Relative normality and product spaces

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Abstract. Arhangel'skiĭ defines in [Topology Appl. 70 (1996), 87–99], as one of various notions on relative topological properties, strong normality of $A$ in $X$ for a subspace $A$ of a topological space $X$, and shows that this is equivalent to normality of $X_A$, where $X_A$ denotes the space obtained from $X$ by making each point of $X \setminus A$ isolated. In this paper we investigate for a space $X$, its subspace $A$ and a space $Y$ the normality of the product $X_A \times Y$ in connection with the normality of $(X \times Y)_{(A \times Y)}$. The cases for paracompactness, more generally, for $\gamma$-paracompactness will also be discussed for $X_A \times Y$. As an application, we prove that for a metric space $X$ with $A \subseteq X$ and a countably paracompact normal space $Y$, $X_A \times Y$ is normal if and only if $X_A \times Y$ is countably paracompact.

Keywords: strongly normal in, normal, $\gamma$-paracompact, product spaces, weak $C$-embedding

Classification: Primary 54B10; Secondary 54B05, 54C20, 54C45, 54D15, 54D20

1. Introduction

Throughout this paper all spaces are assumed to be Hausdorff. Let $\gamma$ denote an infinite cardinal, and $\mathbb{N}$ the set of natural numbers.

Let $X$ be a space and $A$ a subspace of $X$.

As is known, $A$ is said to be $C^\ast$-embedded (respectively $C$-embedded) in $X$ if every bounded real-valued (respectively real-valued) continuous function on $A$ can be extended to a continuous function over $X$.

Next we recall some relative topological properties in Arhangel'skiĭ [2]. We say that $A$ is strongly normal in $X$ if for every pair $E, F$ of disjoint closed subsets of $A$ there exist disjoint open subsets $U$ and $V$ of $X$ such that $E \subseteq U$ and $F \subseteq V$. The subspace $A$ is weakly $C$-embedded in $X$ if for every real-valued continuous function $f$ on $A$ there exists a real-valued function on $X$ which is an extension of $f$ and continuous at each point of $Y$.

For a space $X$ and a subspace $A$ of $X$ let $X_A$ denote the space obtained from the space $X$, with the topology generated by $\{ U \mid U$ is open in $X$ or $U \subseteq X \setminus A \}$. Hence $A$ is a closed subspace of $X_A$ and points in $X \setminus A$ are isolated. As is seen in [2], the space $X_A$ is often useful to describe several relative topological properties. Indeed, the following are shown in [2]: (1) $X_A$ is normal if and only if $A$ is strongly normal in $X$ if and only if $A$ is normal itself, and is weakly $C$-embedded in $X$, (2) $A$ is weakly $C$-embedded in $X$ if and only if $A$ is $C^\ast$-embedded in $X_A$. 
On the other hand, in a joint paper [9] of the first author with Yamazaki the notion of weak C-embedding was characterized by extending disjoint cozero-sets of a subspace to disjoint open sets of the whole space. And it was applied there for a space $X$, a subspace $A$ of $X$ and a space $Y$ to describe weak $C$-embedding of $A \times Y$ in the product $X_A \times Y$; actually, it was shown that if $Y$ is compact Hausdorff, $A \times Y$ is $C^*$-embedded in $X_A \times Y$ if and only if $A \times Y$ is $C^*$-embedded in $(X \times Y)_{(A \times Y)}$, that is, $A \times Y$ is weakly $C$-embedded in $X \times Y$. Being motivated by this result, our main concern in this paper is to study normality of the product $X_A \times Y$ in relation to normality of $(X \times Y)_{(A \times Y)}$ (or, equivalently, strong normality of $A \times Y$ in $X \times Y$). Namely we prove

**Theorem 1.1.** For a space $X$, a subspace $A$ of $X$ and a space $Y$, the product $X_A \times Y$ is normal if and only if $(X \times Y)_{(A \times Y)}$ is normal and the following condition $(\ast)$ holds:

$(\ast)$ for every closed subset $E$ of $X_A \times Y$ disjoint from $A \times Y$ there exists an open subset $U$ of $X_A \times Y$ such that $E \subset U$ and $U \cap (A \times Y) = \emptyset$.

As a corollary to this result we have that for a space $X$, a subspace $A$ of $X$ and a compact Hausdorff space $Y$, $X_A \times Y$ is normal if and only if $(X \times Y)_{(A \times Y)}$ is normal. Moreover, using condition $(\ast)$ above we prove analogous results for $\gamma$-collectionwise normality or $\gamma$-paracompactness. In particular, the case $\gamma = \omega$ is applied to obtain further the following theorem; putting $A = X$, we have the well-known theorem due to Morita [14] (for the proof see [10]) and Rudin and Starbird [16].

**Theorem 1.2.** Let $X$ be a metric space, $A$ a subspace of $X$ and $Y$ a normal and countably paracompact space. Then $X_A \times Y$ is normal if and only if $X_A \times Y$ is countably paracompact.

For undefined notation and terminology see Engelking’s book [6].

2. Preliminaries

The following theorem due to Arhangel’skii [2] mentioned in the introduction is useful.

**Theorem 2.1** ([2]). For a subspace $A$ of a space $X$, the following statements are equivalent:

1. $X_A$ is normal,
2. $A$ is strongly normal in $X$,
3. $A$ is normal and $A$ is weakly $C$-embedded in $X$.

Weak $C$-embedding was characterized in [9] as follows.
Theorem 2.2 ([9]). Let $A$ be a subspace of a space $X$. Then $A$ is weakly $C$-embedded in $X$ if and only if for every pair $G_0$, $G_1$ of disjoint cozero-sets in $A$ there exist disjoint open subsets $H_0$, $H_1$ of $X$ such that $G_i \subset H_i$ ($i = 0, 1$).

By this result we see that if either $A$ is dense in $X$ or $A$ is $z$-embedded in $X$, then $A$ is weakly $C$-embedded in $X$ ([5], [9]); a subspace $A$ of a space $X$ is said to be $z$-embedded in $X$ if every zero-set $Z$ of $A$ can be written as $Z = Z' \cap A$ with a zero-set $Z'$ of $X$. It is known that every cozero-set of a space or a Lindelöf subspace of a Tychonoff space is $z$-embedded. Also, observe the following implications:

$C^*$-embedding $\Rightarrow$ $z$-embedding $\Rightarrow$ weak $C$-embedding.

The next two results show when a subspace $A \times Y$ is weakly $C$-embedded in $X \times Y$, a subspace $A$ of $X$ and a metric space $Y$. The first one is essentially due to Kodama [11].

Theorem 2.3 ([11]). Let $X$ be a normal space, $A$ a closed subspace of $X$ and $Y$ a metric space. If $A \times Y$ is normal and countably paracompact, then $A \times Y$ is $z$-embedded in $X \times Y$, hence, weakly $C$-embedded in $X \times Y$.

In case $A \times Y$ is not assumed to be normal, we have the following.

Theorem 2.4. Let $A$ be an arbitrary subspace of a hereditarily normal space $X$, and $Y$ a metric space. Then $A \times Y$ is weakly $C$-embedded in $X \times Y$.

Proof: We show that any two disjoint open sets of $A \times Y$ are separated by disjoint open sets of $X \times Y$, which implies weak $C$-embedding of $A \times Y$ in $X \times Y$ by Theorem 2.2. Let $G_0$ and $G_1$ be disjoint open sets of $A \times Y$. Let $B = \bigcup_{n \in \mathbb{N}} B_n$ be a $\sigma$-locally finite open base for $Y$, where each $B_n$ is locally finite. Let $B_n = \{B_{n\lambda} \mid \lambda \in \Lambda_n\}$. Define for $n \in \mathbb{N}$ and $\lambda \in \Lambda_n$

$$H^0_{n\lambda} = \bigcup\{O \mid O \text{ is open in } A, O \times \overline{B_{n\lambda}} \subset G_0\}.$$ 

Then $H^0_{n\lambda}$ and $p_A((A \times \overline{B_{n\lambda}}) \cap G_1)$ are disjoint open subsets of $A$. Since $X$ is hereditarily normal, there exists an open set $W^0_{n\lambda}$ of $X$ such that

$$H^0_{n\lambda} \subset W^0_{n\lambda}, \quad \overline{W^0_{n\lambda}} \cap p_A((A \times \overline{B_{n\lambda}}) \cap G_1) = \emptyset.$$ 

For each $n \in \mathbb{N}$ let us put $U^0_n = \bigcup\{W^0_{n\lambda} \times B_{n\lambda} \mid \lambda \in \Lambda_n\}$. Then $U^0_n$ is an open set of $X \times Y$ and we have $G_0 \subset \bigcup_{n \in \mathbb{N}} U^0_n$ and $\overline{U^0_n} \cap G_1 = \emptyset$ for every $n \in \mathbb{N}$. Similarly, we can find an open set $U^1_n$ of $X \times Y$ for each $n \in \mathbb{N}$ so that $G_1 \subset \bigcup_{n \in \mathbb{N}} U^1_n$ and $\overline{U^1_n} \cap G_0 = \emptyset$ for every $n \in \mathbb{N}$. Hence, as is well-known, $G_0$ and $G_1$ are separated by open sets of $X \times Y$. This completes the proof.

It was shown in [9] that every subspace of a space $X$ is weakly $C$-embedded in $X$ if and only if $X$ is hereditarily normal.

In connection with Theorems 2.3 and 2.4, let us observe the following two examples.
Example 2.5. (1) (Michael [12]) Let \( \mathbb{R} \), \( \mathbb{Q} \) and \( \mathbb{P} \) be the real line, the set of rationals and the set of irrationals, respectively. Then \( \mathbb{R} \times \mathbb{Q} \) is known as the Michael line, and it is hereditarily normal. Since \( \mathbb{Q} \times \mathbb{P} \) is Lindelöf, it is \( z \)-embedded in \( \mathbb{R} \times \mathbb{Q} \times \mathbb{P} \), but is not \( C^* \)-embedded as was shown by Morita [15].

(2) (Vaughan [17]) Let \( D(\omega_1) \) denote the set \( \omega_1 \) with the discrete topology. Let \( \hat{D}(\omega_1) \) denote the space obtained from the space \( \omega_1+1 \) with the usual order topology by letting all points except \( \omega_1 \) be isolated. That is, \( \hat{D}(\omega_1) = (\omega_1+1)\setminus\{\omega_1\} \).

Let \( X = \square_{\omega_1} \hat{D}(\omega_1) \) denote the box product of countably many copies of \( \hat{D}(\omega_1) \), and \( Y = D(\omega_1)^{\omega} \) denote the usual product of countably many copies of \( D(\omega_1) \).

Then \( X \) is hereditarily paracompact and \( Y \) is metrizable. Put

\[ A = X \setminus Y, \quad \Delta(Y) = \{(x, x) \mid x \in Y\}. \]

Then \( A \times Y \) and \( \Delta(Y) \) are disjoint closed sets of \( X \times Y \) and cannot be separated by open sets, which shows \( X \times Y \) is not normal ([17]).

By Theorem 2.4 we see that \( A \times Y \) is weakly \( C \)-embedded in \( X \times Y \). Since \( A \) contains a closed subset homeomorphic to \( X \), \( A \times Y \) is not normal. Hence, in view of Theorem 2.3, it may be of interest to see whether \( A \times Y \) is \( z \)-embedded in \( X \times Y \), but this is unknown to the authors. However, we can show further that \( A \times Y \) is not \( C^* \)-embedded in \( X \times Y \). To prove this, first note that \( Y \cong (\text{is homeomorphic to}) \ Y^2 \). Hence, if we show the fact below, by the same argument of Morita [15] we can conclude that \( A \times Y \) is not \( C^* \)-embedded in \( X \times Y \).

**Fact.** \( \Delta(Y) \) is a zero-set of \( X \times Y \).

**Proof:** Since the box topology is stronger than the usual topology, it suffices to show that \( \Delta(Y) \) is a zero-set of \( \hat{D}(\omega_1)^{\omega} \times D(\omega_1)^{\omega} \).

For each point \( \langle x, y \rangle \in \hat{D}(\omega_1)^{\omega} \times D(\omega_1)^{\omega} \setminus \Delta(Y) \), define

\[ n(x, y) = \min\{k \mid x_k \neq y_k\}. \]

Put

\[ H_m = \{\langle x, y \rangle \in \hat{D}(\omega_1)^{\omega} \times D(\omega_1)^{\omega} \setminus \Delta(Y) \mid n(x, y) = m\}. \]

Then we have

\[ \hat{D}(\omega_1)^{\omega} \times D(\omega_1)^{\omega} \setminus \Delta(Y) = \bigcup_{m \in \mathbb{N}} H_m, \]

\[ m \neq m' \Rightarrow H_m \cap H_{m'} = \emptyset. \]

**Claim.** \( H_m \) is an open and closed subset of \( \hat{D}(\omega_1)^{\omega} \times D(\omega_1)^{\omega} \).

**Proof of Claim:** Let \( \langle x, y \rangle \in H_m \). Since \( n(x, y) = m \), we have \( x_1 = y_1, \ldots, x_{m-1} = y_{m-1} < \omega_1 \).
Case (i). $x_m > y_m$. Put

$$U = \{x_1\} \times \cdots \times \{x_{m-1}\} \times (y_m, \omega_1] \times \hat{D}(\omega_1) \times \cdots,$$

$$V = \{y_1\} \times \cdots \times \{y_{m-1}\} \times \{y_m\} \times \hat{D}(\omega_1) \times \cdots.$$ 

Then $\langle x, y \rangle \in U \times V \subset H_m$.

Case (ii). $x_m < y_m$. Put

$$U = \{x_1\} \times \cdots \times \{x_{m-1}\} \times \{x_m\} \times \hat{D}(\omega_1) \times \cdots,$$

$$V = \{y_1\} \times \cdots \times \{y_{m-1}\} \times \{y_m\} \times \hat{D}(\omega_1) \times \cdots.$$ 

Then $\langle x, y \rangle \in U \times V \subset H_m$.

Hence, in either case $H_m$ is open in $\hat{D}(\omega_1) \times D(\omega_1)$.

For each $\langle y, y \rangle \in \Delta(Y)$, put

$$U = \{y_1\} \times \cdots \times \{y_m\} \times \hat{D}(\omega_1) \times \cdots,$$

$$V = \{y_1\} \times \cdots \times \{y_m\} \times D(\omega_1) \times \cdots.$$ 

Then $(U \times V) \cap H_m = \emptyset$. Hence, $\Delta(Y) \cap \overline{H_m} = \emptyset$, which shows that $H_m$ is closed in $X \times Y$.

It follows that $H_m$ is a cozero-set, therefore, $\bigcup_{m \in \mathbb{N}} H_m$ is a cozero-set of $\hat{D}(\omega_1) \times D(\omega_1)$. Hence $\Delta(Y)$ is a zero-set of $X \times Y$. This completes the proof. \hfill \Box

3. Proof of Theorem 1.1

First we prove

**Lemma 3.1.** Let $X$ be a space, $A$ a subspace of $X$ and $Y$ a space. If $X_A \times Y$ is normal, then $(X \times Y)_{(A \times Y)}$ is normal.

**Proof:** Let $E$ and $F$ be disjoint closed subsets of $A \times Y$. Then they are closed also in $X_A \times Y$ and disjoint. Hence, there exist disjoint open subsets $U$ and $V$ of $X_A \times Y$ such that $E \subset U$ and $F \subset V$. Define $U' = \text{Int}_{(X \times Y)} U$ and $V' = \text{Int}_{(X \times Y)} V$, where $\text{Int}_Z W$ denotes the interior of $W$ in the space $Z$. Then $U'$ and $V'$ are disjoint open in $X \times Y$ and so in $(X \times Y)_{(A \times Y)}$, and we have $E \subset U'$ and $F \subset V'$. Hence, $A \times Y$ is strongly normal in $X \times Y$. Hence by Theorem 2.1 $(X \times Y)_{(A \times Y)}$ is normal. This completes the proof. \hfill \Box
Remark. \((\mathbb{R} \times \mathbb{P})(\mathbb{Q} \times \mathbb{P})\) is normal, but \(\mathbb{R}_Q \times \mathbb{P}\) is not normal. The converse of the lemma, therefore, need not hold.

Proof of Theorem 1.1: From Lemma 3.1 the “only if” part easily follows. To prove the “if” part, assume that \(X \times Y\) is normal and condition (*) holds. Let \(E, F\) be a pair of disjoint closed subsets of \(X \times Y\). Since \(X \times Y\) is strongly normal, there exist disjoint open subsets \(U, V\) of \(X \times Y\) such that \(E \cap (X \times Y) \subseteq U\) and \(F \cap (X \times Y) \subseteq V\). Put \(D = (E \setminus U) \cup (F \setminus V)\). Then \(D\) is a closed subset of \(X \times Y\) and \(D \cap (X \times Y) = \emptyset\). Then by (*), there exists an open subset \(W\) of \(X \times Y\) such that \(X \times Y \subseteq W\) and \(\overline{W} \cap D = \emptyset\).

Put \(U_1 = U \cap W\) and \(V_1 = V \cap W\). Then we have
\[
(A \times Y) \cap E \subseteq U_1, \quad \overline{U_1} \cap F = \emptyset, \quad \text{and} \quad (A \times Y) \cap F \subseteq V_1, \quad \overline{V_1} \cap E = \emptyset.
\]
Then \(E \setminus U_1\) and \(F \setminus V_1\) are disjoint closed subsets of \((X \setminus A) \times Y\). Since \((X \setminus A) \times Y\) is normal, there exist disjoint open subsets \(U_2\) and \(V_2\) of \((X \setminus A) \times Y\) such that \(E \setminus U_1 \subseteq U_2\) and \(F \setminus V_1 \subseteq V_2\). Therefore, \(U_1 \cup (U_2 \setminus \overline{U_1})\) and \(V_1 \cup (V_2 \setminus \overline{V_1})\) are disjoint open subsets of \(X \times Y\), which satisfy \(E \subseteq U_1 \cup (U_2 \setminus \overline{U_1})\) and \(F \subseteq V_1 \cup (V_2 \setminus \overline{V_1})\). Hence \(X \times Y\) is normal. This completes the proof. \(\square\)

The following is proved in Burke and Pol [4].

Theorem 3.2 ([4]). Let \(A\) and \(X\) be subsets of \(\mathbb{R}\) with \(A \subseteq X\) and let \(Y\) be a metric space. Then \(X \times Y\) is normal if and only if condition (*) holds.

Since \(X \times Y\) is a metric space, \((X \times Y)(X \times Y)\) is normal. Therefore, this theorem immediately follows from Theorem 1.1.

The following result was formulated in [9] without proof.

Theorem 3.3 ([9]). Let \(A\) be a subset of a space \(X\) and \(Y\) be a compact Hausdorff space. Then \(X \times Y\) is normal if and only if \((X \times Y)(X \times Y)\) is normal.

Proof: Since the projection \(p_X : X \times Y \rightarrow X\) is a closed map, condition (*) in Theorem 1.1 is easily satisfied. Hence the theorem follows. \(\square\)

Recall that a space \(X\) is \(\gamma\)-collectionwise normal if for every discrete collection \(\{E_\alpha \mid \alpha < \gamma\}\) of closed subsets there exists a disjoint collection \(\{G_\alpha \mid \alpha < \gamma\}\) of open subsets such that \(E_\alpha \subseteq G_\alpha\) for each \(\alpha < \gamma\).

A subspace \(A\) of a space \(X\) is said to be strongly \(\gamma\)-collectionwise normal in \(X\) if for every discrete collection \(\{E_\alpha \mid \alpha < \gamma\}\) of closed subsets of \(A\) there is a disjoint collection \(\{U_\alpha \mid \alpha < \gamma\}\) of open subsets of \(X\) such that \(E_\alpha \subseteq U_\alpha\) for each \(\alpha < \gamma\) ([9]).

It was proved in [9] that \(X_A\) is \(\gamma\)-collectionwise normal if and only if \(A\) is strongly \(\gamma\)-collectionwise normal in \(X\). With this result similarly to Theorem 1.1 we can prove the following.
**Theorem 3.4.** For a space $X$, a subspace $A$ of $X$ and a space $Y$, $X_A \times Y$ is $\gamma$-collectionwise normal if and only if $(X \times Y)_{(A \times Y)}$ is $\gamma$-collectionwise normal and condition (*) in Theorem 1.1 holds.

A space $X$ is $\gamma$-paracompact if every open cover of $X$ of cardinality not greater than $\gamma$ has a locally finite open refinement.

**Theorem 3.5.** If $X_A \times Y$ is $\gamma$-paracompact, then $(X \times Y)_{(A \times Y)}$ is $\gamma$-paracompact. Furthermore, if $X_A \times Y$ satisfies condition (*) in Theorem 1.1, then the converse holds.

**Proof:** Assume $X_A \times Y$ is $\gamma$-paracompact. Let $U$ be an open cover of $(X \times Y)_{(A \times Y)}$ of cardinality not greater than $\gamma$. Put

$$U' = \{U \in U \mid U \cap (A \times Y) \neq \emptyset\}.$$  

Then $\bigcup\{\text{Int}_{(X \times Y)}U \mid U \in U'\} \supset A \times Y$. Hence $\{X_A \times Y \setminus A \times Y\} \cup U'$ is an open cover of $X_A \times Y$ of cardinality not greater than $\gamma$. Since $X_A \times Y$ is $\gamma$-paracompact, there exists a locally finite open cover $V$ of $X_A \times Y$ which refines $U$. Put $V' = \{V \in V \mid V \cap (A \times Y) \neq \emptyset\}$. Then the collection

$$V' \cup \{(x, y) \mid (x, y) \notin \bigcup V'\}$$

is a locally finite open cover of $(X \times Y)_{(A \times Y)}$ and refines $U$. Hence $(X \times Y)_{(A \times Y)}$ is $\gamma$-paracompact.

To prove the converse under (*), assume that $(X \times Y)_{(A \times Y)}$ is $\gamma$-paracompact and (*) holds. Let $U$ be an open cover of $X_A \times Y$ of cardinality not greater than $\gamma$. Then $U$ is an open cover of $(X \times Y)_{(A \times Y)}$ as well. By assumption there exists a locally finite open cover $V$ of $(X \times Y)_{(A \times Y)}$ refining $U$. Put

$$G = \{(x, y) \in X \times Y \mid V \text{ is locally finite at } (x, y) \text{ in the product } X \times Y\}.$$  

Then $G$ is open in $X \times Y$ and $G \supset A \times Y$. Put $V' = \{G \cap \text{Int}_{(X \times Y)}V \mid V \in V\}$. Then we have $\bigcup V' \supset A \times Y$, and $V'$ refines $U$ and is locally finite at each $(x, y) \in \bigcup V'$ in $X \times Y$. By (*) there exist open subsets $O_1$ and $O_2$ in $X_A \times Y$ such that

$$A \times Y \subset O_1 \subset \overline{O_1} \subset O_2 \subset \overline{O_2} \subset \bigcup V'.$$

For every $x \in X \setminus A$, let $P_x$ be a locally finite open cover of $Y$ such that the collection $\{\{x\} \times P \mid P \in P_x\}$ refines $U$. Then the collection

$$\{(\{x\} \times P) \cap \overline{O_1} \mid x \in X \setminus A, P \in P_x\} \cup \{V \cap O_2 \mid V \in V'\}$$

is a locally finite open cover of $X_A \times Y$ which refines $U$. Thus, $X_A \times Y$ is $\gamma$-paracompact. This completes the proof. 

\[\square\]
4. Proof of Theorem 1.2

First we prove

**Theorem 4.1.** Let $A$ be a subset of a space $X$ and $Y$ a space. Suppose that the product $A \times Y$ is $\gamma$-paracompact. If $X_A \times Y$ is normal, then $X_A \times Y$ is $\gamma$-paracompact.

**Proof:** Assume that $X_A \times Y$ is normal. Then $(X \times Y)_{(A \times Y)}$ is normal by Theorem 1.1. Hence $A \times Y$ is normal and weakly $C$-embedded in $X \times Y$ by Theorem 2.1. Since $A \times Y$ is $\gamma$-paracompact, by [9, Lemma 4.6] $(X \times Y)_{(A \times Y)}$ is $\gamma$-paracompact. Since $X_A \times Y$ satisfies (*), $X_A \times Y$ is $\gamma$-paracompact by Theorem 3.5. This completes the proof. □

**Corollary 4.2.** Let $A$ be a subset of a space $X$ and $Y$ a space. Suppose that the product $A \times Y$ is countably paracompact. If $X_A \times Y$ is normal, then $X_A \times Y$ is countably paracompact.

**Proof of Theorem 1.2:** Let $A$ be a subspace of a metric space $X$, and $Y$ a normal and countably paracompact space. To prove the “only if” part, assume $X_A \times Y$ is normal. Since $A \times Y$ is closed in $X_A \times Y$, $A \times Y$ is also normal. Hence by Morita, Rudin-Starbird’s theorem ([14], [16]), $A \times Y$ is countably paracompact. Hence $X_A \times Y$ is countably paracompact by Corollary 4.2.

To prove the converse, assume that $X_A \times Y$ is countably paracompact. Then similarly to above we have that $A \times Y$ is countably paracompact and normal. Then by [11] $A \times Y$ is $z$-embedded in $A \times \beta Y$, where $\beta Y$ is the Čech-Stone compactification of $Y$. Since $X_A \times \beta Y$ is paracompact, $A \times \beta Y$ is $C$-embedded in $X_A \times \beta Y$. It follows that $A \times Y$ is $z$-embedded in $X_A \times Y$, and hence it is weakly $C$-embedded in $X_A \times Y$. This easily implies that $A \times Y$ is weakly $C$-embedded in $X \times Y$. Hence $(X \times Y)_{(A \times Y)}$ is normal.

We next show that property (*) in Theorem 1.1 is satisfied. Let $\{\mathcal{B}_n\}$ be a sequence of locally finite open covers of $X$ such that $\{\text{St}(x, \mathcal{B}_n) | n \in \mathbb{N}\}$ is a neighborhood base at each point $x$ in $X$. Let $\mathcal{B}_n = \{B_{n\alpha} | \alpha \in \Omega_n\}$. Let us put

$$W(\alpha_1, \ldots, \alpha_n) = \bigcap \{B_{i\alpha_i} | i = 1, \ldots, n\}, \text{ for } \alpha_i \in \Omega_i; \ i = 1, \ldots, n.$$  

To prove (*), let $E$ be a closed subset of $X_A \times Y$ such that $E \cap (A \times Y) = \emptyset$. Put

$$G(\alpha_1, \ldots, \alpha_n) = \bigcup \{O | O \text{ is open in } Y, (W(\alpha_1, \ldots, \alpha_n) \times O) \cap E = \emptyset\}.$$  

Then we have

$$G(\alpha_1, \ldots, \alpha_n) \subset G(\alpha_1, \ldots, \alpha_n, \alpha_{n+1})$$
for \( \alpha_i \in \Omega_i, \ i = 1, \ldots, n, n + 1, \) and
\[
\{(W(\alpha_1, \ldots, \alpha_n) \cap A) \times G(\alpha_1, \ldots, \alpha_n) \mid \alpha_i \in \Omega_i, \ i = 1, \ldots, n; \ n \in \mathbb{N}\}
\]
covers \( A \times Y \). Since \( A \times Y \) is normal and countably paracompact, by Morita [13] (see [8]) there exists a cozero-set \( U(\alpha_1, \ldots, \alpha_n) \) of \( Y \) such that
\[
U(\alpha_1, \ldots, \alpha_n) \subset G(\alpha_1, \ldots, \alpha_n)
\]
and
\[
\{(W(\alpha_1, \ldots, \alpha_n) \cap A) \times U(\alpha_1, \ldots, \alpha_n) \mid \alpha_i \in \Omega_i, \ i = 1, \ldots, n; \ n \in \mathbb{N}\}
\]
covers \( A \times Y \). Put
\[
L = \bigcup \{(W(\alpha_1, \ldots, \alpha_n) \times U(\alpha_1, \ldots, \alpha_n) \mid \alpha_i \in \Omega_i, \ i = 1, \ldots, n; \ n \in \mathbb{N}\}.
\]
Then \( L \) is a cozero-set of \( X \times Y \) and we have \( L \supset A \times Y, \ L \cap E = \emptyset \). Since \( X_A \times Y \) is countably paracompact, by [7] there exists an open subset \( H \) of \( X \times Y \) such that \( A \times Y \subset H \subset \mathcal{P} H \subset L \). Hence \( A \times Y \) and \( E \) are separated by open sets of \( X_A \times Y \). This completes the proof of the theorem. \( \square \)

The proof of the “if” part of Theorem 1.1 yields further the following result which seems of interest in itself.

**Theorem 4.3.** Let \( A \) be a subset of a metric space \( X \) and \( Y \) a normal and \( \gamma \)-paracompact space. Then \( (X \times Y)_{(A \times Y)} \) is \( \gamma \)-paracompact if and only if \( A \times Y \) is normal.

**Proof:** To prove the “if” part, assume that \( A \times Y \) is normal. Since \( Y \) is normal and \( \gamma \)-paracompact, so is \( A \times Y \). Hence \( (A \times Y) \times \Gamma^\gamma \) is normal \( \gamma \)-paracompact, that is, \( A \times \Gamma^\gamma \times Y \) is normal, where \( I = [0,1] \). Hence, as is shown in the proof of Theorem 1.2, \( (X \times (\Gamma^\gamma \times Y))_{(A \times (\Gamma^\gamma \times Y)}) \) is normal. Since \( (X \times (\Gamma^\gamma \times Y))_{(A \times (\Gamma^\gamma \times Y))} \cong ((X \times Y) \times \Gamma^\gamma)_{((A \times Y) \times \Gamma^\gamma)}, ((X \times Y) \times \Gamma^\gamma)_{((A \times Y) \times \Gamma^\gamma)}) \) is normal. Thus, by Theorem 3.3 \( (X \times Y)_{(A \times Y)} \times \Gamma^\gamma \) is normal. Therefore, as is well-known, \( (X \times Y)_{(A \times Y)} \) is \( \gamma \)-paracompact (see [6]). This completes the proof. \( \square \)

**Example 4.4.** The condition “\( X \) is metric” cannot be excluded from Theorem 1.2. In fact, there exist compact spaces \( X \) and \( Y \), and a subset \( A \) of \( X \) such that \( A \times Y \) is normal and countably paracompact and \( X_A \times Y \) is countably paracompact, but not normal. We use Bing’s example \( G \) [3]. Let \( \mathcal{P}(\omega_1) \) be the power set of \( \omega_1 \) and
\[
X = \{f \mid f : \mathcal{P}(\omega_1) \longrightarrow \{0,1\}\}.
\]
For every $\alpha \in \omega_1$, let us define a function $f_\alpha : \mathcal{P} \to \{0, 1\}$ for $P \in \mathcal{P}(\omega_1)$ by
\[
f_\alpha(P) = \begin{cases} 
1 & \text{if } \alpha \in P, \\
0 & \text{if } \alpha \notin P.
\end{cases}
\]
Put $A = \{f_\alpha \mid \alpha < \omega_1\}$. Then Bing’s example $G$ is precisely the space $X_A$. It is well-known that $X_A$ is normal and countably paracompact, but it is not $\omega_1$-collectionwise normal. Let $Y$ be the one-point compactification of the discrete space of Card $A$. Since $X_A$ is countably paracompact, $A \times Y$ is countably paracompact. Since $A$ is $w(Y)$-paracompact, $A \times Y$ is normal. However, since $X_A$ is not $\omega_1$-collectionwise normal, by Alas [1] $X_A \times Y$ is not normal.

References

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(Received March 3, 2003)