Non-existence of some canonical constructions on connections

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Abstract. For a vector bundle functor $H : \mathcal{M}f \to \mathcal{V}B$ with the point property we prove that $H$ is product preserving if and only if for any $m$ and $n$ there is an $\mathcal{FM}_{m,n}$-natural operator $D$ transforming connections $\Gamma$ on $(m, n)$-dimensional fibered manifolds $p : Y \to M$ into connections $D(\Gamma)$ on $Hp : HY \to HM$. For a bundle functor $E : \mathcal{FM}_{m,n} \to \mathcal{FM}$ with some weak conditions we prove non-existence of $\mathcal{FM}_{m,n}$-natural operators $D$ transforming connections $\Gamma$ on $(m, n)$-dimensional fibered manifolds $Y \to M$ into connections $D(\Gamma)$ on $EY \to M$.

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0. Introduction

We recall that a (general) connection on a fibered manifold $p : Y \to M$ is a smooth section $\Gamma : Y \to J^1Y$ of the first jet prolongation of $Y$, which can be also interpreted as the lifting map (denoted by the same symbol)

$$\Gamma : Y \times_M TM \to TY.$$ 

Let $H$ be a bundle functor on the category of smooth manifolds and all smooth maps and let $\Gamma : Y \to J^1Y$ be a connection on the fibered manifold $p : Y \to M$. It is well known that if $H$ preserves products, then $\Gamma$ induces a connection $\mathcal{H}\Gamma$ on $Hp : HY \to HM$. More precisely, there is the canonical flow equivalence $THM = HTM$ and the lifting map of $\mathcal{H}\Gamma$ is of the form

$$\mathcal{H}\Gamma : HY \times_{HM} THM \to THY.$$

We recall that the connection $\mathcal{H}\Gamma$ has been constructed by I. Kolár [2] in the case of higher order velocities functors and then by J. Slovák [6] in the general case.

In the present paper we study the non-existence of natural operators $D$ lifting connections $\Gamma$ on $p : Y \to M$ into connections $D(\Gamma)$ on $Hp : HY \to HM$ for non-product preserving vector bundle functors $H : \mathcal{M}f \to \mathcal{V}B$ with the point property $H(pt) = pt$ ($pt$ is a one-point manifold). If $H$ is without the point property, then such $D$ can exist, see [1].
In Section 1, we prove that a vector bundle functor $H : \mathcal{M}f \to \mathcal{V}B$ with the
point property is product preserving if and only if for any $m$ and $n$ there is an
$\mathcal{F}M_{m,n}$-natural operator $D$ transforming connections $\Gamma$ on $(m, n)$-dimensional
fibered manifolds $p : Y \to M$ into connections $D(\Gamma)$ on $H p : HY \to HM$.

In particular, if $H = T^{(2)} = (J^2(., \mathbb{R})_0)^*$ is the second order vector tangent
bundle functor, we get negative answer to the question (formulated by I. Kolář)
about the existence of natural operators $D$ transforming connections $\Gamma$ on fibered
manifolds $p : Y \to M$ into connections $D(\Gamma)$ on $T^{(2)}p : T^{(2)}Y \to T^{(2)}M$.

In next sections, for a bundle functor $E : \mathcal{F}M_{m,n} \to \mathcal{F}M$ with some weak condition we prove the non-existence of $\mathcal{F}M_{m,n}$-natural operators $D$ transforming
connections $\Gamma$ on $(m, n)$-dimensional fibered manifolds $Y \to M$ into connections
$D(\Gamma)$ on $EY \to M$. This is a generalization of the result of [3, Proposition 45.9].

Unless otherwise specified, we use the terminology and notation from the
book [3]. All manifolds and maps are assumed to be of class $C^\infty$.

1. The case $HY \to HM$

Let $H : \mathcal{M}f \to \mathcal{V}B$ be a vector bundle functor with the point property. Let
$m, n$ be natural numbers.

Define a natural bundle $F : \mathcal{M}f_m \to \mathcal{F}M$ by

$$FM = H(M \times \mathbb{R}^n) \quad \text{and} \quad F\varphi = H(\varphi \times id_{\mathbb{R}^n})$$

for an $\mathcal{M}f_m$-object $M$ and an $\mathcal{M}f_m$-morphism $\varphi$.

If $p_M : M \times \mathbb{R}^n \to M$ is the obvious projection, then $p_M$ is a surjective sub-
mersion, so is $H(p_M)$ ([3]) and hence $GM = ker H(p_M)$ is a regular submanifold.
Define a natural bundle $G : \mathcal{M}f_m \to \mathcal{F}M$ by

$$GM = ker H(p_M) \quad \text{and} \quad G\varphi = \text{the restriction of} \ H(\varphi \times id_{\mathbb{R}^n})$$

for an $\mathcal{M}f_m$-object $M$ and an $\mathcal{M}f_m$-morphism $\varphi$.

We have an $\mathcal{M}f_m$-natural equivalence of natural bundles $GM \times_M HM \cong FM$
given by

$$\Phi(\omega, \tilde{\omega}) = \omega + H(i_{M}^{y})(\tilde{\omega}),$$

where $\omega \in H_{(x, y)}(M \times \mathbb{R}^n) \cap G_x M$, $\tilde{\omega} \in H_y M$, $(x, y) \in M \times \mathbb{R}^n$, $+$ is the sum
in the vector space $H_{(x, y)}(M \times \mathbb{R}^n)$ and $i_{M}^{y} = (id_{M}, y) : M \to M \times \mathbb{R}^n$. The
inverse isomorphism is given by $\Phi^{-1}(\omega) = (\omega - H(i_{M}^{y}\circ p_M)(\omega), H(p_M)(\omega))$, where
$\omega \in H_{(x, y)}(M \times \mathbb{R}^n)$, $(x, y) \in M \times \mathbb{R}^n$.

**Proposition 1.** The natural bundle $G$ is of order 0 if and only if $H(\mathbb{R}^{m+n}) =
H(\mathbb{R}^{m}) \times H(\mathbb{R}^{n})$ modulo a diffeomorphism, i.e. iff $H$ preserves product in dimen-
sion $m$ and $n$.

**Proof:** If the equality holds, then $G_0\mathbb{R}^m = H(\mathbb{R}^n)$ and then $G$ is of order 0. If
$G$ is of order 0, then $G_0(\mathbb{R}^m) = H(t id_{\mathbb{R}^m} \times id_{\mathbb{R}^n})(G_0(\mathbb{R}^m))$ for all $t \neq 0$. Putting
Proposition 2. If $G$ is not of order 0, then there is no $\mathcal{FM}_{m,n}$-natural operator $D$ transforming connections $\Gamma$ on $(m, n)$-dimensional fibered manifolds $p : Y \to M$ into connections $D(\Gamma)$ on $Hp : HY \to HM$.

Proof: Suppose that we have an $\mathcal{FM}_{m,n}$-natural operator $D$ lifting connections $\Gamma$ on $p : Y \to M$ into connections $D(\Gamma) : HY \times_{HM} THM \to THY$ on $Hp : HY \to HM$. Then we can define a natural operator $A : T_{Mf_m} \rightsquigarrow TG$ by

$$A(X)_\omega = T\operatorname{pr}_1(D(\Gamma_M)(\omega, \mathcal{H}X_{0_x})),$$

where $\omega \in G_x M$, $x \in M$, $0_x = 0 \in H_x M$, $\mathcal{H}X$ is the flow lifting of $X$ to $HM$, $\operatorname{pr}_1 : FM \equiv GM \times_M HM \to GM$ is the obvious projection and $\Gamma_M$ is the trivial connection on the trivial bundle $p_M : M \times \mathbb{R}^n \to M$.

Since $\mathcal{H}X_{0_x}$ depends only on $X_x$, $A$ is of order 0.

Since $D(\Gamma_M)$ is a lifting transformation, $A(X)$ covers $X$. Hence

$$A(X) = \mathcal{G}X + \mathcal{V}(X),$$

where $\mathcal{G}X$ is the flow lifting of $X$ to $GM$ and $\mathcal{V}(X)$ is a vertical type operator $T_{Mf_m} \rightsquigarrow TG$. Clearly, $\mathcal{G}$ is of order $\operatorname{ord}(G) \geq 1$ and not of order $\operatorname{ord}(G) - 1$ and $\mathcal{V}$ is of order $\operatorname{ord}(G) - 1$, see Lemma 1 in [5] (or Appendix of the present paper). So, $A$ is not of order 0, which is a contradiction. \qed

Thus we have proved the following general fact.

Theorem 1. A vector bundle functor $H : \mathcal{M}f \to \mathcal{VB}$ with the point property is product preserving if and only if for any $m$ and $n$ there is an $\mathcal{FM}_{m,n}$-natural operator $D$ transforming connections $\Gamma$ on $(m, n)$-dimensional fibered manifolds $p : Y \to M$ into connections $D(\Gamma)$ on $Hp : HY \to HM$.

Any product-preserving vector bundle functor $H : \mathcal{M}f \to \mathcal{VB}$ is equivalent to some vector bundle functor $T^{[s]} : \mathcal{M}f \to \mathcal{VB}$, $T^{[s]} M = TM \otimes \mathbb{R}^s$, $T^{[s]}f = Tf \otimes \text{id}_{\mathbb{R}^s}$, see [3]. So, we have the following classification theorem.

Theorem 1’. Up to natural equivalence the $T^{[s]}$ for $s = 0, 1, 2, \ldots$ are all vector bundle functors $H : \mathcal{M}f \to \mathcal{VB}$ with the point property such that for any $m$ and $n$ there is an $\mathcal{FM}_{m,n}$-natural operator $D$ transforming connections $\Gamma$ on $(m, n)$-dimensional fibered manifolds $p : Y \to M$ into connections $D(\Gamma)$ on $Hp : HY \to HM$.

Open problem: Our conjecture is that a bundle functor $H : \mathcal{M}f \to \mathcal{FM}$ with the point property is product preserving if and only if for any $m$ and $n$ there is an $\mathcal{FM}_{m,n}$-natural operator $D$ transforming connections $\Gamma$ on $(m, n)$-dimensional fibered manifolds $p : Y \to M$ into connections $D(\Gamma)$ on $Hp : HY \to HM$. 

\[t \to 0\] we obtain $G_0(\mathbb{R}^m) = H(\{0\} \times \mathbb{R}^n) = H(\mathbb{R}^n)$. Then $\dim(H_{(0,0)}(\mathbb{R}^m \times \mathbb{R}^n)) = \dim(H_0(\mathbb{R}^m)) + \dim(H_0(\mathbb{R}^n))$, and Proposition 38.14 in [3] completes the proof. \[\square\]
2. The case \( EY \to M \)

**Theorem 2.** Let \( E : \mathcal{FM}_{m,n} \to \mathcal{FM} \) be a bundle functor such that the corresponding natural bundle \( \tilde{E} : \mathcal{M}_f m \to \mathcal{FM}, \tilde{E} M = E(M \times \mathbb{R}^n), \tilde{E} \phi = E(\phi \times \text{id}_{\mathbb{R}^n}) \) is not of order 0. Then there is no \( \mathcal{FM}_{m,n} \)-natural operator \( D \) transforming connections \( \Gamma \) on \((m, n)\)-dimensional fibered manifolds \( Y \to M \) into connections \( D(\Gamma) \) on \( EY \to M \).

**Proof:** Suppose we have such an \( \mathcal{FM}_{m,n} \)-natural operator \( D(\Gamma) \). Then we can define a natural operator \( A : T_{\mathcal{M}_f m} \to T\tilde{E} \) by

\[
A(X)\omega = D(\Gamma_M)(\omega, X_x),
\]

where \( \omega \in \tilde{E}_x M, x \in M, X \) is a vector field on \( M \) and \( \Gamma_M \) is the trivial connection on the trivial bundle \( p_M : M \times \mathbb{R}^n \to M \).

Then \( A \) is of order 0 and \( A(X) \) covers \( X \).

This is a contradiction by the same arguments as at the end of the proof of Proposition 2. \( \square \)

For \( E = J^1 \) we reobtain Proposition 45.9 from [3] without the order assumption.

**Remark 2.** The existence of a connection \( V^F \Gamma \) on a vertical bundle \( V^F Y \to M \) canonically depending on a connection \( \Gamma \) on \( Y \to M \) ([4]) shows that the assumption of Theorem 2 is essential.

3. The case \( EY \to Y \)

**Theorem 3.** Let \( E : \mathcal{FM}_{m,n} \to \mathcal{FM} \) be a bundle functor such that the corresponding natural bundle \( \tilde{E} : \mathcal{M}_f m \to \mathcal{FM}, \tilde{E} M = E(M \times \mathbb{R}^n), \tilde{E} \phi = E(\phi \times \text{id}_{\mathbb{R}^n}) \) is not of order 0. Then there is no \( \mathcal{FM}_{m,n} \)-natural operator \( D \) transforming connections \( \Gamma \) on \((m, n)\)-dimensional fibered manifolds \( Y \to M \) into connections \( D(\Gamma) \) on \( EY \to Y \).

**Proof:** Suppose that such \( D(\Gamma) \) exists. Composing \( D(\Gamma) \) with \( \Gamma \) we obtain a connection on \( EY \to M \) canonically dependent on \( \Gamma \). This contradicts Theorem 2. \( \square \)

We remark that in [5] we proved the following theorem.

**Theorem 4** ([5]). Let \( E : \mathcal{FM}_{m,n} \to \mathcal{FM} \) be a bundle functor such that the corresponding natural bundle \( \overline{E} : \mathcal{M}_f n \to \mathcal{FM}, \overline{E} N = E(\mathbb{R}^m \times N), \overline{E} \phi = E(\text{id}_{\mathbb{R}^m} \times \phi) \) is not of order 0. Then there is no \( \mathcal{FM}_{m,n} \)-natural operator \( D \) transforming connections \( \Gamma \) on \((m, n)\)-dimensional fibered manifolds \( Y \to M \) into connections \( D(\Gamma) \) on \( EY \to Y \).
4. Appendix

Because Lemma 1 from [5] is essential in the proof of Proposition 2, we cite this lemma with the proof here for the reader’s convenience.

Lemma 1 ([5]). Let $G : \mathcal{M}f_n \to \mathcal{F} \mathcal{M}$ be a natural bundle of order $r \geq 1$. Then any natural operator $\mathcal{V} : T\mathcal{M}f_n \to TG$ of vertical type is of order $r - 1$.

Proof: ([5]) Let $X_1, X_2 \in \mathcal{X}(N)$ be two vector fields with $j^r_x^{-1}(X_1) = j^r_x^{-1}(X_2)$, $x \in N$. Let $w \in G_x N$. Because of the regularity of $\mathcal{V}$ we can assume that $X_1(x) \neq 0$. There is an $x$-preserving local diffeomorphism $\varphi : N \to N$ such that $j^r_x \varphi = \text{id}$ and $\varphi_* X_1 = X_2$ near $x$, see [3]. Then $\mathcal{V}(X_2)(w) = \mathcal{V}(\varphi_* X_1)(w) = TG_x(\varphi) \circ \mathcal{V}(X_1) \circ G_x(\varphi^{-1})(w) = \mathcal{V}(X_1)(w)$ since $G_x(\varphi) = \text{id}$ as $G$ is of order $r$ and $j^r_x \varphi = \text{id}$. □

References


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