An inequality in Orlicz function spaces with Orlicz norm

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Abstract. We use Simonenko quantitative indices of an $N$-function $\Phi$ to estimate two parameters $q_\Phi$ and $Q_\Phi$ in Orlicz function spaces $L^\Phi[0,\infty)$ with Orlicz norm, and get the following inequality: $\frac{B_\Phi}{A_\Phi-1} \leq q_\Phi \leq \frac{A_\Phi}{A_\Phi-1}$, where $A_\Phi$ and $B_\Phi$ are Simonenko indices. A similar inequality is obtained in $L^\Phi[0,1]$ with Orlicz norm.

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1. Introduction

Definition 1.1. A function $M : \mathbb{R} \rightarrow \mathbb{R}$ is called an $N$-function, if

(i) $M$ is continuous, convex and even;
(ii) $M(u) > 0$ for $u \neq 0, M(0) = 0$;
(iii) $\lim_{u \to 0} M(u)/u = 0, \lim_{u \to \infty} M(u)/u = \infty$.

Let

$$\Phi(u) = \int_0^{|u|} \phi(t) \, dt \text{ and } \Psi(u) = \int_0^{|v|} \psi(s) \, ds$$

be a pair of complementary $N$-functions. The Orlicz function space is defined as follows: $L^\Phi[0,1] = \{x(t) : x(t) \text{ is measurable on } [0,1] \text{ and } \rho_\Phi(\lambda x(t)) \, dt < \infty \text{ for some } \lambda > 0\}$, where $\rho_\Phi(x(t)) = \int_{[0,1]} \Phi(x(t)) \, dt$; $L^\Phi[0,\infty) = \{x(t) : x(t) \text{ is measurable on } [0,\infty), \rho_\Phi(\lambda x(t)) \, dt < \infty \text{ for some } \lambda > 0\}$, and $\rho_\Phi(x(t)) = \int_{[0,\infty)} \Phi(x(t)) \, dt$. We define the Orlicz norm on the Orlicz space as

$$\|x\|_\Phi = \inf_{k>0} \frac{1}{k} [1 + \rho_\Phi(kx)] .$$

An $N$-function $\Phi(u)$ is said to satisfy the $\triangle_2$-condition for small $u$ (in symbol $\Phi \in \triangle_2(0)$), if there exists $u_0 > 0$ and $C > 0$, such that $\Phi(2u) \leq C\Phi(u)$ for $0 \leq u \leq u_0$. $\Phi(u)$ is said to satisfy the $\triangle_2$-condition for large $u$ (in symbol $\Phi \in \triangle_2(\infty)$), if there exists $u_0 > 0$ and $C > 0$ such that $\Phi(2u) \leq C\Phi(u)$ for $u \geq u_0$. $\Phi(u)$ is said to satisfy the $\triangle_2$-condition for all $u \geq 0$ (in symbol $u \in \triangle_2$), if there exist $C > 0$ such that $\Phi(2u) \leq C\Phi(u)$ for $u \geq 0$. An $N$-function
\( \Phi(u) \) is said to satisfy the \( \nabla_2 \text{-condition for small } u \) (for large \( u \), for all \( u \geq 0 \)), in symbol \( \Phi \in \nabla_2(0) \) (\( \Phi \in \nabla_2(\infty) \), \( \Phi \in \nabla_2 \)), if its complementary \( \mathcal{N} \)-function \( \Psi \in \nabla_2(0) \) (\( \Psi \in \nabla_2(\infty) \), \( \Psi \in \Delta_2 \)).

The basic results on Orlicz spaces can be found in Krasnosel’skii and Rutickii [2], Lindenstrauss and Tzafriri [3], Rao and Ren [6], Chen [1].

The Simonenko indices of an \( \mathcal{N} \)-function \( \Phi \) are defined as

\[
A_\Phi = \inf_{t > 0} \frac{t \phi(t)}{\Phi(t)}, \quad B_\Phi = \sup_{t > 0} \frac{t \phi(t)}{\Phi(t)}.
\]

Simonenko introduced these indices in [9] and [8], and we can find a detailed description in Maligranda [4].

Clearly, \( 1 \leq A_\Phi \leq B_\Phi \leq \infty \).

**Proposition 1.1.** Let \( \Phi \) be an \( \mathcal{N} \)-function. Then \( \Phi \in \nabla_2 \iff 1 < A_\Phi \); \( \Phi \in \Delta_2 \iff B_\Phi < \infty \).

The proof of the proposition can be found in Krasnosel’skii and Rutickii [2, p. 24–26].

**Lemma 1.2.** Let \( \Phi \) and \( \Psi \) be a pair of complementary \( \mathcal{N} \)-functions. Then

\[
\frac{1}{A_\Phi} + \frac{1}{B_\Psi} = 1.
\]

The proof of Lemma 1.2 can be found in Simonenko [9] or Rao & Ren [6].

The next lemma can be found in [1], [10] or [5].

**Lemma 1.3.** Let \( \Phi(u) = \int_0^{|u|} \phi(t) \, dt \) and \( \Psi(v) = \int_0^{|v|} \psi(s) \, ds \) be a pair of complementary \( \mathcal{N} \)-functions. We denote

\[
k_x^* = \inf\{k > 0 : \rho_\Psi[\phi(k|x|)] \geq 1\}, \quad k_x^{**} = \sup\{k > 0 : \rho_\Psi[\phi(k|x|)] \leq 1\}.
\]

Then \( k \in [k_x^*, k_x^{**}] \) if and only if

\[
||x||_\Phi = \frac{1}{k}[1 + \rho_\Phi(kx)].
\]

2. Main results

Y. Yan estimated the two parameters \( Q_\Phi \) and \( q_\Phi \) in the Orlicz sequence space \( l^\Phi \), and got the following result (see [11], [7] or [13]).
Proposition 2.1. Let $\Phi$ and $\Psi$ be a pair of complementary $N$-functions. Then

$$\frac{b^*_\Phi}{b^*_\Phi - 1} \leq q_\Phi \leq \frac{a^*_\Phi}{a^*_\Phi - 1},$$

where

$$a^*_\Phi = \inf \left\{ \frac{t \phi(t)}{\Phi(t)} : 0 < t \leq \psi[\Psi^{-1}(1)] \right\},$$

$$b^*_\Phi = \sup \left\{ \frac{t \phi(t)}{\Phi(t)} : 0 < t \leq \psi[\Psi^{-1}(1)] \right\}.$$

The upper estimate in (3) can also be found in [12]. Now we establish a similar inequality in the Orlicz function space with Orlicz norm. Firstly, we have

Theorem 2.1. Let $\Phi, \Psi$ be a pair of complementary $N$-functions. For $L^\Phi[0, \infty)$, we denote

$$Q_\Phi = \sup_{\|x\|_\Phi = 1} k^{**}_x = \sup_{\|x\|_\Phi = 1} \left\{ k > 0 : \|x\|_\Phi = \frac{1}{k}(1 + \rho_\Phi(kx)) \right\},$$

$$q_\Phi = \inf_{\|x\|_\Phi = 1} k^*_x = \inf_{\|x\|_\Phi = 1} \left\{ k > 0 : \|x\|_\Phi = \frac{1}{k}(1 + \rho_\Phi(kx)) \right\}.$$

Then

$$A_\Psi = \frac{B_\Phi}{B_\Phi - 1} \leq q_\Phi \leq Q_\Phi \leq \frac{A_\Phi}{A_\Phi - 1} = B_\Psi,$$

where $A_\Phi, B_\Phi, A_\Psi$ and $B_\Psi$ are defined by (1).

Proof: The left and right equations in (4) follow from Lemma 1.2. Now we prove

$$q_\Phi \geq \frac{B_\Phi}{B_\Phi - 1}.$$

For $\Phi \notin \Delta_2$, by Proposition 1.1, we have $B_\Phi = \infty$ or $A_\Psi = 1$. The result is obvious.

For $\Phi \in \Delta_2$, we only prove that for every $x \in L^\Phi[0, \infty)$ which satisfies $\|x\|_\Phi = 1$, we have $k^*_x \geq \frac{B_\Phi}{B_\Phi - 1}$. Firstly, we have $\rho_\Psi(\phi(k^*_x|x(t)|)) \geq 1$. In fact, if $\Phi \in \Delta_2$, then $\rho_\Phi((k^*_x + 1)x] < \infty$. So

$$\rho_\Psi(\phi((k^*_x + 1)|x(t)|)) \leq \rho_\Psi(\phi((k^*_x + 1)|x(t)|) + \rho_\Phi((k^*_x + 1)|x(t)|))$$

$$= \int_G (k^*_x + 1)|x(t)| \cdot \phi((k^*_x + 1)|x(t)|) \, dt$$

$$\leq B_\Phi \rho_\Phi((k^*_x + 1)|x(t)|) < \infty.$$
Choose \( k^*_x < k_n < k^*_x + 1 \) such that \( k_n \downarrow k^*_x \). By the right continuity of \( \phi \) and Lebesgue dominated convergence theorem, we have

\[
\rho_\Psi(\phi(k^*_x|x(t)|)) = \lim_{n \to \infty} \rho_\Psi(\phi(k_n|x(t)|)) \geq 1.
\]

For every \( x \in L^\Phi[0, \infty) \) which satisfies \( \|x\|_\Phi = 1 \), we have

\[
1 + \rho_\Phi(k^*_x x) \leq \rho_\Psi(\phi(k^*_x|x(t)|)) + \rho_\Phi(k^*_x|x(t)|) \\
= \int_{[0,\infty)} \Psi\{\phi[(k^*_x|x(t)|)]\} \, dt + \int_{[0,\infty)} \Phi(k^*_x|x(t)|) \, dt \\
= \int_{[0,\infty)} k^*_x|x(t)|\phi(k^*_x|x(t)|) \, dt \\
\leq B_\Phi \int_{[0,\infty)} \Phi(k^*_x|x(t)|) \, dt = B_\Phi \rho_\Phi(k^*_x x).
\]

This implies

\[
(6) \quad \rho_\Phi(k^*_x x) \geq \frac{1}{B_\Phi - 1}.
\]

By Lemma 1.3, we get

\[
1 = \|x\|_\Phi = \frac{1}{k^*_x} \{1 + \rho_\Phi(k^*_x x)\}.
\]

So \( \rho_\Phi(k^*_x x) = k^*_x - 1 \). By (6)

\[
k^*_x \geq \frac{B_\Phi}{B_\Phi - 1}.
\]

Next, we prove

\[
(7) \quad Q_\Phi \leq \frac{A_\Phi}{A_\Phi - 1}.
\]

If \( \Phi \notin \nabla_2 \), then \( A_\Phi = 1 \) or \( B_\Psi = \infty \). The result is obvious.

If \( \Phi \in \nabla_2 \), then \( A_\Phi > 1 \). For every \( x \in L^\Phi[0, \infty) \) which satisfies \( \|x\|_\Phi = 1 \), and for any \( k \in [k^*_x, k^*_x] \), we have

\[
1 = \|x\|_\Phi = \frac{1}{k}[1 + \rho_\Phi(k x)].
\]

For any \( 0 < \epsilon < 1 < k \), we have

\[
(8) \quad 1 = \|x\|_\Phi = \inf_{t > 0} \frac{1}{t}[1 + \rho_\Phi(t x)] \leq \frac{1}{k - \epsilon}[1 + \rho_\Phi((k - \epsilon)x)].
\]
By the definition of \( k^{**}_x \) and \( k - \varepsilon < k^{**}_x \), we have

\[
1 + \rho_{\Phi}[(k - \varepsilon)x] \geq \rho_{\Psi}\{\phi[(k - \varepsilon)x]\} + \rho_{\Phi}[(k - \varepsilon)x]
\]

\[
= \int_{[0, \infty)} (k - \varepsilon)x(t)\phi[(k - \varepsilon)x(t)] \, dt 
\]

\[
\geq A_{\Phi} \rho_{\Phi}((k - \varepsilon)x(t)).
\]

Therefore by (8) and (9), we have

\[
1 \geq (A_{\Phi} - 1) \rho_{\Phi}((k - \varepsilon)x(t)) \geq (A_{\Phi} - 1)(k - \varepsilon - 1)
\]

or

\[
k - \varepsilon \leq \frac{A_{\Phi}}{A_{\Phi} - 1}.
\]

Since \( \varepsilon \) is arbitrary, we have

\[
k \leq \frac{A_{\Phi}}{A_{\Phi} - 1}.
\]

This implies (7) since \( x \) and \( k \) are arbitrary. \( \square \)

**Corollary 2.1.** (i) If \( \Phi \in \nabla_2 \), then \( Q_{\Phi} < \infty \); (ii) If \( \Phi \in \triangle_2 \), then \( q_{\Phi} > 1 \).

For \( 0 \neq x \in L_{\Phi}[0, 1] \), we still denote

\[
k^{*}_x = \inf\{k > 0 : \rho_{\Psi}[\phi(kx)] \geq 1\},
\]

\[
k^{**}_x = \sup\{k > 0 : \rho_{\Psi}[\phi(kx)] \leq 1\},
\]

\[
Q_{\Phi} = \sup_{\|x\|_{\Phi} = 1} k^{**}_x = \sup_{\|x\|_{\Phi} = 1} \left\{ k > 0 : \|x\|_{\Phi} = \frac{1}{k}(1 + \rho_{\Phi}(kx)) \right\},
\]

\[
q_{\Phi} = \inf_{\|x\|_{\Phi} = 1} k^{*}_x = \inf_{\|x\|_{\Phi} = 1} \left\{ k > 0 : \|x\|_{\Phi} = \frac{1}{k}(1 + \rho_{\Phi}(kx)) \right\}.
\]

Let \( \varepsilon_0 = \min\{\frac{1}{2\phi(1)}, 1\} \). Denote

\[
A_{\Phi}^* = \inf\left\{ \frac{t\phi(t)}{\Phi(t)} : t \in [\varepsilon_0, \infty) \right\},
\]

\[
B_{\Phi}^* = \sup\left\{ \frac{t\phi(t)}{\Phi(t)} : t \in [\varepsilon_0, \infty) \right\}.
\]

Obviously, \( \varepsilon_0 \phi(\varepsilon_0) \leq \frac{\phi(\varepsilon_0)}{2\phi(1)} \leq \frac{1}{2} \).
Theorem 2.2. If $\Phi, \Psi$ is a pair of complementary $N$-functions, then

$$\frac{B^*_\Phi - \varepsilon_0 \phi(\varepsilon_0)}{B^*_\Phi - 1} \leq q_\Phi \leq \frac{A^*_\Phi + A^*_\Phi \Phi(\varepsilon_0)}{A^*_\Phi - 1}.$$ 

Proof: Firstly, we prove $q_\Phi \geq \frac{B^*_\Phi - \varepsilon_0 \phi(\varepsilon_0)}{B^*_\Phi - 1}$. If $\Phi \notin \triangle_2(\infty)$, then $B^*_\Phi = \infty$, and the result is clear. If $\Phi \in \triangle_2(\infty)$, then $B^*_\Phi < \infty$. By the proof of Theorem 2.1, for $x \in L^\Phi[0,1]$ with $\|x\|_\Phi = 1$, we have $\rho_\Psi(\phi(k^*_x x)) \geq 1$. So

$$1 + \rho_\Phi(k^*_x x) \leq \rho_\Psi(\phi(k^*_x x)) + \rho_\Phi(k^*_x x)$$

$$= \int_{[0,1]} k^*_x |x(t)|\phi(k^*_x |x(t)|) \, dt$$

$$\leq \int_{G_1 = \{t : k^*_x |x(t)| < \varepsilon_0\}} \varepsilon_0 \phi(\varepsilon_0) \, dt + \int_{G \setminus G_1} k^*_x |x(t)|\phi(k^*_x |x(t)|) \, dt$$

$$< \varepsilon_0 \phi(\varepsilon_0) + B^*_\Phi \rho_\Phi(k^*_x x).$$

Therefore

$$1 - \varepsilon_0 \phi(\varepsilon_0) \leq (B^*_\Phi - 1) \rho_\Phi(k^*_x x).$$

Noting that $\rho_\Phi(k^*_x x) = k^*_x - 1$, we have

$$1 - \varepsilon_0 \phi(\varepsilon_0) \leq k^*_x - 1,$$

i.e.

$$k^*_x \geq \frac{B^*_\Phi - \varepsilon_0 \phi(\varepsilon_0)}{B^*_\Phi - 1}.$$ 

Since $x$ is arbitrary,

$$q_\Phi \geq \frac{B^*_\Phi - \varepsilon_0 \phi(\varepsilon_0)}{B^*_\Phi - 1}.$$ 

Next we prove $Q_\Phi \leq \frac{A^*_\Phi (1 + \Phi(\varepsilon_0))}{A^*_\Phi - 1}$. If $\Phi \notin \nabla_2(\infty)$, the result is obvious. If $\Phi \in \nabla_2(\infty)$, then $\forall x \in S(L^\Phi[0,1])$, $\forall k \in [k^*_x, k^{**}_x]$ and $0 < \varepsilon < 1$, we get

$$1 + \rho_\Phi[(k - \varepsilon)x] \geq \rho_\Psi\{\phi[(k - \varepsilon)|x|]\} + \rho_\Phi[(k - \varepsilon)x]$$

$$= \int_{[0,1]} (k - \varepsilon)|x(t)|\phi((k - \varepsilon)|x(t)|) \, dt$$

$$\geq \int_{\{t \in [0,1] : (k - \varepsilon)|x(t)| \geq \varepsilon_0\}} (k - \varepsilon)|x(t)|\phi((k - \varepsilon)|x(t)|) \, dt$$

$$\geq A^*_\Phi \int_{\{(k - \varepsilon)|x(t)| \geq \varepsilon_0\}} \Phi((k - \varepsilon)|x(t)|) \, dt$$

$$\geq A^*_\Phi \{\rho_\Phi[(k - \varepsilon)x(t)] - \int_{\{t \in [0,1] : (k - \varepsilon)|x(t)| \varepsilon_0\}} \Phi((k - \varepsilon)|x(t)|) \, dt\}$$

$$\geq A^*_\Phi \{\rho_\Phi[(k - \varepsilon)x(t)] - \Phi(\varepsilon_0)\}.$$
So
\[ 1 + A_\Phi^* \Phi(\varepsilon_0) \geq (A_\Phi^* - 1) \rho((k - \varepsilon)x(t)) \geq (A_\Phi^* - 1)(k - \varepsilon - 1), \]
i.e.
\[ k \leq \frac{A_\Phi^*[1 + \Phi(\varepsilon_0)]}{A_\Phi^* - 1} + \varepsilon. \]
Therefore,
\[ k \leq \frac{A_\Phi^*[1 + \Phi(\varepsilon_0)]}{A_\Phi^* - 1}. \]
Since \( x \in S(L^\Phi[0,1]) \) is arbitrary,
\[ Q_\Phi \leq \frac{A_\Phi^*(1 + \Phi(\varepsilon_0))}{A_\Phi^* - 1}. \]

\[ \square \]

**Corollary 2.2** (S.T. Chen [1, p. 21]).

(i) If \( \Phi \in \Delta_2(\infty) \), then \( q_\Phi > 1 \).

(ii) If \( \Phi \in \nabla_2(\infty) \), then \( Q_\Phi < \infty \).

From the proof of Theorem 2.2, we know Theorem 2.2 is true for any \( 0 < \varepsilon < \varepsilon_0 \).
Letting \( \varepsilon \) to tend to 0, we get

**Corollary 2.3.** Let \( \Phi, \Psi \) be a pair of complementary \( \mathcal{N} \)-functions. Then

(10) \[ A_\Psi = \frac{B_\Phi}{B_\Phi - 1} \leq q_\Phi \leq Q_\Phi \leq \frac{A_\Phi}{A_\Phi - 1} = B_\Psi, \]

where \( A_\Phi, B_\Phi, A_\Psi \) and \( B_\Psi \) are defined by (1).

**Example 1.** For the \( \mathcal{N} \)-function \( \Phi(u) = |u|^p \), which generates \( L^p[0,\infty) \), we have \( A_\Phi = B_\Phi = p \). By Theorem 2.1 and Corollary 2.3, we have \( q_\Phi = Q_\Phi = \frac{p}{p-1} \).

**Example 2.** For the \( \mathcal{N} \)-function \( \Phi(u) = e^{\lvert u \rvert} - |u| - 1 \), we have

(11) \[ 1 \leq q_\Phi \leq Q_\Phi \leq 2. \]

Indeed, \( F_\Phi(t) = \frac{t(e^t - 1)}{e^t - t - 1} \) is increasing in \((0, +\infty)\). So \( A_\Phi = \lim_{t \to 0^+} F_\Phi(t) = 2 \) and \( B_\Phi = \lim_{t \to +\infty} F_\Phi(t) = \infty \). Therefore (11) follows from Theorem 2.1 and Corollary 2.3.

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References


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