ such that $\xi$ is tangent to $M$. Then

$$A_{\xi\xi}Y + Ph(X,Y) = 0,$$

$$\nabla^+_{\nabla^{-}_{Y}X}X = \phi \nabla^{-}_{Y}X - \beta \eta(X)\phi Y$$

for any $X,Y \in TM$.

Lemma 2.6. Let $M$ be an anti-invariant submanifold of a $\alpha$-Sasakian $\overline{M}$ such that $\xi$ is tangent to $M$. Then

$$\nabla_{X}\xi = 0, \quad h(X,\xi) = \alpha \phi X$$

for all $X \in TM$.

Lemma 2.7. Let $M$ be an anti-invariant submanifold of a $\beta$-Kenmotsu manifold $\overline{M}$ such that $\xi$ is tangent to $M$. Then

$$\nabla_{X}\xi = \beta (X - \eta(X)\xi),$$

$$h(X,\xi) = 0$$

for all $X \in TM$.

Now from Lemma 2.2, we have

$$\nabla_{\xi}PN - P\nabla^+_{\xi}N = A_{Q\xi}\xi.$$

Also

$$g(A_{Q\xi}\xi, X) = g(h(X,\xi), QN) = \alpha g(\phi X, QN) = 0.$$  

Thus

$$(\nabla_{\xi}P)(N) = \nabla_{\xi}PN - P\nabla^+_{\xi}N = 0,$$

and similarly

$$(\nabla_{\xi}Q)(N) = \nabla^+_{\xi}QN - Q\nabla^+_{\xi}N = 0.$$
Hence, we have

**Proposition 2.9.** Let $M$ be an anti-invariant submanifold of a trans-Sasakian manifold $\overline{M}$ such that $\xi$ is tangent to $M$. Then

(a) $\xi$ is parallel vector field along $M$ and $h(\xi, \xi)$ vanishes in the direction of $\xi$.

(b) $P$ and $Q$ are parallel along $\xi$.

Now, suppose that $\dim M = m = n + 1$. Then $Q = 0$ and from Lemma 2.2, we have

$$\overline{R}(X,Y)PN = PR_{\perp}(X,Y)N,$$

for vector field $X, Y$ tangent to $M$, where $R_{\perp}$ is the curvature tensor on the normal bundle. Thus $\overline{R} = 0$ implies that $R_{\perp} = 0$. Conversely, if $R_{\perp} = 0$ then $\overline{R}(X,Y)PN = 0$ and also $\overline{R}(X,Y)\xi = 0$. Hence $\overline{R}(X,Y) = 0$.

Thus we have

**Proposition 2.9.** Let $\overline{M}$ be a $(2n + 1)$-dimensional trans-Sasakian manifold and $M^{n+1}$ be an anti-invariant submanifold $\overline{M}^{2n+1}$ with $\xi$ tangent to $M^{n+1}$. Then $\overline{R} = 0$ if and only if $R_{\perp} = 0$.

Next, we prove

**Proposition 2.10.** Let $M^{n+1}$ be an anti-invariant submanifold of a trans-Sasakian manifold $\overline{M}^{2n+1}$ such that $\xi$ is tangent to $M^{n+1}$. Then $M$ cannot be totally umbilical when $n \geq 1$.

**Proof.** Suppose $M$ is totally umbilical. Then $h(X,Y) = g(X,Y)H$, where $H$ is the mean curvature vector. From (2.7), we have $h(\xi, \xi) = 0$ which implies that $g(\xi, \xi)H = 0$ and therefore $M$ is minimal and hence totally geodesic. Thus, we have $h(X, \xi) = 0$ and consequently $\alpha \phi X = 0$, which is a contradiction as $n > 1$. Hence $M$ is not totally umbilical, whereby proving the result.

**Proposition 2.11.** Let $M$ be an anti-invariant submanifold of a trans-Sasakian manifold $\overline{M}$ with $\xi$ tangent to $M^n$. Then we have

$$\nabla_X F(X, \xi) = -\alpha, \quad (2.8)$$

$$\nabla_X \eta(X) = \beta \quad (2.9)$$
where $F$ is the fundamental 2-form given by

$$F(X, Y) = g(X, \phi Y).$$

**Proof.** From (1.3) and (1.4), we have

$$(\overline{\nabla}_X F)(Y, Z) = -\alpha \{ g(X, Z)\eta(Y) - g(X, Y)\eta(Z) \} - \beta [g(X, \phi Z)\eta(Y) - g(X, \phi Y)\eta(Z) ]$$

and

$$(\nabla_X \eta)(Y) = -\alpha g(\phi Y, Y) - \beta [g(X, Y) - \eta(X)\eta(Y)].$$

so that our assertion follows from the above equations.

**Proposition 2.12.** Let $M$ be an anti-invariant submanifold of a trans-Sasakian manifold $\overline{M}$ with $\xi$ tangent to $M$. If $A_N X = 0$ for any $N \in T^\perp_M$ then $\phi(T^\perp_M)$ is parallel with respect to the normal connection.

**Proof.** Using Gauss and Weingarten formulae and Equation (1.3), we have

\[
\nabla^\perp_X \phi Y = \nabla^\perp_X \phi Y + A_{\phi Y} X = (\nabla^\perp_X \phi)(Y) + \phi \nabla^\perp_X Y + A_{\phi Y} X
\]

\[
= \alpha \{ g(X, Y)\xi - \eta(Y)X \} + \beta \{ g(\phi X, Y)\xi - \eta(Y)\phi X \} + \phi \nabla^\perp_X Y + A_{\phi Y} X.
\]

Since $A_N = 0$ for any $N \in T^\perp_M$, we have

\[
g(\nabla^\perp_X \phi Y, N) = -\beta \eta(Y)g(\phi X, N) - g(\overline{\nabla}_X Y, \phi N)
\]

\[
= -\beta \eta(Y)g(\phi X, N) + g(\phi \nabla^\perp_X Y, N) + g(h(X, Y), N)
\]

\[
= \beta \eta(Y)g(X, \phi N) - g(\nabla_X Y, \phi N) + g(h(X, Y), \phi N)
\]

\[
= -g(A_{\phi X} Y, Y) = 0
\]

as $\phi N \in T^\perp_M$ for any $N \in T^\perp_M$, which proves the result.

### 3. Anti-invariant submanifold of trans-Sasakian manifold when $\xi$ is normal to $M$

In this section, we assume that $\xi$ is normal to $M$. Then $\xi_t = 0$ and from (1.8), we get...
\[-X = P\phi X, \quad Q\phi X = 0, \quad PQN = 0,\]
\[-N + \eta(N)\xi = \phi PN + Q^2 N\]

for any \(X \in TM, \ N \in T^1M\).

Now suppose that \(\overline{M}^{2n+1}\) is a trans-Sasakian manifold. Then differentiating \(\phi X, \ phi N\) and \(\xi\) covariantly and using (1.3), (1.7) and Gauss and Weingarten formulae, we have the following:

**Lemma 3.1.** Let \(M\) be an anti-invariant submanifold of a trans-Sasakian manifold \(\overline{M}\) such that \(\xi\) is normal to \(M\). Then

\[A_{\phi}X = P h(X, Y),\]  
(3.1)
\[\nabla_{\phi}^{\perp} \phi Y = -\alpha g(X, Y)\xi + \phi \nabla_{\phi}^\perp Y + Q h(X, Y)\]  
(3.2)

for any \(X, Y \in TM\).

**Lemma 3.2.** Let \(M\) be an anti-invariant submanifold of a trans-Sasakian manifold \(\overline{M}\) such that \(\xi\) is normal to \(M\). Then

\[\nabla X PN - A_{PN}X - PV X N + \alpha \eta(N)X = 0,\]  
(3.3)
\[h(X, PN) + \nabla_{\phi}^{\perp} QN - Q\nabla_{\phi}^{\perp} N + \phi A_{N}X = \beta g(\phi X, N)\xi - \eta(N)\phi X\]  
(3.4)

for any \(X \in TM, \ N \in T^1M\).

**Lemma 3.3.** Let \(M\) be an anti-invariant submanifold of a trans-Sasakian manifold \(\overline{M}\) such that \(\xi\) is normal to \(M\). Then

\[-A_{\xi}X = \beta X,\]  
(3.5)
\[\nabla_{\phi}^{\perp} \xi = -\alpha \phi X\]  
(3.6)

for \(X \in TM\).

We now prove the following.

**Proposition 3.4.** If \(M\) is an anti-invariant submanifold of a trans-Sasakian manifold \(\overline{M}\) such that \(\xi\) is normal to \(M\), then the curvature tensor of the normal bundle annihilates \(\xi\).
Proof. From (3.2) and (3.6), we get

\[
\nabla_X^\perp(\nabla_X^\perp \xi) = \nabla_X^\perp(-\alpha \phi X) = -\alpha (\nabla_X^\perp \phi X)
\]

\[
= -\alpha \left\{ -\alpha g(X, Y) \xi + \alpha \nabla_Y^\perp X + Qh(X, Y) \right\}
\]

\[
= \alpha^2 g(X, Y) \xi - \alpha \phi \nabla_Y^\perp X - \alpha Qh(X, Y)
\]

for \( X, Y \in TM \).

Now,

\[
R^\perp(X, Y)\xi = \nabla_X^\perp \nabla_Y^\perp \xi - \nabla_Y^\perp \nabla_X^\perp \xi - \nabla_{[X, Y]}^\perp \xi
\]

\[
= \nabla_X^\perp (-\alpha \phi Y) - \nabla_Y^\perp (-\alpha \phi Y) - \alpha \phi [X, Y]
\]

\[
= -\alpha \left\{ \nabla_X^\perp (\phi Y) - \nabla_Y^\perp (\phi X) - \phi [X, Y] \right\}
\]

which, in view of (3.2), gives that

\[
R^\perp(X, Y)\xi = 0
\]

whereby proving the result.

Suppose that \( m = n \) and hence \( Q = 0 \). Then from (3.1)–(3.6), we have

**Lemma 3.5.** Let \( M \) be an anti-invariant submanifold of a trans-Sasakian manifold \( \overline{M} \) such that \( \xi \) is normal to \( M \). Then

\[
A\phi X = \phi h(X, Y), \quad \nabla_X^\perp (\phi Y) = -\alpha g(X, Y) + \phi \nabla_X Y, \quad \quad (3.7)
\]

\[
\alpha \eta N X - \phi \nabla_X^\perp N = \nabla_X \phi N, \quad \quad (3.8)
\]

\[
A\phi X = -\beta X, \quad \nabla_X^\perp \xi = -\alpha \phi X \quad \quad (3.9)
\]

for any \( X, Y \in TM \).

**Proposition 3.6.** Let \( M \) be an anti-invariant submanifold of a trans-Sasakian manifold \( \overline{M} \) such that \( \xi \) is normal to \( M \). Then the connection in the normal bundle is trivial if and only if \( M \) is of constant curvature \( -\alpha^2 \).

Proof. Using \( \nabla_X^\perp (\phi Y) = -\alpha g(X, Y) + \phi \nabla_X Y \) and \( \nabla_X^\perp \xi = -\alpha \phi X \).
we have
\[
R^\perp(X, Y)\phi Z = \nabla^\perp_Y \left( \nabla^\perp_X \phi Z \right) - \nabla^\perp_X \left( \nabla^\perp_Y \phi Z \right) - \nabla^\perp_{[X,Y]} \phi Z \\
= -\alpha g(Y, Z)\nabla^\perp_X \xi + \nabla^\perp_X (\phi \nabla^\perp_Y Z) + \alpha g(X, Z) \nabla^\perp_Y \xi \\
- \nabla^\perp_Y (\phi \nabla^\perp_X Z) + \alpha g([X,Y], Z) \xi - \phi \nabla^\perp_{[X,Y]} Z
\]
or,
\[
R^\perp(X, Y)\phi Z = \alpha^2 g(Y, Z)\phi X + \nabla^\perp_X \left( \phi \nabla^\perp_Y Z \right) - \alpha^2 g(X, Z)\phi Y \\
- \nabla^\perp_Y (\phi \nabla^\perp_X Z) + \alpha g([X,Y], Z) \xi - \phi \nabla^\perp_{[X,Y]} Z. \\
\text{(3.10)}
\]
Also,
\[
\phi R(X, Y)Z + \alpha^2 \{ g(Y, Z)\phi X - g(X, Z)\phi Y \} \\
= \nabla^\perp_X (\phi \nabla^\perp_Y Z) + \alpha g(X, \nabla^\perp_Y Z) \xi - \nabla^\perp_Y (\phi \nabla^\perp_X Z) - \alpha g(Y, \nabla^\perp_X Z) \xi \\
- \phi \nabla^\perp_{[X,Y]} Z + \alpha^2 \{ g(Y, Z)\phi X - g(X, Z)\phi Y \}
\]
which on further simplification, gives
\[
\phi R(X, Y)Z + \alpha^2 \{ g(Y, Z)\phi X - g(X, Z)\phi Y \} = R^\perp(X, Y)\phi Z \\
\text{(3.11)}
\]
for any \( X, Y, Z \in TM \).

From (3.11) we find that if the connection of the normal bundle is trivial, that is, \( R^\perp = 0 \). Then \( M^\perp \) is of constant curvature \( -\alpha^2 \).

Conversely, if \( M^\perp \) is of constant curvature \( -\alpha^2 \), then from (3.11) we have \( R^\perp(X, Y)\phi Z = 0 \). Moreover, from Proposition 3.4, we have \( R^\perp(X, Y)\xi = 0 \), which completes the proof.

From the above result, we have

**Corollary 3.7.** Let \( M \) be an anti-invariant submanifold of a \( \beta \)-Kenmotsu manifold \( \overline{M} \) such that \( \xi \) is normal to \( M \). Then the connection in the normal bundle is trivial if and only if \( M \) is of zero constant curvature.
References

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