Cancellative actions

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Abstract. The following problem is considered: when can the action of a cancellative semigroup $S$ on a set be extended to a simply transitive action of the universal group of $S$ on a larger set.

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Introduction

The following problem arose in [4]. Let $S$ be a cancellative semigroup and $G(S)$ be its universal group. Assume that $S$ can be embedded in $G(S)$. When can the action of $S$ on a set $X$ be extended to a simply transitive action of $G(S)$ on some set $Y \supseteq X$? When $S$ is commutative the solution of this problem is easy but leads to concepts that are of great importance for finitely generated commutative semigroups [4].

Here we consider the general case of an arbitrary semigroup $S$ which acts on a set $X$. In Section 1 we use the universal group $G(S)$ of $S$, and the canonical homomorphism $\gamma : S \rightarrow G(S)$, to construct a set $Y$, a mapping $\iota : X \rightarrow Y$, and an action of $G(S)$ on $Y$ which extends the action of $S$ in the sense that $\iota(s \cdot x) = \gamma(s) \cdot \iota(x)$ for all $s$ and $x$, and has a universal property. This leads in Section 2 to necessary and sufficient conditions for extending the action of $S$ on $X$ to a simply transitive action of $G(S)$ on some set $Z \supseteq X$, or to a simply transitive action of $G(S)$ on $Y$. A later article will show that the latter conditions are equivalent to explicit sets of implications.

We do not assume that $S$ is a monoid. But, if $S$ is a monoid, then we may assume that it acts on $X$ as a monoid ($1 \cdot x = x$ for all $x \in X$), since otherwise its action cannot be extended to a group action.

Recall that a semigroup is a set with an associative operation, which we write as a multiplication. A semigroup $S$ is cancellative when $xz = yz$ implies $x = y$, and $zx = zy$ implies $x = y$ (for all $x, y, z \in S$). A left semigroup action of a semigroup $S$ on a set $X$ is a mapping $(s, x) \mapsto s \cdot x$ of $S \times X$ into $X$. Then $S$ acts simply on $X$ when $s \cdot x = t \cdot x$ implies $x = t$; $S$ acts transitively on $X$ when, for every $x, y \in X$, there exists some $s \in S$ such that $s \cdot x = y$. 
1. Universal actions

This section takes place in the category \( \text{Act} \) of semigroup acts. The objects of \( \text{Act} \) are all ordered triples \((S, X, . . .)\) of a semigroup \( S \), a set \( X \), and a left semigroup action \( . \) of \( S \) on \( X \); then \( X \) is an \( S \)-set and \((S, X, . . .)\) is an \( S \)-act. In \( \text{Act} \), a morphism from \((S, X, . . .)\) to \((T, Y, . . .)\) is an ordered pair \((\varphi, f)\) of a semigroup homomorphism \( \varphi : S \to T \) and a mapping \( f : X \to Y \) such that \( f(s.x) = \varphi(s).f(x) \) for all \( s \in S \) and \( x \in X \); if \( \varphi \) and \( f \) are injective, then the action of \( T \) on \( Y \) extends the action of \( S \) on \( X \). Composition and identity morphisms are componentwise.

1. When \( \varphi : S \to T \) is a semigroup homomorphism, every \( S \)-act has a universal \( T \)-act:

**Proposition 1.1.** Let \((S, X, . . .)\) be a semigroup act and \( \varphi : S \to T \) be a homomorphism. There exist a set \( Y \), an action \( . \) of \( T \) on \( Y \), and a mapping \( \iota : X \to Y \) such that \((\varphi, \iota) : (S, X, . . .) \to (T, Y, . . .)\) is a morphism and, for every morphism \((\varphi, \alpha) : (S, X, . . .) \to (T, Z, . . .)\), there exists a unique action-preserving mapping \( \beta : Y \to Z \) such that \( \beta \circ \iota = \alpha \).

\[
\begin{array}{ccc}
X & \xrightarrow{\iota} & Y \\
\downarrow{\alpha} & & \downarrow{\beta} \\
& & Z
\end{array}
\]

**Proof:** We construct \( Y \) as a tensor product of \( S \)-sets (as introduced in [5]): namely, \( Y = T^1 \otimes_S X \), where \( S \) acts on \( T^1 \) on the right by \( t \cdot s = t \varphi(s) \). The details are as follows. Let \( \sim \) be the smallest equivalence relation on the set \( T^1 \times X \) such that

1. for all \( t, u, v \in T^1 \) and \( x, y \in X \), \((u, x) \sim (v, y)\) implies \((tu, x) \sim (tv, y)\);
2. for all \( s \in S \) and \( x \in X \), \((\varphi(s), x) \sim (1, s \cdot x)\).

This exists since an intersection of equivalence relations with properties (1) and (2) again has properties (1) and (2). A more detailed description of \( \sim \) is given in Lemma 1.2 below.

We show that \( Y = (T^1 \times X)/\sim \) serves. Let \( \text{cls}(t, x) \) denote the \( \sim \)-class of \((t, x)\). The mapping \( \iota : X \to Y \) is given by

\[ \iota(x) = \text{cls}(1, x). \]

By (1), an action \( . \) of \( T^1 \) on \( Y \) is well defined by

\[ t \cdot \text{cls}(u, x) = \text{cls}(tu, x). \]
This is a monoid action since \( 1 \cdot \text{cls}(u, x) = \text{cls}(u, x) \) and
\[
\tau \cdot (\text{cls}(v, x)) = \text{cls}(\tau v, x) = \text{cls}(tuv, x) = \tau \cdot \text{cls}(v, x).
\]
In particular, \( T \) acts on \( Y \). Also
\[
\iota(s \cdot x) = \text{cls}(1, s \cdot x) = \text{cls}(\varphi(s), x) = \varphi(s) \cdot \iota(x)
\]
by (2). Thus \((T, Y, \cdot)\) is an object of \( \text{Act} \) and \((\varphi, \iota)\) is a morphism.

Let \((\varphi, \alpha) : (S, X, \cdot) \to (T, Z, \cdot)\) be a morphism. The mapping \( \alpha \) induces a mapping \( \overline{\alpha} : T^1 \times X \to Z \) defined by
\[
\overline{\alpha}(t, x) = t \cdot \alpha(x)
\]
(with \( \overline{\alpha}(1, x) = \alpha(x) \) if \( t = 1 \in T^1 \)). If \( \overline{\alpha}(u, x) = \overline{\alpha}(v, y) \), then \( u \cdot \alpha(x) = v \cdot \alpha(y) \) and
\[
\overline{\alpha}(tu, x) = tu \cdot \alpha(x) = t \cdot (u \cdot \alpha(x))
\]
\[
= t \cdot (v \cdot \alpha(y)) = tv \cdot \alpha(y) = \overline{\alpha}(tv, y).
\]
Also
\[
\overline{\alpha}(\varphi(s), x) = \varphi(s) \cdot \alpha(x) = \alpha(s \cdot x) = 1 \cdot \alpha(s \cdot x) = \overline{\alpha}(1, s \cdot x)
\]
by the choice of \( \alpha \). Thus the equivalence relation induced by \( \overline{\alpha} \) satisfies (1) and (2). It therefore contains \( \sim: (t, x) \sim (u, y) \) implies \( \overline{\alpha}(t, x) = \overline{\alpha}(u, y) \). Hence a mapping \( \beta : Y \to Z \) is well defined by
\[
\beta(\text{cls}(t, x)) = \overline{\alpha}(t, x) = t \cdot \alpha(x).
\]
In particular \( \beta(\iota(x)) = \beta(1 \cdot \text{cls}(1, x)) = 1 \cdot \alpha(x) = \alpha(x) \) and \( \beta \circ \iota = \alpha \). If moreover \( y = \text{cls}(u, x) \in Y \), then
\[
\beta(t \cdot y) = \beta(\text{cls}(tu, x)) = tu \cdot \alpha(x) = t \cdot (u \cdot \alpha(x)) = t \cdot \beta(y).
\]
Thus \( \beta \) is action-preserving.

If conversely \( \overline{\beta}' : Y \to Z \) is action-preserving and \( \overline{\beta}' \circ \iota = \alpha \), then
\[
\overline{\beta}'(t \cdot \text{cls}(t, x)) = \beta'(t \cdot 1 \cdot \text{cls}(1, x)) = \beta'(t \circ \iota(x))
\]
\[
= t \cdot \beta'(\iota(x)) = t \cdot \alpha(x) = \beta(\text{cls}(t, x));
\]
hence \( \beta \) is unique. \( \square \)

We give a more precise description of \( \sim \) (which would work more generally in any tensor product of \( S \)-sets). For this it is convenient to regard the elements of \( T^1 \times X \) as the vertices of a directed graph, in which there is a labelled edge \( (t, s \cdot x) \xrightarrow{s} (t \varphi(s), x) \) for every \( (t, x) \in T^1 \times X \) and \( s \in S^1 \). In particular there is an identity edge \( (t, x) \xrightarrow{1} (t, x) \) for every \( (t, x) \in T^1 \times X \). We note two properties:

If \( a \xrightarrow{s'} b \xrightarrow{s''} c \), then \( a \xrightarrow{s's''} c \): indeed, if \( a = (t, -) \) and \( c = (-, x) \), then \( b = (t \varphi(s'), s''. x) \), so that \( a = (t, s'. (s''. x)) \), \( c = (t \varphi(s') \varphi(s''), x) \), and \( a \xrightarrow{s's''} c \).

If \( (u, s \cdot x) \xrightarrow{s} (u \varphi(s), x) \), then \( (tu, s \cdot x) \xrightarrow{s} (tu \varphi(s), x) \).
Lemma 1.2. In $T^1 \times X$, $a \sim b$ if and only if

$$a = a_0 \xleftarrow{s_1} a_1 \xrightarrow{s_2} a_2 \cdots a_{2n-2} \xleftarrow{s_{2n-1}} a_{2n-1} \xrightarrow{s_{2n}} a_{2n} = b$$

for some $n \geq 0$, $a_0, \ldots, a_{2n} \in T^1 \times X$, and $s_1, s_2, \ldots, s_{2n} \in S^1$.

Proof: Let $a \mathcal{C} b$ if and only if

$$a = a_0 \xleftarrow{s_1} a_1 \xrightarrow{s_2} a_2 \cdots a_{2n-2} \xleftarrow{s_{2n-1}} a_{2n-1} \xrightarrow{s_{2n}} a_{2n} = b$$

for some $n \geq 0$, $a_0, \ldots, a_{2n} \in T^1 \times X$, and $s_1, s_2, \ldots, s_{2n} \in S^1$. It is immediate that $\mathcal{C}$ is reflexive (let $n = 0$), symmetric, and transitive. Also $(u, x) \mathcal{C} (v, y)$ implies $(tu, x) \mathcal{C} (tv, y)$, since $(u, s \cdot x) \xrightarrow{s} (u \varphi(s), x)$ implies $(tu, s \cdot x) \xrightarrow{s} (tu \varphi(s), x)$; and $(\varphi(s), x) \mathcal{C} (1, s \cdot x)$, since $(\varphi(s), x) \xleftarrow{s} (1, s \cdot x) \xrightarrow{1} (1, s \cdot x)$. Thus $\mathcal{C}$ is an equivalence relation with properties (1) and (2).

If conversely $\mathcal{A}$ is an equivalence relation with properties (1) and (2), then $(t \varphi(s), x) \mathcal{A} (t, s \cdot x)$ for all $t, s$, and $x$; hence $(t, s \cdot x) \xrightarrow{s} (t \varphi(s), x)$ implies $(t, s \cdot x) \mathcal{A} (t \varphi(s), x)$, and $a \mathcal{C} b$ implies $a \mathcal{A} b$. Therefore $\mathcal{C}$ coincides with $\sim$. $\square$

2. Proposition 1.1 implies that every semigroup act has a universal group act in $\text{Act}$. First recall that every semigroup $S$ has a universal group in the category of semigroups and homomorphisms: that is, there exist a group $G(S)$ and a homomorphism $\gamma : S \rightarrow G(S)$, such that every homomorphism $\varphi$ of $S$ into a group $G$ factors uniquely through $\gamma$ ($\varphi = \psi \circ \gamma$ for some unique homomorphism $\psi : G(S) \rightarrow G$). For instance let $F$ be the free monoid on the set $S \cup S'$, where $S'$ is disjoint from $S$ and comes with a bijection $s \rightarrow s'$ of $S$ onto $S'$. Let $\iota : S \cup S' \rightarrow F$ be the canonical mapping. Let $\mathcal{C}$ be the smallest congruence on $F$ such that $\iota(st) \mathcal{C} \iota(s) \iota(t)$, $\iota(s) \mathcal{C} 1$, and $\iota(s') \mathcal{C} 1$, for all $s, t \in S$; then $F/\mathcal{C}$ and the canonical mapping $S \rightarrow F \rightarrow F/\mathcal{C}$ serve as $G(S)$ and $\gamma$. The existence of a universal group also follows from the Adjoint Functor Theorem.

Proposition 1.3. Let $(S, X, \cdot)$ be a semigroup act. Let $G(S)$ be the universal group of $S$ and $\gamma : S \rightarrow G(S)$ be the canonical homomorphism. The universal $G(S)$-set $Y$ of $X$ and its canonical morphism $(\gamma, \iota) : (S, X, \cdot) \rightarrow (G(S), Y, \cdot)$ have the following universal property: for every morphism $(\varphi, \alpha) : (S, X, \cdot) \rightarrow (G, Z, \cdot)$, where $G$ is a group, there exists a unique morphism $(\psi, \beta) : (G(S), Y, \cdot) \rightarrow (G, Z, \cdot)$ such that $(\psi, \beta) \circ (\gamma, \iota) = (\varphi, \alpha)$.
**Proof:** By Proposition 1.1, \((\gamma, \iota) : (S, X, \cdot) \rightarrow (G(S), Y, \cdot)\) is a morphism and, for every morphism \((\gamma, \alpha) : (S, X, \cdot) \rightarrow (G(S), Z, \cdot)\), there exists a unique action-preserving mapping \(\beta : Y \rightarrow Z\) such that \(\beta \circ \iota = \alpha\). We now prove the stronger universal property in the statement.

Let \(G\) be a group and \((\varphi, \alpha) : (S, X, \cdot) \rightarrow (G, Z, \cdot)\) be a morphism. Since \(G(S)\) is the universal group of \(S\) there exists a unique homomorphism \(\psi : G(S) \rightarrow G\) such that \(\psi \circ \gamma = \varphi\). The action of \(G\) on \(Z\) then induces an action of \(G(S)\) on \(Z\), given by

\[
g \cdot z = \psi(g) \cdot z
\]

for all \(g \in G(S)\) and \(z \in Z\). Then

\[
\alpha(s \cdot x) = \varphi(s) \cdot \alpha(x) = \psi(\gamma(s)) \cdot \alpha(x) = \gamma(s) \cdot \alpha(x)
\]

for all \(s \in S\) and \(x \in X\), and \((\gamma, \alpha) : (S, X, \cdot) \rightarrow (G(S), Z, \cdot)\) is a morphism of acts. Hence there is a unique mapping \(\beta : Y \rightarrow Z\) such that \(\beta \circ \iota = \alpha\) and \(\beta\) preserves the action of \(G(S)\). This last condition states that

\[
\beta(g \cdot y) = g \cdot \beta(y) = \psi(g) \cdot \beta(y)
\]

for all \(g \in G(S)\) and \(y \in Y\), i.e. \((\psi, \beta) : (G(S), Y, \cdot) \rightarrow (G, Z, \cdot)\) is a morphism of acts. Thus there is a unique morphism \((\psi, \beta) : (G(S), Y, \cdot) \rightarrow (G, Z, \cdot)\) such that \((\psi, \beta) \circ (\gamma, \iota) = (\varphi, \alpha)\). 

\[\square\]

3. In Proposition 1.3 (up to isomorphism of acts) \(Y = (G(S) \times X) / \sim\), where \(\sim\) is the smallest equivalence relation on the set \(G(S) \times X\) such that (1) \((g, x) \sim (h, y)\) implies \((kg, x) \sim (kh, y)\) and (2) \((\gamma(s), x) \sim (1, s \cdot x)\), for all \(g, h, k \in G(S), x, y \in X\), and \(s \in S\); then \(g \cdot \text{cls}(h, x) = \text{cls}(gh, x)\) and \(\iota(x) = \text{cls}(1, x)\). Lemma 1.2 then leads to a better description of \(\sim\).

When \(x, y \in X\), a connected sequence from \(x\) to \(y\) is a triple of sequences \(x_0, x_1, \ldots, x_n \in X, s_1, \ldots, s_n \in S^1, t_1, \ldots, t_n \in S^1\) (where \(n \geq 0\)) such that

\[
x = x_0, x_n = y, and \quad t_1 \cdot x_0 = s_1 \cdot x_1, t_2 \cdot x_1 = s_2 \cdot x_2, \ldots, t_n \cdot x_{n-1} = s_n \cdot x_n
\]

holds in \(X\) (with \(^1.x = x\) in case \(S\) is not a monoid). The group value of a connected sequence \(x_0, x_1, \ldots, x_n \in X, s_1, \ldots, s_n \in S^1, t_1, \ldots, t_n \in S^1\) is

\[
\gamma(t_1)^{-1} \gamma(s_1) \gamma(t_2)^{-1} \gamma(s_2) \ldots \gamma(t_n)^{-1} \gamma(s_n) \in G(S)
\]

(with \(\gamma(1) = 1 \in G(S)\) in case \(S\) is not a monoid).
Lemma 1.4. \((g, x) \sim (h, y)\) if and only if there exists a connected sequence from \(x\) to \(y\) with group value \(g^{-1}h\).

Proof: Assume \((g, x) \sim (h, y)\). By Lemma 1.2 there exist \(n \geq 0\), \((g_0, x_0)\), \((g_1, x_1)\), \ldots, \((g_{2n}, x_{2n})\) \(\in G(S) \times X\), and \(s_1, \ldots, s_{2n} \in S^1\) such that \((g_0, x_0) = (g, x)\), \((g_{2n}, x_{2n}) = (h, y)\), and

\[
(g_0, x_0) \xleftarrow{s_1} (g_1, x_1) \xrightarrow{s_2} (g_2, x_2) \xleftarrow{s_3} \ldots \xrightarrow{s_{2n-2}} (g_{2n-2}, x_{2n-2}) \xleftarrow{s_{2n-1}} (g_{2n-1}, x_{2n-1}) \xrightarrow{s_{2n}} (g_{2n}, x_{2n}).
\]

Then \(g_0 = g_1 \gamma(s_1)\), \(s_1 \cdot x_0 = x_1 = s_2 \cdot x_2\), and \(g_2 = g_1 \gamma(s_2) = g_0 \gamma(s_1)^{-1} \gamma(s_2)\). Similarly \(s_3 \cdot x_2 = s_4 \cdot x_4\), \(g_4 = g_2 \gamma(s_3)^{-1} \gamma(s_4)\), \ldots, \(s_{2n-1} \cdot x_{2n-2} = s_{2n} \cdot x_{2n}\), and \(g_{2n} = g_{2n-2} \gamma(s_{2n-1})^{-1} \gamma(s_{2n})\). Hence \(x_0, x_1, \ldots, x_{2n} \in X\), \(s_1, s_3, \ldots, s_{2n-1} \in S^1\), is a connected sequence from \(x\) to \(y\), whose group value is \(g_0^{-1} g_{2n} = g^{-1}h\), since

\[
g_{2n} = g_0 \gamma(s_1)^{-1} \gamma(s_2) \gamma(s_3)^{-1} \gamma(s_4) \ldots \gamma(s_{2n-1})^{-1} \gamma(s_{2n}).
\]

The converse is similar. Let \(x_0, x_1, \ldots, x_n \in X\), \(s_1, \ldots, s_n \in S^1\), \(t_1, \ldots, t_n \in S^1\) be a connected sequence from \(x\) to \(y\) with group value \(g^{-1}h\). Let

\[
y_1 = t_1 \cdot x_0 = s_1 \cdot x_1, \quad y_2 = t_2 \cdot x_1 = s_2 \cdot x_2, \quad \ldots, \quad y_n = t_n \cdot x_{n-1} = s_n \cdot x_n
\]

and \(g_0 = g\), \(h_1 = g_0 \gamma(t_1)^{-1}\), \(g_1 = h_1 \gamma(s_1)\), \(h_2 = g_1 \gamma(t_2)^{-1}\), \(g_2 = h_2 \gamma(s_2)\), \ldots, \(h_n = g_{n-1} \gamma(t_{n-1})^{-1}\), \(g_n = h_n \gamma(s_n)\). Then \(g_n = h\), since

\[
g_n = g_0 \gamma(t_1)^{-1} \gamma(s_1) \gamma(t_2)^{-1} \gamma(s_2) \ldots \gamma(t_{n-1})^{-1} \gamma(s_n) = g g^{-1}h.
\]

Moreover,

\[
(g_0, x_0) \xleftarrow{t_1} (h_1, y_1) \xrightarrow{s_1} (g_1, x_1) \xleftarrow{t_2} \ldots \xrightarrow{s_{n-1}} (g_{n-1}, x_{n-1}) \xleftarrow{t_n} (h_n, y_n) \xrightarrow{s_n} (g_n, x_n).
\]

Hence \((g, x) \sim (h, y)\). \qed

It is convenient to write \(x \overset{g}{\longrightarrow} y\) when \(x, y \in X\) and there is a connected sequence from \(x\) to \(y\) with group value \(g \in G(S)\). We note the following properties.
Lemma 1.5. \( x \xrightarrow{1} x \) and \( s \cdot x \xrightarrow{\gamma(s)} x \) for every \( x \in X \) and \( s \in S \). If \( x \xrightarrow{g} y \), then \( y \xrightarrow{g^{-1}} x \). If \( x \xrightarrow{g} y \) and \( y \xrightarrow{h} z \), then \( x \xrightarrow{gh} z \).

Proof: When \( x \in X \), then \( x \xrightarrow{1} x \) since there is a connected sequence with \( n = 0 \) (also, \((1, x) \sim (1, x)) \). More generally, when \( s \in S^1 \), then \( s \cdot x = x_0, x_1 = x \in X, s_1 = s \in S^1, t_1 = 1 \in S^1 \) is a connected sequence from \( s \cdot x \) to \( x \) with group value \( \gamma(s) \); hence \( s \cdot x \xrightarrow{\gamma(s)} x \).

If \( x = x_0, x_1, \ldots, x_n = y \in X, s_1, \ldots, s_n \in S^1, t_1, \ldots, t_n \in S^1 \) is a connected sequence from \( y \) to \( x \) with group value

\[
g = \gamma(t_1)^{-1} \gamma(s_1) \gamma(t_2)^{-1} \gamma(s_2) \cdots \gamma(t_n)^{-1} \gamma(s_n),
\]

then \( y = x_n, x_{n-1}, \ldots, x_0 = x \in X, t_n, \ldots, t_1 \in S^1, s_n, \ldots, s_1 \in S^1 \) is a connected sequence from \( x \) to \( y \) with group value

\[
\gamma(s_n)^{-1} \gamma(t_n) \gamma(s_{n-1})^{-1} \gamma(t_{n-1}) \cdots \gamma(s_1)^{-1} \gamma(t_1) = g^{-1}.
\]

Hence \( x \xrightarrow{g} y \) implies \( y \xrightarrow{g^{-1}} x \).

If finally \( x = x_0, x_1, \ldots, x_m = y \in X, s_1, \ldots, s_m \in S^1, t_1, \ldots, t_m \in S^1 \) is a connected sequence from \( x \) to \( y \) with group value

\[
g = \gamma(t_1)^{-1} \gamma(s_1) \gamma(t_2)^{-1} \gamma(s_2) \cdots \gamma(t_m)^{-1} \gamma(s_m),
\]

and \( y = y_0, y_1, \ldots, y_n = z \in X, u_1, \ldots, u_n \in S^1, v_1, \ldots, v_n \in S^1 \) is a connected sequence from \( y \) to \( z \) with group value

\[
h = \gamma(v_1)^{-1} \gamma(u_1) \gamma(v_2)^{-1} \gamma(u_2) \cdots \gamma(v_n)^{-1} \gamma(u_n),
\]

then \( x = x_0, x_1, \ldots, x_m = y_0, y_1, \ldots, y_n \in X, s_1, \ldots, s_m, u_1, \ldots, u_n \in S^1, t_1, \ldots, t_n, v_1, \ldots, v_n \in S^1 \) is a connected sequence from \( x \) to \( z \) with group value

\[
\gamma(t_1)^{-1} \gamma(s_1) \gamma(t_2)^{-1} \gamma(s_2) \cdots \gamma(t_n)^{-1} \gamma(s_n)
\]

\[
\gamma(v_1)^{-1} \gamma(u_1) \gamma(v_2)^{-1} \gamma(u_2) \cdots \gamma(v_n)^{-1} \gamma(u_n) = gh.
\]

Hence \( x \xrightarrow{g} y \) and \( y \xrightarrow{h} z \) implies \( x \xrightarrow{gh} z \).

When \( S \) acts on \( X \), the relation

\[ x \equiv y \text{ if and only if there exists a connected sequence from } x \text{ to } y \]
is by Lemma 1.5 an equivalence relation on $X$; we call its equivalence classes the \textit{connected components} of $X$. We write the quotient set $X/\equiv$ (the set of all connected components of $X$) as a family $(C_i)_{i \in I}$.

We say that the action of $S$ on $X$ is \textit{connected} when there is only one connected component (when for every $x, y \in X$ there exists a connected sequence from $x$ to $y$); we also say that the $S$-set $X$ is connected. This is weaker than the usual transitivity conditions in, say, [3]. The connected components of any $S$-set are themselves connected $S$-sets.

4. We now give an alternate construction of the universal group act, in which the orbits of $Y$ (under the action of $G(S)$) are constructed from the connected components of $X$. First we note:

\textbf{Proposition 1.6.} In the universal group act $(G(S), Y, \ldots)$ of $(S, X, \ldots)$, $\iota(x)$ and $\iota(y)$ lie in the same orbit if and only if $x$ and $y$ lie in the same connected component of $S$.

\textbf{Proof:} Let $x, y \in X$. If $\iota(x)$ and $\iota(y)$ lie in the same orbit, then $\text{cls}(1, x) = g \cdot \text{cls}(1, y) = \text{cls}(g, y)$ for some $g \in G(S)$ and there exists a connected sequence from $x$ to $y$, by Lemma 1.4. If conversely there exists a connected sequence from $x$ to $y$, and $g \in G(S)$ is its group value, then $\iota(x) = \text{cls}(1, x) = \text{cls}(g, y) = g \cdot \iota(y)$, by Lemma 1.4, so that $\iota(x)$ and $\iota(y)$ lie in the same orbit. \hfill \Box

Stabilizers and orbits in $Y$ can be retrieved from $X$ as follows.

\textbf{Lemma 1.7.} Let $C$ be a connected component of $X$ and $c \in C$. Then

$$H(C) = \{ h \in G(S) \mid c \xrightarrow{h} c \}$$

is a subgroup of $G(S)$; for every $x \in C$, \{ $g \in G(S) \mid x \xrightarrow{g} c$ \} is a left coset of $H(C)$.

\textbf{Proof:} $H = H(C)$ is a subgroup of $G(S)$ by Lemma 1.5. Let $x \xrightarrow{g} c$. If $c \xrightarrow{h} c$, then $x \xrightarrow{gh} c$. If conversely $x \xrightarrow{g'} c$, then $c \xrightarrow{g^{-1}} x \xrightarrow{g'} c$, $g^{-1}g' \in H$, and $g' \in gH$; thus \{ $g' \in G(S) \mid x \xrightarrow{g'} c$ \} = $gH$. \hfill \Box

Recall that, when $H$ is a subgroup of a group $G$, then the left cosets of $H$ constitute a set $G/H$, on which $G$ acts by left multiplication: $g' \cdot gH = g'gH$.

\textbf{Proposition 1.8.} Let $(S, X, \ldots)$ be a semigroup act, $(G(S), Y, \ldots)$ be its universal group act, and $(C_i)_{i \in I}$ be its connected components. For any cross-section $(c_i)_{i \in I}$ of $\equiv$, $Y$ is (up to an isomorphism of $G(S)$-acts) the disjoint union

$$Y = \bigcup_{i \in I} (G(S)/H(C_i) \times \{i\}),$$
with $g.(g'(H(C_i), i) = (gg'(H(C_i), i)$ and $\iota(x) = (gH(C_i), i)$ when $x \in C_i$ and $x \xrightarrow{g} c_i$.

**Proof:** We need a cross-section $(c_i)_{i \in I}$ of $\equiv$ (with $c_i \in C_i$) to define $H(C_i) = \{ h \in G(S) \mid c_i \xrightarrow{h} c_i \}$. Let

$$Z = \bigcup_{i \in I} (G(S)/H(C_i) \times \{i\}),$$

with $g.(g'(H(C_i), i) = (gg'(H(C_i), i)$ as in the statement. Then $Z$ is a $G(S)$-set. By Lemma 1.7, a mapping $\alpha : X \rightarrow Z$ is well-defined by

$$\alpha(x) = (gH(C_i), i) \text{ when } x \in C_i \text{ and } x \xrightarrow{g} c_i.$$  

Let $x \in X$, $x \in C_i$, and $s \in S$. Then $s.x \xrightarrow{\gamma(s)} x$ by Lemma 1.5, in particular $s.x \in C_i$. If $x \xrightarrow{g} c_i$, so that $\alpha(x) = (gH(C_i), i)$, then $s.x \xrightarrow{\gamma(s)g} c_i$ and

$$\alpha(s.x) = (\gamma(s)gH(C_i), i) = \gamma(s).\alpha(x).$$

Thus $(\gamma, \alpha) : (S, X, \cdot) \rightarrow (G(S), Z, \cdot)$ is a morphism of acts.

By Proposition 1.1, there exists an action-preserving mapping $\beta : Y \rightarrow Z$ such that $\beta \circ \iota = \alpha$. We show that $\beta$ is bijective. Since $\beta$ is action-preserving, we have

$$\beta(\cls(g, x)) = \beta(g.\cls(1, x)) = \beta(g.\iota(x)) = g.\beta(\iota(x)) = g.\alpha(x)$$

for all $x \in X$ and $g \in G(S)$. Now $\alpha(c_i) = (H(C_i), i)$, since $c_i \xrightarrow{1} c_i$; hence $(gH(C_i), i) = g.\alpha(c_i)$ and $\beta$ is surjective.

Now assume that $\beta(\cls(g, x)) = \beta(\cls(h, y))$. Let $x \in C_i$, $x \xrightarrow{a} c_i$ and $y \in C_j$, $y \xrightarrow{b} c_j$, so that $\alpha(x) = (aH(C_i), i)$ and $\alpha(y) = (bH(C_j), j)$. We have

$$(gaH(C_i), i) = g.\alpha(x) = h.\alpha(y) = (hbH(C_j), j),$$

so that $i = j$ (x and y lie in the same connected component) and $gaH(C_i) = hbH(C_j)$. Hence $aH(C_i) = g^{-1}hbH(C_i)$, there exists $x \xrightarrow{g^{-1}hb} c_i$ by Lemma 1.7, and $x \xrightarrow{g^{-1}hb} c_i \xrightarrow{b^{-1}} y$ yields $x \xrightarrow{g^{-1}h} y$ and $\cls(g, x) = \cls(h, y)$, by Lemma 1.4. Thus $\beta$ is injective. \hfill $\square$

5. A notable particular case occurs when $S$ acts on itself by left multiplication.
Proposition 1.9. When a semigroup $S$ acts on itself by left multiplication, the connected components of $S$ are left ideals, and $\equiv$ is the smallest congruence $C$ on $S$ such that $S/C$ is a right zero semigroup.

Proof: For all $x, y \in S$ the equality $x \cdot y = xy = 1$. $xy$ shows that $y \equiv xy$; hence the $\equiv$-classes $(L_i)_{i \in I}$ are left ideals. In particular $L_i \subseteq L_j$ for all $i, j$; hence $\equiv$ is a congruence and $S/\equiv$ is a right zero semigroup ($ab = b$ for all $a, b \in S/\equiv$).

Conversely let $C$ be a right zero semigroup congruence on $S$. If $x, y \in S$ and $s, t \in S^1$, then $sx = ty$ implies $x \in C$ $sx \in C$ $ty \in C$; therefore $\equiv$ is contained in $C$. □

Proposition 1.9 goes back to Dubreil [2]. A semigroup $S$ may be called left connected when $S$, as an $S$-set under left multiplication, has only one connected component. For example, every monoid is left connected ($s \sim 1$ for every $s$ since $1 \cdot s = s \cdot 1$). (On the other hand, nontrivial right zero bands, and free semigroups with two or more generators, are not left connected.) Proposition 1.9 implies that every semigroup is a right zero band of left-connected semigroups. Additional results on band decompositions, including right zero band decompositions, can be found in [1].

Lemma 1.10. When $S$ acts on itself by left multiplication, every connected sequence from $x$ to $y$ has group value $\gamma(x)^{-1}\gamma(y)$.

Proof: As in the proof of Lemma 1.4, let $x_0, x_1, \ldots, x_n \in X$, $s_1, \ldots, s_n \in S$, $t_1, \ldots, t_n \in S$ be a connected sequence from $x$ to $y$ with group value $g$. Then

$$
t_1x_0 = s_1x_1, \quad t_2x_1 = s_2x_2, \quad \ldots, \quad t_nx_{n-1} = s_nx_n
$$

and $\gamma(t_1)^{-1}\gamma(s_1)^{-1}\gamma(t_2)^{-1}\gamma(s_2)^{-1} \ldots \gamma(t_n)^{-1}\gamma(s_n)^{-1} = g$. In $G(S)$ we have

$$
\gamma(t_1)^{-1}\gamma(s_1)^{-1} = \gamma(x_0)^{-1}\gamma(x_1)^{-1}, \quad \gamma(t_2)^{-1}\gamma(s_2)^{-1} = \gamma(x_1)^{-1}\gamma(x_2)^{-1}, \\
\ldots, \quad \gamma(t_n)^{-1}\gamma(s_n)^{-1} = \gamma(x_{n-1})^{-1}\gamma(x_n)^{-1}.
$$

Hence

$$
g = \gamma(t_1)^{-1}\gamma(s_1)^{-1}\gamma(t_2)^{-1}\gamma(s_2)^{-1} \ldots \gamma(t_n)^{-1}\gamma(s_n)^{-1} = \gamma(x_0)^{-1}\gamma(x_n)^{-1} \gamma(x)^{-1}. \\
\square
$$

Lemma 1.10 implies that $H(C) = \{1\}$ for every connected component $C$ of $S$. Proposition 1.8 then yields:

Corollary 1.11. Let $S$ act on itself by left multiplication. The universal group act of $S$ is isomorphic to a disjoint union of copies of $G(S)$, one for every connected component of $S$, on which $G(S)$ acts by left multiplication.

If in particular $S$ is left connected (e.g. if $S$ is a monoid), then the universal group act of $S$ is isomorphic to $G(S)$, acting on itself by left multiplication.
2. Simply transitive actions

1. We now turn to the general problem posed in the beginning: can the action of \( S \) on a set \( X \) be extended to a simply transitive action of \( G(S) \)? that is, is there an action-preserving injection \( \alpha : X \rightarrow Z \), where \( G(S) \) acts simply and transitively on \( Z \)?

We note some necessary conditions.

**Proposition 2.1.** Let \((S, X, .)\) be a semigroup act, \((G(S), Y, .)\) be its universal group act, and \((\gamma, \alpha) : (S, X, .) \rightarrow (G(S), Z, .)\) be a morphism of acts, so that \( \alpha = \beta \circ \iota \). If \( \alpha \) is injective, then \( \iota \) is injective. If \( G(S) \) acts simply on \( Z \), then \( G(S) \) acts simply on \( Y \). If \( X \neq \emptyset \) and \( G(S) \) acts simply and transitively on \( Z \), then \( \beta : Y \rightarrow Z \) is surjective; moreover, for every \( z \in Z \), \( \beta^{-1}(z) \) contains a single element of every orbit of \( Y \).

**Proof:** By Proposition 1.1 there is a unique action-preserving mapping \( \beta : Y \rightarrow Z \) such that \( \alpha = \beta \circ \iota \). If \( \alpha \) is injective, then so is \( \iota \).

If \( G(S) \) acts simply on \( Z \) and \( g \cdot y = h \cdot y \) for some \( y \in Y \), then \( g \cdot \beta(y) = \beta(g \cdot y) = \beta(h \cdot y) = h \cdot \beta(y) \) and \( g = h \); thus \( G(S) \) acts simply on \( Y \).

If \( X \neq \emptyset \) and \( G(S) \) acts simply and transitively on \( Z \), then \( Y \neq \emptyset \) and, for any \( z \in Z \) and \( y \in Y \), we have \( z = g \cdot \beta(y) = \beta(g \cdot y) \) for some unique \( g \in G(S) \); thus \( \beta \) is surjective, and \( \beta^{-1}(z) \) contains exactly one element \( g \cdot y \) of the orbit of \( y \).

When \( \alpha \) is injective and \( G(S) \) acts simply and transitively on \( Z \), Proposition 2.1 implies that \( \beta \) is made of bijections from every orbit of \( Y \) onto \( Z \).

As in Section 1, let \((C_i)_{i \in I}\) be the family of connected components of \( X \), and let \((c_i)_{i \in I}\) be a cross-section of \( \equiv \) (with \( c_i \in C_i \)). Let \( V_i \) be the set of all \( g \in G(S) \) such that \( g \) is the group value of a connected sequence from some \( x \in C_i \) to \( c_i \):

\[
V_i = \{ g \in G(S) \mid x \xrightarrow{g} c_i \text{ for some } x \in C_i \}.
\]

By Lemma 1.7, \( V_i \) is a union of left cosets of \( H(C_i) \).

**Lemma 2.2.** In Proposition 2.1, let \( \alpha \) be injective and \( G(S) \) act simply and transitively on \( Z \). Let \( p \in Z \). Then

\[
\alpha(x) = \delta(x) \cdot p
\]

defines an injective mapping \( \delta : X \rightarrow G(S) \). Moreover

\[
\delta(C_i) = V_i \delta(c_i)
\]

for every connected component \( C_i \) of \( X \) and \( c_i \in C_i \).

**Proof:** \( \delta \) is well-defined: since \( G(S) \) act simply and transitively on \( Z \) there is for every \( x \in X \) a unique \( \delta(x) \in G(S) \) such that \( \alpha(x) = \delta(x) \cdot p \). Then \( \delta \) is injective, since \( \alpha \) is injective.
If \( x \xrightarrow{g} c_i \), then \((1, x) \sim (g, c_i)\) by Lemma 1.4 and \( \iota(x) = g \cdot \iota(c_i) \). Applying \( \beta \) yields

\[
\alpha(x) = \beta(\iota(x)) = \beta(g \cdot \iota(c_i)) = g \cdot \beta(\iota(c_i)) = g \cdot \alpha(c_i).
\]

Hence \( \delta(x) \cdot p = g\delta(c_i) \cdot p \) and \( \delta(x) = g\delta(c_i) \). Therefore \( \delta(C_i) = V_i \delta(c_i) \). \( \square \)

We say that the connected components of \( S \) have disjoint images in \( G(S) \) (relative to a cross-section of \( \equiv \)) if there exist \( g_i \in G(S) \) such that the sets \( V_i g_i \) are disjoint. If the action of \( S \) on a set \( X \) can be extended to a simply transitive action of \( G(S) \), then (relative to any cross-section of \( \equiv \)) the connected components of \( S \) have disjoint images in \( G(S) \), by Lemma 2.2.

**Theorem 2.3.** Let \((S, X, \cdot)\) be a semigroup act and \((G(S), Y, \cdot)\) be its universal group act. The action of \( S \) on \( X \) can be extended to a simply transitive action of \( G(S) \) on some set \( Z \supseteq X \) if and only if \( \iota \) is injective, \( G(S) \) acts simply on \( Y \), and, relative to some cross-section of \( \equiv \), the connected components of \( S \) have disjoint images in \( G(S) \).

**Proof:** These conditions are necessary by Proposition 2.1 and Lemma 2.2. Conversely, assume that \( \iota \) is injective, \( G(S) \) acts simply on \( Y \), and, relative to a cross-section \((c_i)_{i \in I}\) of \( \equiv \), the connected components of \( S \) have disjoint images in \( G(S) \): the sets \( V_i g_i \) are disjoint for some \( g_i \in G(S) \).

Construct \( \alpha : X \rightarrow G(S) \) as follows. Let \( x \in C_i \). When \( x \xrightarrow{g} c_i \), then \((1, x) \sim (g, c_i)\) by Lemma 1.4 and \( \iota(x) = \text{cls}(1, x) = \text{cls}(g, c_i) = g \cdot \iota(c_i) \). Since \( G(S) \) acts simply on \( Y \), \( g \) depends only on \( x \) (all connected sequences from \( x \) to \( c_i \) have the same group value). Therefore a mapping \( \alpha : X \rightarrow G(S) \) is well-defined by

\[
\alpha(x) = gg_i \quad \text{when} \quad x \in C_i \quad \text{and} \quad x \xrightarrow{g} c_i.
\]

Let \( x, y \in X \). If \( x \) and \( y \) lie in different connected components \( C_i \) and \( C_j \), then \( \alpha(x) \neq \alpha(y) \), since the sets \( V_i g_i \) and \( V_j g_j \) are disjoint. Now let \( x \) and \( y \) lie in the same connected component \( C_i \). Let \( x \xrightarrow{g} c_i \) and \( y \xrightarrow{h} c_i \). If \( \alpha(x) = \alpha(y) \), then \( g = h \),

\[
\iota(x) = g \cdot \iota(c_i) = \iota(y)
\]

by Lemma 1.4, and \( x = y \) since \( \iota \) is injective. Thus \( \alpha \) is injective.

Now \( G(S) \) acts simply and transitively on itself by left multiplication. We show that \((\gamma, \alpha) : (S, X, \cdot) \rightarrow (G(S), Z, \cdot)\) is a morphism of acts. Let \( x \in X \) and \( s \in S \). Let \( x \in C_i \) and \( x \xrightarrow{g} c_i \). By Lemma 1.5, \( s \cdot x \xrightarrow{\gamma(s)} x \), \( s \cdot x \xrightarrow{\gamma(s)g} c_i \), and

\[
\alpha(s \cdot x) = \gamma(s) gg_i = \gamma(s) \alpha(x).
\]

\( \square \)

2. The following results complete Theorem 2.3.
Proposition 2.4. In the universal group act \((G(S), Y, .)\) of \((S, X, .)\), \(\iota(x) = \iota(y)\) if and only if there exists a connected sequence from \(x\) to \(y\) with group value \(1\). If \(\iota\) is injective, then \(S\) acts by injections.

Proof: \(\iota(x) = \iota(y)\) if and only if \((1, x) \sim (1, y)\), so the first part of the statement follows from Lemma 1.4. Now assume that \(\iota\) is injective. If \(s \cdot x = s \cdot y\), then \(x_0 = x, x_1 = y, s_1 = s, t_1 = s\) is a connected sequence from \(x\) to \(y\) with group value \(1\); hence \(x = y\); thus \(S\) acts by injections.

Proposition 2.5. In the universal group act \((G(S), Y, .)\) of \((S, X, .)\), \(G(S)\) acts simply on \(Y\) if and only if, for every \(x \in X\), every connected sequence from \(x\) to \(x\) has group value \(1\).

Proof: This follows from Proposition 1.8, but we give a direct proof. If \(x \xrightarrow{g} x\), then \((1, x) \sim (g, x)\) by Lemma 1.4 and 1. \(\text{cls}(1, x) = g \cdot \text{cls}(1, x)\); if \(G(S)\) acts simply on \(Y\) this implies \(g = 1\). Conversely let \(g \cdot \text{cls}(k, x) = h \cdot \text{cls}(k, x)\). Then \((gk, x) \sim (hk, x)\); by Lemma 1.4, there is a connected sequence from \(x\) to \(x\) with group value \((gk)^{-1} (hk)\). If all such sequences have group value \(1\), then \(gk = hk\) and \(g = h\); thus \(G(S)\) acts simply on \(Y\).

Propositions 2.4 and 2.5 will be made more explicit in Section 4.

If \(X\) is connected, then the universal \(G(S)\)-set \(Y\) of \(X\) serves in Theorem 2.3:

Proposition 2.6. In the universal group action \((G(S), Y, .)\) of \((S, X, .)\), \(G(S)\) acts transitively on \(Y\) if and only if \(X\) is connected.

Proof: This follows from Proposition 1.6, and from Proposition 1.8, but can be shown directly as follows. Let \(x, y \in X\). If \(G(S)\) acts transitively on \(Y\), then \(\text{cls}(1, x) = g \cdot \text{cls}(1, y) = \text{cls}(g, y)\) for some \(g \in G(S)\) and there exists a connected sequence from \(x\) to \(y\), by Lemma 1.4; thus \(X\) is connected. Conversely let \(\text{cls}(h, x), \text{cls}(k, y) \in Y\). If \(X\) is connected, there exists a connected sequence from \(x\) to \(y\) and \((h, x) \sim (g, y)\) for some \(g \in G(S)\), by Lemma 1.4; then \(kg^{-1} \cdot \text{cls}(h, x) = kg^{-1} \cdot \text{cls}(g, y) = \text{cls}(k, y)\). Thus \(G(S)\) acts transitively on \(Y\).

References


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