On a class of discontinuous operators in Hilbert spaces

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Abstract. We construct a class of discontinuous operators in infinite-dimensional separable Hilbert spaces, answering a natural question which arises in comparing a fixed point theorem of Altman and Shinbrot ([1], [4]) with its improvement obtained by Ricceri ([2], [3]).

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In [4], M. Shinbrot gave a proof of the following fixed point theorem which was previously announced (without proof) by M. Altman in [1]:

Theorem A. Let \((H, \langle \cdot, \cdot \rangle)\) be a separable real Hilbert space, and \(\Psi : H \to H\) a sequentially weakly continuous operator. Assume that there is some \(r > 0\) such that
\[
\langle \Psi(x), x \rangle \leq r^2
\]
for all \(x \in H\) satisfying \(\|x\| = r\).
Then, there exists \(x^* \in H\) such that \(x^* = \Psi(x^*)\) and \(\|x\| \leq r\).

In [3] (see also [2]), B. Ricceri obtained an extension of Theorem A to a class of discontinuous operators. His result was as follows:

Theorem B. Let \((H, \langle \cdot, \cdot \rangle)\) be an infinite-dimensional separable real Hilbert space; \(V\) the linear hull of an orthonormal base \(\{e_n\}\) of \(H\); \(X \subseteq H\) a closed, bounded, convex set, with \(0 \in \text{int}(X)\). Further, let \(\Psi : X \to H\) be an operator satisfying the following conditions:

(i) for each \(y \in V\), the set
\[
\{x \in X \cap V : \langle x - \Psi(x), y \rangle \leq 0\}
\]
is finitely closed (that is, its intersection with any finite-dimensional linear subspace of \(H\) is closed);
(ii) for each \(n \in \mathbb{N}\), the set
\[
\{x \in X : \langle x - \Psi(x), e_n \rangle = 0\}
\]
is weakly closed;  
(iii) for each \( x \in V \cap \partial X \), one has \[
\langle \Psi(x), x \rangle \leq \|x\|^2.
\]

Then, there exists \( x^* \in X \) such that \( x^* = \Psi(x^*) \).

It is clear that the most natural (though less general) way to check (i) and (ii) is to assume that, for each \( n \in \mathbb{N} \), the functional \( x \to \langle \Psi(x), e_n \rangle \) be sequentially weakly continuous in \( X \). To see this, take into account that, since \( H \) is separable and \( X \) is weakly compact, the relative weak topology on \( X \) can be deduced by a metric.

On the other hand, the most natural condition ensuring the sequential weak continuity of each functional \( x \to \langle \Psi(x), e_n \rangle \) \((n \in \mathbb{N})\) is the sequential weak continuity of the operator \( \Psi \), just as required in Theorem A.

Then, it is natural to ask whether there exist operators \( \Psi : X \to H \) which, though not sequentially weakly continuous, satisfy condition (iii) and, at the same time, are such that, for each \( n \in \mathbb{N} \), the functional \( x \to \langle \Psi(x), e_n \rangle \) is sequentially weakly continuous.

The aim of this paper is to provide an affirmative answer to such a question.

Our main result is as follows:

**Theorem 1.** Let \( (H, \langle \cdot, \cdot \rangle) \) be an infinite-dimensional separable real Hilbert space and \( \{e_n\} \) an orthonormal base of \( H \). Put 
\[
Y = \{x \in H : \langle x, e_1 \rangle = 0\}.
\]

Then, there exists an operator \( \Phi : H \to H \) which has the following properties:

(a) \( Y \subseteq \Phi^{-1}(0) \);
(b) for each \( n \in \mathbb{N} \), the functional \( x \to \langle \Phi(x), e_n \rangle \) is weakly continuous;
(c) \( \langle \Phi(x), x \rangle = 0 \) for all \( x \in H \);
(d) \( \limsup_{\|x\| \to 0} \|\Phi(x)\| = +\infty \).

**Proof:** For each \( n \in \mathbb{N} \), define \( \alpha_n : \mathbb{R} \to \mathbb{R} \) by
\[
\alpha_n(t) = \begin{cases} 
  t^{-4} & \text{if } |t| > n^{-\frac{1}{2}}, \\
  n^2 & \text{if } (2n)^{-\frac{1}{2}} \leq |t| \leq n^{-\frac{1}{2}}, \\
  2^\frac{1}{2} n^\frac{5}{2} |t| & \text{if } |t| < (2n)^{-\frac{1}{2}}.
\end{cases}
\]

Note that each function \( \alpha_n \) is continuous and non-negative. Moreover, for each \( n \in \mathbb{N}, t \in \mathbb{R} \), one has
\[
\alpha_n(t) \leq \alpha_{n+1}(t)
\]
as well as
\[ \sup_{n \in \mathbb{N}} \alpha_n(t) < +\infty. \]

Now, put
\[ \varphi_n(t) = (\alpha_n(t) - \alpha_{n-1}(t))^{\frac{1}{2}} \]
with \( \alpha_0(t) = 0 \). Also, for each \( x \in H, n \in \mathbb{N} \), set
\[ \gamma_n(x) = \begin{cases} 
-\varphi_{n+\frac{1}{2}}(\langle x, e_1 \rangle)\langle x, e_{n+1} \rangle & \text{if } n \text{ is odd}, \\
\varphi_{n+\frac{1}{2}}(\langle x, e_1 \rangle)\langle x, e_{n-1} \rangle & \text{if } n \text{ is even}.
\end{cases} \]

Fix \( x \in H \). Clearly, the series
\[ |\langle x, e_2 \rangle|^2 + |\langle x, e_1 \rangle|^2 + |\langle x, e_4 \rangle|^2 + |\langle x, e_3 \rangle|^2 + \ldots \]
is convergent and the sequence
\[ |\varphi_1(\langle x, e_1 \rangle)|^2, |\varphi_1(\langle x, e_1 \rangle)|^2, |\varphi_2(\langle x, e_1 \rangle)|^2, |\varphi_2(\langle x, e_1 \rangle)|^2 \ldots \]
is bounded. So, by a classical result, the series
\[
|\gamma_1(x)|^2 + |\gamma_2(x)|^2 + |\gamma_3(x)|^2 + |\gamma_4(x)|^2 + \ldots \\
= |\varphi_1(\langle x, e_1 \rangle)|^2|\langle x, e_2 \rangle|^2 + |\varphi_1(\langle x, e_1 \rangle)|^2|\langle x, e_1 \rangle|^2 \\
+ |\varphi_2(\langle x, e_1 \rangle)|^2|\langle x, e_4 \rangle|^2 + |\varphi_2(\langle x, e_1 \rangle)|^2|\langle x, e_3 \rangle|^2 + \ldots
\]
is convergent. Then, by the Riesz-Fischer theorem, for each \( x \in H \), the series
\[ \gamma_1(x)e_1 + \gamma_2(x)e_2 + \gamma_3(x)e_3 + \gamma_4(x)e_4 + \ldots \]
is convergent in \( H \). For each \( x \in H \), put
\[ \Phi(x) = \sum_{n=1}^{\infty} \gamma_n(x)e_n. \]

So, for each \( n \in \mathbb{N} \), one has
\[ \gamma_n(x) = \langle \Phi(x), e_n \rangle. \]

Let us now prove that the operator \( \Phi : H \to H \) just defined has properties (a)-(d). Property (a) follows at once observing that \( \varphi_n(0) = \gamma_n(0) = 0 \) for all \( n \in \mathbb{N} \). Concerning (b), the weak continuity of each functional \( \gamma_n \) follows at once from
the continuity of $\varphi_n$ and the weak continuity of any continuous linear functional on $H$. For each $x \in H$, one has

\[
\langle \Phi(x), x \rangle = \sum_{n=1}^{\infty} \gamma_n(x) \langle x, e_n \rangle
\]

\[
= -\varphi_1(\langle x, e_1 \rangle) \langle x, e_2 \rangle \langle x, e_1 \rangle + \varphi_1(\langle x, e_1 \rangle) \langle x, e_1 \rangle \langle x, e_2 \rangle
\]

\[
- \varphi_2(\langle x, e_1 \rangle) \langle x, e_3 \rangle \langle x, e_2 \rangle + \varphi_2(\langle x, e_1 \rangle) \langle x, e_3 \rangle \langle x, e_2 \rangle + \ldots
\]

Observe that $\sum_{n=1}^{2k} \gamma_n(x) \langle x, e_n \rangle = 0$ for each $k \in \mathbb{N}$, and so $\langle \Phi(x), x \rangle = 0$. That is, (c) is satisfied. Finally, let us check that (d) is satisfied too. To this end, fix $M > 0$ and $r \in ]0,1[$. We shall prove that there is $x \in H$, with $\|x\|^2 = r$, such that $\|\Phi(x)\|^2 > M$. Fix $p \in \mathbb{N}$, with $p > Mr^{-\frac{3}{2}}$. For each $n \in \mathbb{N}$, put

\[
\eta_n = \begin{cases} 
\left(\frac{r}{2p}\right)^{\frac{1}{2}} & \text{if } n \leq 2p, \\
0 & \text{if } n > 2p.
\end{cases}
\]

Finally, set

\[
x = \sum_{n=1}^{\infty} \eta_n e_n.
\]

Clearly, $\|x\|^2 = r$. Also, one has

\[
\|\Phi(x)\|^2 = \frac{r}{p} \sum_{n=1}^{p} \varphi_n \left(\left(\frac{r}{2p}\right)^{\frac{1}{2}}\right) = \frac{r}{p} \alpha_p \left(\left(\frac{r}{2p}\right)^{\frac{1}{2}}\right) = r^3 p > M.
\]

This concludes the proof. \(\square\)

**Remark 1.** Observe that, by (d), the operator $\Phi$ is even discontinuous with respect to the strong topology.

Applying Theorem B, via Theorem 1, we then get the following extension of Theorem A:

**Theorem 2.** Let $(H, \langle \cdot, \cdot \rangle)$ be an infinite-dimensional separable real Hilbert space, $X \subseteq H$ a closed, bounded, convex set, with $0 \in \text{int}(X)$, and $\Psi : X \to H$ a sequentially weakly continuous operator such that

\[
\langle \Psi(x), x \rangle \leq \|x\|^2
\]

for all $x \in \partial X$.

Then, for each operator $\Phi : H \to H$ as in Theorem 1, the operator $\Phi + \Psi$ is not sequentially weakly continuous and admits a fixed point in $X$.

From Theorem 2, in particular, we get the following surjectivity result:
Theorem 3. Let $\Phi : H \to H$ be an operator as in Theorem 1. Then, the operator $x \to x - \Phi(x)$ is surjective.

Proof: Fix $y \in H$ and choose $r > \|y\|$. Let $X = \{x \in H : \|x\| \leq r\}$, and put $\Psi(x) = y$ for all $x \in X$. Then, since, for each $x \in \partial X$, one has

$$\langle \Psi(x), x \rangle \leq \|y\| \|x\| \leq \|x\|^2,$$

one can apply Theorem 2, and so there exists $x^* \in X$ such that $x^* = y + \Phi(x^*)$, as claimed. \hfill \Box

We conclude observing that, when $\Psi : H \to H$ is an affine operator, Theorem B coincides substantially with Theorem A. In fact, we have the following result:

Theorem 4. Let $H$, $\{e_n\}$, and $X$ be as in Theorem B, and let $\Psi : H \to H$ be a linear operator such that, for each, the set

$$\{x \in X : \langle x - \Psi(x), e_n \rangle = 0\}$$

is closed.

Then, $\Psi$ is continuous.

Proof: First, observe that, if $A \subseteq H$ is a linear subspace such that $A \cap X$ is closed, then $A$ is closed. Indeed, fix $r > 0$ so that $\{x \in H : \|x\| \leq r\} \subseteq X$. Let $x \in \overline{A} \setminus \{0\}$, and let $\{x_n\}$ be any sequence in $A \setminus \{0\}$ converging to $x$. Then, the sequence $\left\{\frac{rx_n}{\|x_n\|}\right\}$ lies in $A \cap X$ and converges to $\frac{rx}{\|x\|}$. Since $A \cap X$ is closed, it follows that $\frac{rx}{\|x\|} \in A \cap X$, and so $x \in A$, as claimed. Consequently, by assumption, for each $n \in \mathbb{N}$, the hyperplane

$$\{x \in H : \langle x - \Psi(x), e_n \rangle = 0\}$$

is closed, and hence the functional $x \to \langle x - \Psi(x), e_n \rangle$ is continuous. Then, by Osgood’s lemma, there is a non-empty open set $\Omega \subset H$ such that

$$\sup_{x \in \Omega} \sup_{n \in \mathbb{N}} \sum_{i=1}^{n} |\langle x - \Psi(x), e_i \rangle|^2 < +\infty.$$

On the other hand, by Parseval’s identity, we have

$$\sup_{n \in \mathbb{N}} \sum_{i=1}^{n} |\langle x - \Psi(x), e_i \rangle|^2 = \|x - \Psi(x)\|^2$$

and so

$$\sup_{x \in \Omega} \|x - \Psi(x)\| < +\infty.$$

From this, of course, the continuity of $\Psi$ follows. \hfill \Box
References


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