Perimeter preservers of nonnegative integer matrices

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Abstract. We investigate the perimeter of nonnegative integer matrices. We also characterize the linear operators which preserve the rank and perimeter of nonnegative integer matrices. That is, a linear operator $T$ preserves the rank and perimeter of rank-1 matrices if and only if it has the form $T(A) = P(A \circ B)Q$, or $T(A) = P(A^t \circ B)Q$ with appropriate permutation matrices $P$ and $Q$ and positive integer matrix $B$, where $\circ$ denotes Hadamard product.

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1. Introduction and preliminaries

Nonnegative integer matrices are combinatorially interesting matrices. So it has been a subject of many research works (see [5]). In [1], Beasley and Pullman defined the perimeter of a Boolean rank-1 matrix in order to characterize the linear operators that preserve Boolean rank. In this paper, we consider the nonnegative integer matrices of rank-1 and their perimeters. We also characterize the linear operators that preserve the rank and perimeter of the rank-1 matrices over nonnegative integers.

Let $\mathbb{Z}_+$ be a semiring of nonnegative integers and let $M_{m,n}(\mathbb{Z}_+)$ denote the set of all $m \times n$ matrices with entries in $\mathbb{Z}_+$. The rank or factor rank [2], $r(A)$, of a nonzero matrix $A \in M_{m,n}(\mathbb{Z}_+)$ is defined as the least integer $k$ for which there exist $m \times k$ and $k \times n$ matrices $B$ and $C$ with $A = BC$. The rank of a zero matrix is zero. If $A \in M_{m,n}(\mathbb{Z}_+)$ has rank 1, there exist nonzero vectors $u \in M_{m,1}(\mathbb{Z}_+)$ and $v \in M_{n,1}(\mathbb{Z}_+)$ such that $A = uv^t$. The perimeter [1] of this rank 1 matrix $A$, $p(A)$ is defined as $|u| + |v|$ for arbitrary factorization $A = uv^t$, where $|u|$ denotes the number of nonzero entries in $u$. It is clear that the perimeter of a rank 1 matrix is uniquely determined by the given matrix. Let $A \circ B$ denote the Hadamard (or Schur) product, the $(i, j)$ entry of $A \circ B$ is $a_{ij}b_{ij}$.

A matrix in $M_{m,n}(\mathbb{Z}_+)$ is called a cell [3] if it has exactly one nonzero entry, that being a 1. We denote the cell whose nonzero entry is in the $(i, j)$th position by $E_{ij}$. Let $E_{m,n} = \{E_{ij} | 1 \leq i \leq m, 1 \leq j \leq n\}$. For $A = [a_{ij}]$ in $M_{m,n}(\mathbb{Z}_+)$, we define $A^* = [a_{ij}^*]$ to be the $m \times n$ $(0,1)$-matrix whose $(i, j)$th entry is 1 if and only if $a_{ij} > 0$. 
It follows from the definition that \( p(A) = p(A^*) \) and \((AB)^* = A^*B^*\), \((B + C)^* = B^* + B C^*\), where \(1 + B 1 = 1\) is Boolean arithmetic, for all \(A \in \mathcal{M}_{m,n}(\mathbb{Z}+)\) and all \(B, C \in \mathcal{M}_{n,r}(\mathbb{Z}+)\).

If \(A\) and \(B\) are in \(\mathcal{M}_{m,n}(\mathbb{Z}+)\), we say that \(A\) dominates \(B\) (written \(B \leq A\) or \(A \geq B\)) if \(a_{ij} = 0\) implies \(b_{ij} = 0\) for all \(i,j\) ([4]). Then we can obtain the fact that \(A \geq B\) if and only if \((A + B)^* = A^*\) for any \(m \times n\) matrices \(A\) and \(B\).

2. Perimeter preservers

A mapping \(T : \mathcal{M}_{m,n}(\mathbb{Z}+) \rightarrow \mathcal{M}_{m,n}(\mathbb{Z}+)\) is called a linear operator if \(T\) satisfies

\[ T(\alpha A + \beta B) = \alpha T(A) + \beta T(B) \]

for all \(A, B \in \mathcal{M}_{m,n}(\mathbb{Z}+)\) and for all \(\alpha, \beta \in \mathbb{Z}+\).

In this section, we will characterize the linear operators that preserve both the rank and the perimeter of every rank-1 matrix in \(\mathcal{M}_{m,n}(\mathbb{Z}+)\).

Suppose \(T\) is a linear operator on \(\mathcal{M}_{m,n}(\mathbb{Z}+)\). Then

1. \(T\) is a \((P, Q, B)\)-operator if there exist permutation matrices \(P \in \mathcal{M}_{m,m}(\mathbb{Z}+)\), \(Q \in \mathcal{M}_{n,n}(\mathbb{Z}+)\) and a positive matrix \(B \in \mathcal{M}_{m,n}(\mathbb{Z}+)\) with \(r(B) = 1\) such that \(T(A) = P(A \circ B)Q\) for all \(A\) in \(\mathcal{M}_{m,n}(\mathbb{Z}+)\), or \(m = n\) and \(T(A) = P(A^t \circ B)Q\) for all \(A\) in \(\mathcal{M}_{m,n}(\mathbb{Z}+)\);
2. \(T\) preserves rank 1 if \(r(T(A)) = 1\) whenever \(r(A) = 1\) for all \(A \in \mathcal{M}_{m,n}(\mathbb{Z}+)\);
3. \(T\) preserves perimeter \(k\) of rank-1 matrices if \(p(T(A)) = k\) whenever \(p(A) = k\) for all \(A \in \mathcal{M}_{m,n}(\mathbb{Z}+)\) with \(r(A) = 1\).

**Theorem 2.1.** If \(T\) is a \((P, Q, B)\)-operator on \(\mathcal{M}_{m,n}(\mathbb{Z}+)\), then \(T\) preserves both rank and perimeter of every rank-1 matrix.

**Proof:** Since the operators Hadamard product, transpose and permutational equivalence preserve the rank and perimeter of every rank-1 matrix, the theorem follows. \(\square\)

We note that an \(m \times n\) matrix has perimeter 2 if and only if it is a positive integer multiple of a cell. We say that \(A\) is a row (column) matrix if \(A\) has nonzero entries only in one row (column, respectively). Thus we have the following lemma:

**Lemma 2.2.** Let \(T\) be a linear operator on \(\mathcal{M}_{m,n}(\mathbb{Z}+)\). If \(T\) preserves rank 1 and perimeter 2 of every rank-1 matrix, then the following statements hold:

1. there exist positive integers \(u_{ij}, i = 1, \ldots, m, j = 1, \ldots, n\), and a mapping \(f : \mathbb{E}_{m,n} \rightarrow \mathbb{E}_{m,n}\) such that for \(A = [a_{ij}]\), \(T(A) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} u_{ij} f(E_{ij})\);
2. \(T\) maps a row (column) matrix to a row (column) matrix or if \(m = n\), a row (column) matrix to a column (row) matrix.
Proof: (1) Since $T$ preserves perimeter 2, $T$ maps a cell into a positive integer multiple of a cell.

(2) If not, then there exist two distinct cells $E_{ij}, E_{ih}$ in some $i$th row such that $T(E_{ij})$ and $T(E_{ih})$ lie in two different rows and different columns. Then the rank of $E_{ij} + E_{ih}$ is 1 but that of $T(E_{ij} + E_{ih}) = T(E_{ij}) + T(E_{ih})$ is 2. Therefore $T$ does not preserve rank 1, a contradiction.

An example follows of a linear operator that preserves rank 1 and perimeter 2 of a rank-1 matrix, but the operator does not preserve perimeter 3 and is not a $(P, Q, B)$-operator.

Example 2.3. Let $T : \mathcal{M}_{2,2}(\mathbb{Z}_+) \rightarrow \mathcal{M}_{2,2}(\mathbb{Z}_+)$ be defined by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a + b + c + d) \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. $$

It is easy to verify that $T$ is a linear operator which preserves rank 1 and perimeter 2. But $T$ does not preserve perimeter 3 and hence it is not a $(P, Q, B)$-operator.

Lemma 2.4. Let $T$ be a linear operator on $\mathcal{M}_{m,n}(\mathbb{Z}_+)$. Suppose that $T$ preserves rank 1 and perimeters 2 and $p \geq 3$ of every rank-1 matrix. Then

(1) $T$ maps two distinct cells in a row (or column) into positive multiples of two distinct cells in a row or in a column;

(2) for the case $m = n$, if $T$ maps some $R_i$ into a row (column) matrix then $T$ maps every row matrix into a row (column) matrix, and if $T$ maps some $C_j$ into a row (column) matrix then $T$ maps every column matrix into a row (column) matrix.

Proof: (1) Suppose $T(E_{ij}) = \alpha E_{rl}$ and $T(E_{ih}) = \beta E_{rl}$ for some cells $E_{ij} \neq E_{ih}$ and some positive integers $\alpha, \beta \in \mathbb{Z}_+$. Then $T$ maps the $i$th row of a matrix $A$ into $r$th row or $l$th column by Lemma 2.2. Without loss of generality, we assume the former. Thus for any rank-1 matrix $A$ with perimeter $p \geq 3$ which dominates $E_{ij} + E_{ih}$, we can show that $T(A)$ has perimeter at most $p - 1$, a contradiction. Thus $T$ maps two distinct cells in a row into two distinct cells in a row or in a column.
(2) If not, then there exist rows $R_i$ and $R_j$ such that $T^*(R_i) \subseteq R_r$ and $T^*(R_j) \subseteq C_s$ for some $r, s$. Consider a rank-1 matrix $D = E_{ip} + E_{iq} + E_{jp} + E_{jq}$ with $p \neq q$. Then we have

$$T(D) = T(E_{ip} + E_{iq}) + T(E_{jp} + E_{jq}) = (\alpha_1 E_{rp'} + \alpha_2 E_{rq'}) + (\beta_1 E_{p'q} + \beta_2 E_{q'r})$$

for some $p' \neq q'$ and $p'' \neq q''$ and some positive integers $\alpha_i, \beta_i \in \mathbb{Z}_+$. Therefore $r(T(D)) \neq 1$ and $T$ does not preserve rank 1, a contradiction. Hence $T$ maps each row of $A$ into a row (or a column) of $T(A)$. Similarly, $T$ maps each column of $A$ into a column (or a row) of $T(A)$. □

Now we have an interesting example:

**Example 2.5.** Consider a linear operator $T$ on $\mathcal{M}_{m,n}(\mathbb{Z}_+)$ with $m \geq 3$ and $n \geq 4$ such that

$$T(A) = B = [b_{ij}]$$

where $A = [a_{ij}]$ in $\mathcal{M}_{m,n}(\mathbb{Z}_+)$, $b_{ij} = 0$ if $i \geq 2$ and $b_{1j} = \sum_{i=1}^{m} a_{ir}$ with $r \equiv i + (j - 1) \pmod{n}$ and $1 \leq r \leq n$. Then $T$ maps each row and each column into the first row with some positive integer multiplication. And $T$ preserves both rank and perimeters $2, 3$ and $n + 1$ of rank-1 matrices. But $T$ does not preserve perimeters $k$ ($k \geq 4$ and $k \neq n + 1$) of rank-1 matrices: For if $4 \leq k \leq n$, then we can choose a $2 \times (k - 2)$ submatrix with perimeter $k$ which is mapped to distinct $k$ positions in the first row of $B$ under $T$. Then this $1 \times k$ submatrix has perimeter $k + 1$. Therefore $T$ does not preserve perimeter $k$ of rank-1 matrices. □

**Lemma 2.6.** Let $T$ be a linear operator defined by

$$T(A) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} u_{ij} f(E_{ij})$$

for some function $f : \mathbb{E}_{m,n} \to \mathbb{E}_{m,n}$ and for some positive integers $u_{ij}$, $i = 1, \ldots, m$, $j = 1, \ldots, n$. If $T$ preserves both rank and perimeters $2$ and $k$ ($k \geq 4, k \neq n + 1$) of rank-1 matrices, then the corresponding map $f$ is a bijection on $\mathbb{E}_{m,n}$.

**Proof:** By Lemma 2.2, $T(E_{ij}) = b_{ij} E_{rl}$ for some $E_{rl} \in \mathbb{E}_{m,n}$ and some positive integer $b_{ij} \in \mathbb{Z}_+$. Without loss of generality, we may assume that $T$ maps the $i$th row of a matrix into the $r$th row with positive integer multiplication. Suppose $f(E_{ij}) = f(E_{pq})$ for some distinct pairs $E_{ij}, E_{pq} \in \mathbb{E}_{m,n}$. Then we have $T(E_{ij}) = b_{ij} E_{rl}$ and $T(E_{pq}) = c_{pq} E_{rl}$ for some positive integers $b_{ij}, c_{pq} \in \mathbb{Z}_+$. If $i = p$ or $j = q$, then we have contradictions by Lemma 2.4. So let $i \neq p$ and $j \neq q$.

If $4 \leq k \leq n$, we will show that we can choose a $2 \times (k - 2)$ submatrix from the $i$th and $p$th row whose image under $T$ has a $1 \times k$ submatrix in the $r$th row as follows: Since $T(E_{ij}) = b_{ij} E_{rl}$ and $T(E_{pq}) = c_{pq} E_{rl}$, $T$ maps the $i$th row and
the $p$th row into the $r$th row. But $T$ maps distinct cells in each row (or column) to distinct cells by Lemma 2.4. Now, choose $E_{ij}, E_{pq}$ but do not choose $E_{iq}, E_{pq}$. Since there is a cell $E_{ij}$, $E_{pq}$ in the $p$th row such that $f(E_{ih}) = f(E_{pq})$ but $f(E_{ij}) \neq f(E_{pq})$, we choose the $2 \times 2$ submatrix $E_{ij} + E_{ih} + E_{pq} + E_{ph}$ whose image under $T$ is a $1 \times 4$ submatrix in the $r$th row. And we can choose a cell $E_{ps}$ ($s \neq q, j, h$) such that $f(E_{ps}) \neq f(E_{pq}), f(E_{pq}), f(E_{ph})$. Then we have a $2 \times 3$ submatrix $E_{ij} + E_{ih} + E_{is} + E_{pq} + E_{ph} + E_{ps}$ whose image under $T$ is a $1 \times 5$ submatrix in the $r$th row. Similarly, we can choose a $2 \times (k - 2)$ submatrix whose image under $T$ is a $1 \times k$ submatrix in the $r$th row. This shows that $T$ does not preserve the perimeter $k$ of a rank-1 matrix, a contradiction.

If $k = n + k' \geq n + 2$, consider the matrix

$$D = \sum_{s=1}^{n} E_{is} + \sum_{t=1}^{n} E_{pt} + \sum_{h=1}^{k'-2} \sum_{g=1}^{n} E_{hg}$$

with rank 1 and perimeter $n + k' = k$. Then $T$ maps the $i$th and $p$th row of $D$ into the $r$th row with positive integer multiplication by Lemma 2.4. Thus the perimeter of $T(D)$ is less than $n + k' = k$, a contradiction.

Hence $f(E_{ij}) \neq f(E_{pq})$ for any two distinct cells $E_{ij}, E_{pq} \in \mathbb{E}_{m,n}$. Therefore $f$ is a bijection.

We obtain the following characterization theorem for linear operators preserving the rank and the perimeter of rank-1 matrices over nonnegative integers.

**Theorem 2.7.** Let $T$ be a linear operator on $\mathcal{M}_{m,n}(\mathbb{Z}_+).$ Then the following are equivalent:

1. $T$ is a $(P, Q, B)$-operator;
2. $T$ preserves both rank and perimeter of rank-1 matrices;
3. $T$ preserves both rank and perimeters 2 and $k$ ($k \geq 4, k \neq n + 1$) of rank-1 matrices.

**Proof:** (1) implies (2) by Theorem 2.1. It is obvious that (2) implies (3). We now show that (3) implies (1). Assume (3). Then $T$ induces a bijection $f : \mathbb{E}_{m,n} \rightarrow \mathbb{E}_{m,n}$ by Lemma 2.6. By Lemma 2.4, there are two cases; (a) $T^*$ maps $\mathcal{R}$ onto $\mathcal{R}$ and maps $\mathcal{C}$ onto $\mathcal{C}$ or (b) $T^*$ maps $\mathcal{R}$ onto $\mathcal{C}$ and $\mathcal{C}$ onto $\mathcal{R}$.

Case (a). We note that $T^*(R_i) = R_{\sigma(i)}$ and $T^*(C_j) = C_{\tau(j)}$ for all $i, j$, where $\sigma$ and $\tau$ are permutations of $\{1, \ldots, m\}$ and $\{1, \ldots, n\}$, respectively. Let $P$ and $Q$ be the permutation matrices corresponding to $\sigma$ and $\tau$, respectively. Then for any $E_{ij} \in \mathbb{E}_{m,n}$, we can write $T(E_{ij}) = b_{ij} E_{\sigma(i)\tau(j)}$ for some positive integer $b_{ij} \in \mathbb{Z}_+$. Now we claim that $B = (b_{ij})$ has rank 1. For, consider an $m \times n$ matrix $J$, all of whose entries are 1’s. Then we have

$$T(J) = T \left( \sum_{i=1}^{m} \sum_{j=1}^{n} E_{ij} \right) = \sum_{i=1}^{m} \sum_{j=1}^{n} T(E_{ij}) = \sum_{i=1}^{m} \sum_{j=1}^{n} b_{ij} E_{\sigma(i)\tau(j)} = PBQ.$$
Since $J$ has rank 1, it follows that $r(T(J)) = 1$ and hence $r(B) = 1$ since permutational equivalences preserve rank. Therefore for any $A = [a_{ij}]$ in $\mathcal{M}_{m,n}(\mathbb{Z}^+)$, we have

$$T(A) = T \left( \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} E_{ij} \right) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} T(E_{ij})$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} b_{ij} E_{\sigma(i)\tau(j)} = P(A \circ B)Q.$$  

Thus $T$ is a $(P, Q, B)$-operator.

Case (b). We note that $m = n$, $T^*(R_i) = C_{\sigma(i)}$ and $T^*(C_j) = R_{\tau(j)}$ for all $i, j$, where $\sigma$ and $\tau$ are permutations of $\{1, \ldots, m\}$. By an argument similar to case (a), we obtain that $T(A)$ is of the form $T(A) = P(A^t \circ B)Q$. Thus $T$ is a $(P, Q, B)$-operator.  

We say that a linear operator $T$ on $\mathcal{M}_{m,n}(\mathbb{Z}^+)$ strongly preserves perimeter $k$ of rank-1 matrices if $p(T(A)) = k$ if and only if $p(A) = k$.

Consider a linear operator $T$ on $\mathcal{M}_{2,2}(\mathbb{Z}^+)$ defined by

$$T \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = (a + b + c + d) \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then $T$ preserves both rank and perimeter 2 of rank-1 matrices but does not strongly preserve perimeter 2.

**Theorem 2.8.** Let $T$ be a linear operator on $\mathcal{M}_{m,n}(\mathbb{Z}^+)$. Then $T$ preserves both rank and perimeter of rank-1 matrices if and only if it preserves perimeter 3 and strongly preserves perimeter 2 of rank-1 matrices.

**Proof:** Suppose $T$ preserves perimeter 3 and strongly preserves perimeter 2 of rank-1 matrices. Then $T$ maps each row of a matrix into a row or a column (if $m = n$) with positive integer multiplication. Since $T$ strongly preserves perimeter 2, $T$ maps each cell onto a positive integer multiple of a cell. This means that $T$ induces a bijection $f$ on $E_{m,n}$. Thus $T$ preserves both rank and perimeter of rank-1 matrices by a method similar to that in the proof of Theorem 2.7.

The converse is immediate.  

Thus we have characterizations of the linear operators that preserve both rank and perimeter of rank-1 matrices over nonnegative integers.
References


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