Lyapunov measures on effect algebras

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Abstract. We prove a Lyapunov type theorem for modular measures on lattice ordered effect algebras.

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1. Introduction

The celebrated Lyapunov’s theorem says that the range of a non-atomic finite dimensional measure $\mu$ on a $\sigma$-algebra is convex. In general, this is not true if $\mu$ is infinite dimensional. On the other hand, Knowles showed that when $\mu$ is properly non-injective with values in a locally convex linear space, then its range is still convex. In [11], De Lucia and Wright, after introducing a notion of a convex set, generalize Knowles’ result to the case when $\mu$ is group-valued.

In noncommutative measure theory it is known (see [5, Example 3.7]) that there are examples of nonatomic $\mathbb{R}^n$-valued measures on effect algebras which do not have a convex range. Nevertheless, in [5] it is proved (see 3.12) that a Lyapunov type theorem holds for $\mathbb{R}^n$-valued modular measures on lattice ordered effect algebras. Moreover, in [2], the result of [11] has been extended to modular functions on complemented lattices. Then a natural question arises: Is it possible to extend the result of [11] to modular measures on effect algebras?

In this paper we give an affirmative answer to this question, introducing the notion of a pseudo non-injective measure (see Definition 4.1) in an effect algebra which is equivalent to the notion of properly non-injective measures in the Boolean case.

We recall that effect algebras have been introduced by D.J. Foulis and M.K. Bennett in 1994 (see [7]) for modelling unsharp measurement in a quantum mechanical system. They are a generalization of many structures which arise in quantum physics (see [6]) and in Mathematical Economics (see [14] and [9]), in particular of orthomodular lattices in noncommutative measure theory (e.g. see [12]) and MV-algebras in fuzzy measure theory.
2. Preliminaries

We will fix some notations. First we will give the definition of a D-poset. Examples of D-posets can be found in [10] and [13].

Definition 2.1. Let \((L, \leq)\) be a partial ordered set (a poset for short). A partial binary operation \(\ominus\) on \(L\) such that \(b \ominus a\) is defined iff \(a \leq b\) is called a difference on \((L, \leq)\) if the following conditions are satisfied for all \(a, b, c \in L\):

1. if \(a \leq b\) then \(b \ominus a \leq b\) and \(b \ominus (b \ominus a) = a\),
2. if \(a \leq b \leq c\) then \(c \ominus b \leq c \ominus a\) and \((c \ominus a) \ominus (c \ominus b) = b \ominus a\).

Definition 2.2. Let \((L, \leq, \ominus)\) be a poset with difference and let 0 and 1 be the least and greatest elements in \(L\), respectively. The structure \((L, \leq, \ominus)\) is called a difference poset (D-poset for short), or a difference lattice (D-lattice for short) if \(L\) is a lattice.

An alternative structure to a D-poset is that of an effect algebra introduced by Foulis and Bennett in [7]. These two structures, D-posets and effect algebras, are equivalent as shown in [13, Theorem 1.3.4].

We recall that a D-lattice is complete (\(\sigma\)-complete) if every set (countable set) has a supremum and an infimum.

If \(a \in L\), we set \(a^\perp = 1 \ominus a\).

We say that \(a\) and \(b\) are orthogonal if \(a \leq b^\perp\) and we write \(a \downarrow b\). If \(a \downarrow b\), we set \(a \oplus b = (a^\perp \ominus b)^\perp\). If \(a_1, \ldots, a_n \in L\) we define inductively \(a_1 \oplus \cdots \oplus a_n = (a_1 \oplus \cdots \oplus a_{n-1}) \ominus a_n\) if the right-hand side exists. The sum is independent of any permutation of the elements. We say that \(\{a_1, \ldots, a_n\}\) is orthogonal if \(a_1 \oplus \cdots \oplus a_n\) exists. We say that a family \(\{a_\alpha\}_{\alpha \in A}\) is orthogonal if every finite subfamily is orthogonal. If \(\{a_\alpha\}_{\alpha \in A}\) is orthogonal, we define \(\bigoplus_{\alpha \in A} a_\alpha := \sup\{\bigoplus_{\alpha \in F} a_\alpha : F \subset A \text{ finite}\}\) if the left-hand side exists.

If \((G, +)\) is an abelian group, a function \(\mu : L \to G\) is called modular if, for every \(a, b \in L\), \(\mu(a \vee b) + \mu(a \wedge b) = \mu(a) + \mu(b)\); \(\mu\) is called a measure if, for every \(a, b \in L\), with \(a \downarrow b\), \(\mu(a \oplus b) = \mu(a) + \mu(b)\). It is easy to see that \(\mu\) is a measure iff for every \(a, b \in L\) with \(b \leq a\), \(\mu(a \ominus b) = \mu(a) - \mu(b)\).

A measure \(\mu\) is said to be \(\sigma\)-additive if, for every orthogonal sequence in \(L\) such that \(a = \bigoplus_{n \in \mathbb{N}} a_n\) exists, \(\mu(a) = \sum_{n \in \mathbb{N}} \mu(a_n)\). A measure \(\mu\) is said to be completely additive if for every orthogonal family \(\{a_\alpha\}_{\alpha \in A}\) in \(L\) such that \(a = \bigoplus_{\alpha \in A} a_\alpha\) exists, the family \(\{\mu(a_\alpha)\}_{\alpha \in A}\) is summable in \(G\) and \(\mu(a) = \sum_{\alpha \in A} \mu(a_\alpha)\).

Recall that by 3.1 of [17] every modular function \(\mu : L \to G\) on any lattice generates a lattice uniformity, \(\mathcal{U}(\mu)\), i.e. a uniformity which makes \(\wedge\) and \(\vee\) uniformly continuous.

We say that \(\mathcal{U}(\mu)\) is exhaustive if every monotone sequence \(\{a_n\}\) is a Cauchy sequence. We say that \(\mathcal{U}(\mu)\) is \(\sigma\)-order (order) continuous if every sequence (net)
\{a_n\} which is order converging to \(a\) is converging to \(a\). We say that a modular measure is exhaustive, \(\sigma\)-order (order) continuous iff \(U(\mu)\) is so. By 2.2 of [4], a measure is \(\sigma\)-additive iff it is \(\sigma\)-order continuous.

Throughout this article, \((G, +)\) is an abelian topological Hausdorff group which has not \(\mathbb{Z}_2\) as a subgroup, \(L\) is a \(\sigma\)-complete \(D\)-lattice and \(\mu : L \rightarrow G\) is a \(\sigma\)-additive modular measure.

### 3. Semi-convexity

We shall call \(x \in G\) infinitely divisible if for every \(n \in \mathbb{N}\) there exists \(y \in G\) such that \(2^n y = x\). Since \(\mathbb{Z}_2\) is not a subgroup of \(G\) it is clear that when \(2^n y = x\), \(y\) is uniquely determined. In what follows we shall denote such a \(y\) by \(\frac{1}{2^n} x\). If \(d = \frac{s}{2^n}\) is a dyadic rational number of the real interval \([0, 1]\) and \(x \in G\) is infinitely divisible, we define \(dx\) to be \(sy\), where \(y = \frac{1}{2^n} x\). By [11] the definition of \(dx\) does not depend on the representation of \(d\). Let \(D\) be the set of dyadic rationals in \([0, 1]\). For every infinite divisible \(x \in G\), let \(g_x : D \rightarrow G\) be defined by \(g_x(d) = dx\) for \(d \in D\). If \(t \in [0, 1]\) and \(\lim_{d \to t} g_x(d)\) exists in \(G\), we define \(tx = \lim_{d \to t} g_x(d)\).

If \(M \subset G\), \(M\) is said to be convex if for every \(x, y \in M\) and \(t \in [0, 1]\), \(tx, (1 - t)y\) exist and \(tx + (1 - t)y \in M\).

**Definition 3.1.** A measure \(\mu\) is said to be semiconvex if, for each \(b \in L\), there exists \(c \in L\) such that \(c \leq b\) and \(\mu(b) = 2\mu(c)\).

**Lemma 3.2.** If \(\mu\) is semiconvex, then every element of \(\mu(L)\) is infinitely divisible.

**Proof:** For every \(a \in L\) and \(n \in \mathbb{N}\), there exists \(b \leq a\) such that \(\mu(a) = 2^n \mu(b)\).

**Lemma 3.3.** Suppose that \(\mu\) is semiconvex. Then for every \(a \in L\) and \(d \in D\), there exists \(a_d \leq a\) such that \(\mu(a_d) = d \mu(a)\). Moreover, if \(d_1 < d_2\), then \(a_{d_1} \leq a_{d_2}\).

**Proof:** Let \(a \in L\).

(i) Claim 1: For every \(n \in \mathbb{N}\) there exists an orthogonal family \(\Pi_n = \{a_{n,1}, \ldots, a_{n,2^n}\}\) in \(L\) such that \(\bigoplus_{j=1}^{2^n} a_{n,j} = a\) and, for every \(i \in \{1, \ldots, 2^n\}\) we have:

(a) \(2^n \mu(a_{n,i}) = \mu(a)\),
(b) \(a_{n,2i-1} \oplus a_{n,2i} = a_{n-1,i}\).

This is trivial for \(n = 1\): Since \(\mu\) is semiconvex, we can choose \(a_{1,1} \leq a\) such that \(2\mu(a_{1,1}) = \mu(a)\). Let \(a_{1,2} := a \oplus a_{1,1}\). Then \(a_{1,1} \oplus a_{1,2} = a\) and \(2\mu(a_{1,2}) = 2\mu(a) - 2\mu(a_{1,1}) = \mu(a)\).

By induction, suppose that Claim 1 holds for \(n \in \mathbb{N}\). Since \(\mu\) is semiconvex, for every \(i \in \{1, \ldots, 2^n\}\) we can find \(a_{n+1,2i-1}, a_{n+1,2i}\) in \(L\) such that \(a_{n+1,2i-1} \oplus a_{n+1,2i} = a_{n,i}\) and \(2\mu(a_{n+1,2i-1}) = 2\mu(a_{n+1,2i}) = \mu(a_{n,i})\).
Set $\Pi_{n+1} = \{a_{n+1,1}, a_{n+1,2}, \ldots, a_{n+1,2^n+1}\}$. Then $\Pi_{n+1}$ is orthogonal since
\[ a = \bigoplus_{i=1}^{2n} a_{n,i} = \bigoplus_{i=1}^{2n}(a_{n+1,2i-1} \oplus a_{n+1,2i}) = \bigoplus_{i=1}^{2n+1} a_{n+1,i} \]
and for every $i \in \{1, \ldots, 2^{n+1}\}$ we have $2^{n+1} \mu(a_{n+1,i}) = 2^n \mu(a_{n,i}) = \mu(a)$.

(ii) Now we obtain a family $\{b_{n,s} : n \in \mathbb{N}\}$ with $s \in \{0,1,\ldots,2^n\}$ such that:

1. $b_{n,0} = 0$ and $b_{n,2^n} = a$,
2. $b_{n,i-1} \leq b_{n,i}$,
3. $2^n \mu(b_{n,i}) = \mu(a)$,
4. If $\frac{n}{2^m} = \frac{r}{2^s}$, then $b_{m,r} = b_{n,s}$.

It is sufficient to set $b_{n,0} = 0$ and, for $i \in \{1, \ldots, 2^n\}$, $b_{n,i} = \bigoplus_{j \leq i} a_{n,j}$.

(iii) If $d = \frac{r}{2^m}$, set $a_d = b_{m,r}$. Then by (ii), $a_d \leq a$ and $2^m \mu(a_d) = r \mu(a)$, from which $\mu(a_d) = d \mu(a)$. Moreover, by (ii), if $d_1 < d_2$ then $a_{d_1} \leq a_{d_2}$. \hfill \Box

**Lemma 3.4.** Suppose that $\mu$ is semiconvex. Then for every $a \in L$ and for every $0$-neighborhood $W$ in $G$ there exists $m \in \mathbb{N}$ such that for every $p \in D$ with $p \leq \frac{1}{2^m}$, $\mu(p) \in W$.

**Proof:** Let $a \in L$ and $W$ be a $0$-neighborhood in $G$. Since $\mu$ is semiconvex, we can construct a decreasing sequence $\{a_n\}$ in $L$ such that $a_n \leq a$ and $2^n \mu(a_n) = \mu(a)$ for every $n \in \mathbb{N}$. Let $b_1 := a \ominus a_1$ and for every $n \geq 2$, let $b_n := a_{n-1} \ominus a_n$.

By 3.3 of [1], $\{b_n\}$ is orthogonal and for every $n \in \mathbb{N}$, $2^n \mu(b_n) = 2^n \mu(a_{n-1}) - 2^n \mu(a_n) = 2 \mu(a) - \mu(a) = \mu(a)$. Suppose that for every $m \in \mathbb{N}$ there exists $c_m$ such that $\mu(b_m \wedge c_m) \notin W$. Since $\{b_n\}$ is orthogonal, $\{c_m \wedge b_m\}$ is orthogonal, too. Moreover, by 8.1.2 of [16], $\mu$ is exhaustive. By 2.4 of [3], $\mu$ is exhaustive if and only if $\mu(a_n) \to 0$ for every orthogonal sequence $\{a_n\}$ in $L$. Therefore, we obtain that $\lim_m \mu(b_m \wedge c_m) = 0$, a contradiction. Hence we can choose $m \in \mathbb{N}$ such that $\mu(b_m \wedge b) \in W$ for every $b \in L$. Set $p = \frac{r}{2^m}$, with $p \leq \frac{1}{2^m}$. By 3.3, we can find $c \leq b_m$ such that $\mu(c) = \frac{r}{2^{n-m}} \mu(b_m)$. Then $p \mu(a) = \frac{r}{2^m} \mu(a) = \frac{r}{2^{n-m}} \mu(b_m) = \mu(c) = \mu(c \wedge b_m) \in W$. \hfill \Box

**Lemma 3.5.** Suppose that $\mu$ is semiconvex. Then for every $a \in L$ and every $t \in [0,1]$ there exists $a_t \leq a$ such that $t \mu(a)$ is defined and $t \mu(a) = \mu(a_t)$. Moreover, the map $t \mapsto a_t$ is increasing.

**Proof:** We repeat the same argument as in [2]. It follows from 3.3 that there exists a family of elements of $L \{a_d\}_{d \in D}$ such that $\mu(a_d) = d \mu(a)$ for each $d \in D$ and, also, for $d_1 < d_2$, $a_{d_1} \leq a_{d_2} \leq a$. Let $t \in [0,1] \setminus D$. We define $\alpha_t$, $\beta_t$ by $\alpha_t = \bigvee \{a_d : d \in D \text{ and } d < t\}$ and $\beta_t = \bigwedge \{a_d : d \in D \text{ and } d > t\}$. By using the $\sigma$-order continuity of $\mu$ we find that $\mu(\alpha_t) = \lim_{d \uparrow t} \mu(a_d)$, $\mu(\beta_t) = \lim_{d \downarrow t} \mu(a_d)$. Let $V$ be any symmetric $0$-neighbourhood in $G$. It follows from the construction and from 3.4 that we can find $n \in \mathbb{N}$ and $r \in \{0,1,\ldots,2^n\}$ such that $d = \frac{r}{2^n} < t < \frac{r+1}{2^n} = d'$, $\mu(\beta_t) - \mu(\alpha_{d'}) \in V$, $\mu(\alpha_t) - \mu(a_d) \in V$, and $\frac{1}{2^m} \mu(a) \in V$. Then $(\mu(\beta_t) - \mu(\alpha_t)) \in (\mu(a_{d'}) - \mu(a_d)) + 2V = \frac{1}{2^m} \mu(a) + 2V \subset 3V$. Since the symmetric neighbourhoods form a base for $0$-neighbourhoods, and since
the topology is Hausdorff, \( \mu(\hat{b}_t) = \mu(\alpha_t) \). Hence we can define \( \alpha_t \), for \( t \in [0, 1] \setminus D \) to be \( \alpha_t \). Then it is clear that \( \mu(\alpha_t) = t\mu(a) \) for each \( t \in [0, 1] \).

**Lemma 3.6.** Let \( t \in [0, 1] \) and \( \nu_t : L \rightarrow G \) be defined as \( \nu_t(a) = t\mu(a) \). Then \( \nu_t \) is a modular measure.

**Proof:** Let \( a, b \in L \).

First suppose \( t = \frac{s}{2^n} \in D \). By 3.3 we can find \( a_t, b_t \in L \) with \( a_t \leq a, b_t \leq b \), \( 2^n\mu(a_t) = s\mu(a) \) and \( 2^n\mu(b_t) = s\mu(b) \). Then we have \( 2^n\mu(a_t \lor b_t) + 2^n\mu(a_t \land b_t) = 2^n\mu(a_t) + 2^n\mu(b_t) = s\mu(a) + s\mu(b) = s\mu(a \lor b) + s\mu(a \land b) \), from which \( \nu_t(a \lor b) + \nu_t(a \land b) = \nu_t(a) + \nu_t(b) \).

Now let \( t \notin D \) and choose an increasing sequence \( \{d_n\} \) in \( D \) which converges to \( t \). Then \( t\mu(a \lor b) + t\mu(a \land b) = \lim_n d_n\mu(a \lor b) + \lim_n d_n\mu(a \land b) = t\mu(a) + t\mu(b) \), from which \( \nu_t(a \lor b) + \nu_t(a \land b) = \nu_t(a) + \nu_t(b) \).

In a similar way we prove that \( \nu_t \) is a measure. 

**4. Lyapunov measures**

In this section we set

\[
I(\mu) = \{a \in L : \mu([0, a]) = \{0\}\}
\]

and

\[
N(\mu) = \{(a, b) \in L \times L : \mu \text{ is constant on } [a \land b, a \lor b]\}.
\]

By 3.1 of [17] and 4.3 of [4] \( N(\mu) \) is a congruence relation and the quotient \( \hat{L} = L / N(\mu) \) is a D-lattice. Moreover, the function \( \hat{\mu} : \hat{L} \rightarrow G \) defined as \( \hat{\mu}(\hat{a}) = \mu(a) \) for \( a \in \hat{a} \in \hat{L} \) is trivially a modular measure.

We say that \( \mu \) is closed if \( \hat{L} \) is complete with respect to the uniformity \( U(\hat{\mu}) \) generated by \( \hat{\mu} \).

**Definition 4.1.** We say that \( \mu \) is pseudo non-injective if for every \( a \in L \setminus I(\mu) \) there exist \( b, c \in L \setminus I(\mu), b \perp c, b \oplus c \leq a \) and \( \mu(b) = \mu(c) \).

**Lemma 4.2.**  
(1) \( \mu \) is exhaustive.

(2) \( \mu \) is closed iff \( \mu \) is order continuous and \( (\hat{L}, \preceq) \) is complete.

(3) If \( G \) is metrizable, then \( \mu \) is closed.

(4) If \( \mu \) is order continuous, then \( \mu \) is completely additive.

**Proof:** (1) By 8.1.2 of [16], every \( \sigma \)-order continuous lattice uniformity on a \( \sigma \)-complete lattice is exhaustive.

(2) By (1) and 1.2.6 of [16], the Hausdorff uniformity \( U(\hat{\mu}) \) generated by \( \hat{\mu} \) on \( \hat{L} \) is exhaustive. Then, by 6.3 of [16], \( (\hat{L}, U(\hat{\mu})) \) is complete iff \( U(\hat{\mu}) \) is order continuous and \( (\hat{L}, \preceq) \) is complete. Therefore, if \( \mu \) is closed, we have that \( (\hat{L}, \preceq) \) is complete and \( \hat{\mu} \) is order continuous, too.

Conversely, if \( (\hat{L}, \preceq) \) is complete and \( \mu \) is order continuous, then \( \hat{\mu} \) is order continuous by 7.1.9 of [16], and therefore \( \mu \) is closed.
(3) Since $G$ is metrizable, $\mathcal{U}(\mu)$ is metrizable and, by (1), it is exhaustive. By 8.1.4 of [16] (see also 3.5 and 3.6 of [17]), we get that $(L, \leq)$ is complete and $\mu$ is order continuous. By 7.1.9 of [16], $(\bar{L}, \leq)$ is complete, too. Hence $\mu$ is closed by (2).

(4) Let $\{a_\alpha\}_{\alpha \in A}$ be an orthogonal family in $L$ such that $a = \sup\{\bigoplus_{\alpha \in F} a_\alpha : F \subset A \text{ finite}\}$ exists in $L$. For every finite $F \subset A$, let $a_F = \bigoplus_{\alpha \in F} a_\alpha$. Then $\{a_F : F \subset A, F \text{ finite}\}$ is an increasing net in $L$, with $a = \sup_F a_F$. Since $\mu$ is order continuous, $\mu(a) = \lim_F \mu(a_F)$. On the other hand $\mu(a_F) = \sum_{\alpha \in F} \mu(a_\alpha)$. Thus $\mu(a) = \sum_{\alpha \in A} \mu(a_\alpha)$. □

**Theorem 4.3.** Let $L$ be complete and let $\mu$ be completely additive with $I(\mu) = \{0\}$. Then $\mu$ is semiconvex if and only if $\mu$ is pseudo non-injective.

**Proof:** $\Rightarrow$: Let $a \in L \setminus I(\mu)$.

First, suppose $\mu(a) \neq 0$. Then there exists $b \leq a$ such that $2\mu(b) = \mu(a)$. Put $c := a \ominus b$. Then $b \perp c$, $b \oplus c = a$ and $\mu(b) = \mu(c)$, as $2\mu(c) = 2\mu(a) - 2\mu(b) = \mu(a)$. Moreover, $b, c \notin I(\mu)$, since $\mu(b) = \mu(c) \neq 0$.

Now let $\mu(a) = 0$. As $a \notin I(\mu)$, there exists $d \leq a$ such that $\mu(d) \neq 0$. From above, there exist $b, c \in L \setminus I(\mu)$, $b \perp c$, $b \oplus c \leq d$ and $\mu(b) = \mu(c)$. Obviously, $b \oplus c \leq a$.

$\Leftarrow$: Let $a \neq 0$. We can suppose $\mu(a) \neq 0$.

(i) We will show that $\exists h$, $0 < h \leq a$ such that $\mu(h) = \mu(a)$ and $\mu(k) \neq 0$ for each $0 < k \leq h$.

We can suppose that there exists $b \leq a$, $b \neq 0$ and $\mu(b) = 0$, since otherwise (i) is satisfied with $h = a$.

Recall that in a complete D-lattice, if $\{b_\gamma\}_{\gamma \in \Gamma}$ is an orthogonal family then, for every $\bar{\gamma} \in \Gamma$, the set $\{\gamma \in \Gamma : b_\gamma = b_{\bar{\gamma}}\}$ is finite (see [DP] p.17). Then by Zorn’s lemma we can find an orthogonal family $\{a_\alpha\}_{\alpha \in A}$ with the following properties:

(1) For every $\alpha \in A$, $a_\alpha \neq 0$ and $\mu(a_\alpha) = 0$.

(2) For every finite $F \subset A$, $\bigoplus_{\alpha \in F} a_\alpha \leq a$.

(3) If $\{b_\gamma\}_{\gamma \in \Gamma}$ is an orthogonal family in $L$ with (1) and (2), then for each $\bar{\gamma} \in \Gamma$ the set \{$\alpha \in A : a_\alpha = b_{\bar{\gamma}}$\} is finite and \{$\gamma \in \Gamma : b_\gamma = b_{\bar{\gamma}}$\} is a subset of $\{\alpha \in A : a_\alpha = b_{\bar{\gamma}}\}$. Since $L$ is complete, $e = \bigoplus_{\alpha \in A} a_\alpha$ is well-defined. By (2) we get $e \leq a$. Since $\mu$ is completely additive, we have $\mu(e) = \sum_{\alpha \in A} \mu(a_\alpha) = 0$. Put $h := a \ominus e$. Then $h \leq a$ and $\mu(h) = \mu(a)$.

We will show that, if $0 < b \leq h$, $\mu(b) \neq 0$.

By way of contradiction, assume $b \in L$, $0 < b \leq h$ and $\mu(b) = 0$. Since $b \leq h \leq e \perp \bigoplus_{\alpha \in F} a_\alpha \perp \bigoplus_{\alpha \in F} a_\alpha$ for each finite $F \subset A$, we have, by 4.2 of [7] that every finite subfamily of $\{a_\alpha\}_{\alpha \in A} \cup \{b\}$ is orthogonal. Moreover, if $F \subset A$ is finite, we have $b \bigoplus_{\alpha \in F} a_\alpha \leq h \oplus e = (a \oplus e) \oplus e = a$. Then $\{a_\alpha\}_{\alpha \in A} \cup \{b\}$ gives a contradiction with (3).

Let $h$ be as in (i).
We claim that, if $0 < k \leq h$, then there exist $c, d \in L$ such that $0 < c < d \leq k$ and $2\mu(c) = \mu(d)$.

If $0 < k \leq h$, $\mu(k) \neq 0$ by (i) and, by pseudo non-injectivity, there exist $b_1, b_2 \in L$, $b_1 \perp b_2$, $b_1 \oplus b_2 \leq k$, $b_1 \neq 0$, $b_2 \neq 0$ and $\mu(b_1) = \mu(b_2)$. Then for $c := b_1$ and $d := b_1 \oplus b_2$ we have $0 < c < d \leq k$ as $b_1$ and $b_2$ are not zero and $\mu(d) = \mu(b_1) + \mu(b_2) = 2\mu(c)$.

(ii) Zorn’s lemma ensures the existence of an orthogonal family $\{d_\alpha\}_{\alpha \in A}$ with the following properties:

1. for every $\alpha \in A$, $d_\alpha \neq 0$ and there exists $c_\alpha$ such that $0 < c_\alpha < d_\alpha$ and $2\mu(c_\alpha) = \mu(d_\alpha)$;
2. for every finite $F \subset A$, $\bigoplus_{\alpha \in F} d_\alpha \leq h$;
3. if $\{c_\gamma : \gamma \in \Gamma\}$ is an orthogonal family in $L$ with properties (1) and (2), then for every $\gamma \in \Gamma$ the set $\{\alpha \in A : d_\alpha = c_\gamma\} \neq \emptyset$.

It is easy to see that the set $\{c_\alpha : \alpha \in A\}$ is orthogonal. Put $d = \bigoplus_{\alpha \in A} d_\alpha$ and $c = \bigoplus_{\alpha \in A} c_\alpha$. We get $c \neq 0$, since $c_\alpha \neq 0$ for every $\alpha \in A$. By (2) $d \leq h$.

Moreover, as $\mu(d) = \sum_{\alpha \in A} \mu(d_\alpha) = 2\sum_{\alpha \in A} \mu(c_\alpha) = 2\mu(c)$ and $c \leq d$, we obtain $c < d$.

(iii) We will show that $d = h$.

Suppose $d < h$. Then $h \ominus d \neq 0$. From above, there exist $c_1, c_2 \in L$ with $0 < c_1 < c_2 \leq h \ominus d$ and $\mu(c_2) = 2\mu(c_1)$.

We will check that $\{d_\alpha\}_{\alpha \in A} \cup \{c_2\}$ has the same properties as $\{d_\alpha\}_{\alpha \in A}$.

Since $c_2 \leq h \ominus d \leq d^\perp \leq (\bigoplus_{\alpha \in F} d_\alpha)^\perp$ for every finite $F \subset A$, from 4.2 of [7] it follows that every finite subfamily of $\{d_\alpha\}_{\alpha \in A} \cup \{c_2\}$ is orthogonal and so, the family is orthogonal. Moreover, if $F \subset A$ is finite, then $c_2 \oplus (\bigoplus_{\alpha \in F} d_\alpha) \leq (h \ominus d) \oplus d = h$. Obviously, $c_2$ verifies (1). Then $\{d_\alpha\}_{\alpha \in A} \cup \{c_2\}$ contradicts property (3). Hence $d = h$.

It follows that $\mu(a) = \mu(h) = \mu(d) = 2\mu(c)$. Therefore $\mu$ is semiconvex.

**Theorem 4.4.** Let $\mu$ be closed and pseudo non-injective. Then $\mu(L)$ is convex.

**Proof:** It is clear that we can replace $L$ by $L/N(\mu)$ and $\mu$ by $\hat{\mu}$. Then by 4.2 we can suppose $L$ complete, $\mu$ completely additive and $I(\mu) = \{0\}$. Hence by 4.3 $\mu$ is semiconvex.

Let $b, c \in L$ and $t \in [0,1]$.

First, suppose $b \land c = 0$.

By 3.3 there exist $d, e \in L$ such that $d \leq b$, $e \leq c$, $\mu(d) = t\mu(b)$ and $\mu(e) = (1-t)\mu(c)$. Since $b \land c = 0$, we have $d \land e = 0$. It follows that $t\mu(b) + (1-t)\mu(c) = \mu(d) + \mu(e) = \mu(d \lor e) + \mu(d \land e) = \mu(d \lor e)$.

Now let $b, c \in L$. Put $b_1 := b \ominus (b \land c)$ and $c_1 = c \ominus (b \land c)$. By 1.8.5 of [13] we have $b_1 \land c_1 = 0$. Then, from above, there exist $b_2, c_2 \in L$ with $b_2 \leq b_1$, $c_2 \leq c_1$ and $t\mu(b_1) + (1-t)\mu(c_1) = \mu(b_2 \lor c_2)$.
Since $b = (b \land c) \oplus b_1$ and $c = (b \land c) \oplus c_1$, by 3.6 we obtain $t \mu (b) = t \mu (b_1) + t \mu (b \land c)$ and $(1 - t) \mu (c) = (1 - t) \mu (b \land c) + (1 - t) \mu (c_1)$. It follows that $t \mu (b) + (1 - t) \mu (c) = \mu (b \land c) + t \mu (b_1) + (1 - t) \mu (c_1) = \mu (b \land c) + \mu (b_2 \lor c_2)$.

We claim that $b \land c \perp b_2 \lor c_2$. By 1.8.4 of [13] applied with $c = a \land b$, we obtain $b_1 \lor c_1 = (b \ominus (b \land c)) \lor (c \ominus (b \land c)) = (b \lor c) \ominus (b \land c)$, hence $b_2 \lor c_2 \leq b_1 \lor c_1 \leq 1 \ominus (b \land c) = (b \land c) \perp$.

It follows that $\mu (b \land c) + \mu (b_2 \lor c_2) = \mu ((b \land c) \ominus (b_2 \lor c_2))$ and, therefore, $t \mu (b) + (1 - t) \mu (c) \in \mu (L)$. $\square$

**Corollary 4.5.** Let $\mu$ be closed. Then $\mu$ is pseudo non-injective if and only if for every $a \in L$, $\mu ([0, a])$ is convex.

**Proof:** $\Leftarrow$: From the assumptions we get that $\mu$ is semiconvex. Hence, $\hat{\mu}$ is semiconvex, too. Moreover, since $\mu$ is closed, by 4.2 we have that $L / N(\mu)$ is complete and $\hat{\mu}$ is completely additive. Since $I(\hat{\mu}) = \{0\}$, by 4.3 we have that $\hat{\mu}$ is pseudo non-injective. We see that $\mu$ is pseudo non-injective, too. Let $a \in L \setminus I(\mu)$ and choose $b \leq a$ such that $\mu (b) \neq 0$. Since $\hat{\mu}$ is pseudo non-injective, there exist $\hat{c}, \hat{d}$; $\hat{c}, \hat{d} \neq 0$, $\hat{c} \perp \hat{d}$, $\hat{c} \oplus \hat{d} \leq \hat{b}$ and $\hat{\mu} (\hat{c}) = \hat{\mu} (\hat{d})$. Then there exist $c, d \in L \setminus I(\mu)$, $c \perp d$, $c \oplus d \leq b \leq a$ and $\mu (b) = \mu (c)$.

$\Rightarrow$: As in 4.4 we can suppose $L = L / N(\mu)$. Let $a \in L$ and denote by $\mu_0$ the restriction of $\mu$ to $[0, a]$. Observe that $[0, a]$ is a complete D-lattice and $\mu_0$ is a $\sigma$-order continuous pseudo non-injective modular measure, since $\mathcal{U}(\mu_0)$ coincides with the restriction of $\mathcal{U}(\mu)$ to $[0, a]$ and $N(\mu_0) = N(\mu) \cap ([0, a] \times [0, a])$. Hence by 4.4 we have that $\mu ([0, a])$ is convex. $\square$

**References**


Lyapunov measures on effect algebras


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