On the class of positive almost weak* Dunford-Pettis operators

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Abstract. In this paper, we introduce and study the class of almost weak* Dunford-Pettis operators. As consequences, we derive the following interesting results: the domination property of this class of operators and characterizations of the wDP* property. Next, we characterize pairs of Banach lattices for which each positive almost weak* Dunford-Pettis operator is almost Dunford-Pettis.

Keywords: almost weak* Dunford-Pettis operator; almost Dunford-Pettis operator; weak Dunford-Pettis* property; positive Schur property; order continuous norm

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1. Introduction and notation

Let us recall from [2] that a norm bounded subset $A$ of a Banach lattice $E$ is said to be almost limited if every disjoint weak* null sequence $(f_n)$ of $E'$ converges uniformly on $A$, that is, $\lim_{n \to \infty} \sup_{x \in A} |f_n(x)| = 0$.

An operator $T$ from a Banach lattice $E$ into a Banach space $Y$ is said to be almost Dunford-Pettis if $\|T(x_n)\| \to 0$ in $Y$ for every weakly null sequence $(x_n)$ consisting of pairwise disjoint elements in $E$ [6].

A Banach space $X$ has the Dunford-Pettis* property (DP* property for short), if $x_n \overset{w}{\to} 0$ in $X$ and $f_n \overset{w*}{\to} 0$ in $X'$ imply $f_n(x_n) \to 0$.

A Banach lattice $E$ has
- the positive Schur property, if $\|f_n\| \to 0$ for every weakly null sequence $(f_n) \subset E^+$, equivalently, $\|f_n\| \to 0$ for every weakly null sequence $(f_n) \subset E^+$ consisting of pairwise disjoint terms (see page 16 of [9]);
- the weak Dunford-Pettis* property (wDP* property for short), if every relatively weakly compact set in $E$ is almost limited, equivalently, whenever $f_n(x_n) \to 0$ for every weakly null sequence $(x_n)$ in $E$ and for every disjoint weak* null sequence $(f_n)$ in $E'$ [2].

Recall from [4] that an operator $T$ from a Banach space $X$ into another Banach space $Y$ is called weak* Dunford-Pettis if $f_n(T(x_n)) \to 0$ for every weakly null sequence $(x_n) \subset X$, and every weak* null sequence $(f_n) \subset Y'$. In this paper,
we introduce and study the disjoint version of this class of operators, that we call almost weak* Dunford-Pettis operators (Definition 2.1). It is a class which contains that of weak* Dunford-Pettis (resp. almost Dunford-Pettis).

The main results are some characterizations of almost weak* Dunford-Pettis operators (Theorem 2.3). Next, we derive the following interesting consequences: the domination property of this class of operators (Corollary 2.4), a characterization of wDP* property (Corollary 2.5). After that, we prove that each positive almost weak* Dunford-Pettis operator from a Banach lattice $E$ into a $\sigma$-Dedekind complete Banach lattice $F$ is almost Dunford-Pettis if and only if $E$ has the positive Schur property or the norm of $F$ is order continuous (Theorem 2.7). As consequence, we will give some interesting results (Corollaries 2.8 and 2.9).

To state our results, we need to fix some notations and recall some definitions. A Banach lattice is a Banach space $(E, \| \cdot \|)$ such that $E$ is a vector lattice and its norm satisfies the following property: for each $x, y \in E$ such that $|x| \leq |y|$, we have $\|x\| \leq \|y\|$. If $E$ is a Banach lattice, its topological dual $E'$, endowed with the dual norm, is also a Banach lattice. A norm $\| \cdot \|$ of a Banach lattice $E$ is order continuous if for each generalized sequence $(x_\alpha)$ such that $x_\alpha \downarrow 0$ in $E$, the sequence $(x_\alpha)$ converges to 0 in the norm $\| \cdot \|$, where the notation $x_\alpha \downarrow 0$ means that the sequence $(x_\alpha)$ is decreasing, its infimum exists and $\inf(x_\alpha) = 0$. A Riesz space is said to be $\sigma$-Dedekind complete if every countable subset that is bounded above has a supremum, equivalently, whenever $0 \leq x_n \uparrow \leq x$ implies the existence of $\sup(x_n)$.

We will use the term operator $T : E \rightarrow F$ between two Banach lattices to mean a bounded linear mapping. It is positive if $T(x) \geq 0$ in $F$ whenever $x \geq 0$ in $E$. If $T$ is an operator from a Banach lattice $E$ into another Banach lattice $F$ then its dual operator $T'$ is defined from $F'$ into $E'$ by $T'(f)(x) = f(T(x))$ for each $f \in F'$ and for each $x \in E$. We refer the reader to [1] for unexplained terminology of Banach lattice theory and positive operators.

2. Main results

Next we give the definition of almost weak* Dunford-Pettis operator between Banach lattices, which is a different version of the weak* Dunford-Pettis operator.

**Definition 2.1.** An operator $T$ from a Banach lattice $E$ to a Banach lattice $F$ is almost weak* Dunford-Pettis if $f_n(T(x_n)) \rightarrow 0$ for every weakly null sequence $(x_n)$ in $E$ consisting of pairwise disjoint terms, and for every weak* null sequence $(f_n)$ in $F'$ consisting of pairwise disjoint terms.

For proof of the next theorem, we need the following lemma which is just Lemma 2.2 of Chen in [2].

**Lemma 2.2.** Let $E$ be a $\sigma$-Dedekind complete Banach lattice, and let $(f_n)$ be a weak* convergent sequence of $E'$. If $(g_n)$ is a disjoint sequence of $E'$ satisfying $|g_n| \leq |f_n|$ for each $n$, then the sequences $(g_n), (\|g_n\|), (g_n)^+, (g_n)^-$ are all weak* convergent to zero. In particular, if $(f_n)$ is a disjoint weak* convergent sequence in its own right, then the sequences $(f_n), (\|f_n\|), (f_n)^+, (f_n)^-$ are all weak* null.
Now, for positive operators between two Banach lattices, we give a characterization of almost weak* Dunford-Pettis operators.

**Theorem 2.3.** Let $E$ and $F$ be two Banach lattices such that $F$ is $\sigma$-Dedekind complete. For every positive operator $T$ from $E$ into $F$, the following assertions are equivalent.

1. $T$ is almost weak* Dunford-Pettis operator.
2. For every disjoint weakly null sequence $(x_n) \subset E^+$, and every disjoint weak* null sequence $(f_n) \subset (F^')^+$ it follows that $f_n(T(x_n)) \to 0$.
3. For every disjoint weakly null sequence $(x_n) \subset E^+$, and every weak* null sequence $(f_n) \subset F'$ it follows that $f_n(T(x_n)) \to 0$.
4. For every disjoint weakly null sequence $(x_n) \subset E^+$, and every weak* null sequence $(f_n) \subset (F^')^+$ it follows that $f_n(T(x_n)) \to 0$.
5. For every weakly null sequence $(x_n) \subset E^+$, and every weak* null sequence $(f_n) \subset (F^')^+$ it follows that $f_n(T(x_n)) \to 0$.

**Proof:** (1) $\Rightarrow$ (2) Obvious.

(2) $\Rightarrow$ (3) Assume by way of contradiction that there exists a disjoint weakly null sequence $(x_n) \subset E^+$, and a weak* null sequence $(f_n) \subset F'$ such that $f_n(T(x_n))$ does not converge to 0. The inequality $|f_n(T(x_n))| \leq |f_n(T(x_n))|$ implies $|f_n(T(x_n))|$ does not converge to 0. Then there exist some $\epsilon > 0$ and a subsequence of $|f_n|(T(x_n))$ (which we shall denote by $|f_n|(T(x_n))$ again) satisfying $|f_n|(T(x_n)) > \epsilon$ for all $n$.

On the other hand, since $x_n \to 0$ weakly in $E$, then $T(x_n) \to 0$ weakly in $F$. Now an easy inductive argument shows that there exist a subsequence $(z_n)$ of $(x_n)$ and a subsequence $(g_n)$ of $(f_n)$ such that

$$|g_n|(T(z_n)) > \epsilon$$

and

$$(4^n \sum_{i=1}^{n} |g_i|)(T(z_{n+1})) < \frac{1}{n}$$

for all $n \geq 1$. Put $h = \sum_{i=1}^{\infty} 2^{-n} |g_n|$ and $h_n = (|g_{n+1}| - 4^n \sum_{i=1}^{n} |g_i| - 2^{-n} h)^+$. By Lemma 4.35 of [1] the sequence $(h_n)$ is disjoint. Since $0 \leq h_n \leq |g_{n+1}|$ for all $n \geq 1$ and $(g_n)$ is weak* null in $F'$, then from Lemma 2.2 $(h_n)$ is weak* null in $F'$. From the inequality

$$h_n(T(z_{n+1})) \geq \left(|g_{n+1}| - 4^n \sum_{i=1}^{n} |g_i| - 2^{-n} h\right)(T(z_{n+1}))$$

$$\geq \epsilon - \frac{1}{n} - 2^{-n} h(T(z_{n+1}))$$

we see that $h_n(T(z_{n+1})) \geq \frac{\epsilon}{n}$ must hold for all $n$ sufficiently large (because $2^{-n} h(T(z_{n+1})) \to 0$), which contradicts with our hypothesis (2).

(3) $\Rightarrow$ (4) Obvious.
(4) ⇒ (5) Assume by way of contradiction that there exists a weakly null sequence \((x_n) \subset E^+\) and a weak* null sequence \((f_n) \subset (F')^+\) such that \(f_n(T(x_n))\) does not converge to 0. Then there exists some \(\epsilon > 0\) and a subsequence of \(f_n(T(x_n))\) (which we shall denote by \(f_n(T(x_n))\) again) satisfying \(f_n(T(x_n)) \geq \epsilon\) for all \(n\).

On the other hand, since \((f_n)\) is a weak* null sequence in \((F')\), then \(T'(f_n) \to 0\) weak* in \(E'\). Now an easy inductive argument shows that there exist a subsequence \((z_n)\) of \((x_n)\) and a subsequence \((g_n)\) of \((f_n)\) such that

\[
T'(g_n)(z_n) > \epsilon
\]

and

\[
T'(g_{n+1})(4^n \sum_{i=1}^{n} z_i) < \frac{1}{n}
\]

for all \(n \geq 1\). Put \(z = \sum_{n=1}^{\infty} 2^{-n} z_n\) and \(y_n = (z_{n+1} - 4^n \sum_{i=1}^{n} z_i - 2^{-n} z)^+\). By Lemma 4.35 of [1] the sequence \((y_n)\) is disjoint. Since \(0 \leq y_n \leq z_{n+1}\) for all \(n \geq 1\) and \((z_n)\) is weakly null in \(E\), then from Theorem 4.34 of [1] \((y_n) \to 0\) weakly in \(E\). From the inequality

\[
T'(g_{n+1})(y_n) \geq T'(g_{n+1}) \left( z_{n+1} - 4^n \sum_{i=1}^{n} z_i - 2^{-n} z \right) \geq \epsilon - \frac{1}{n} - 2^{-n} T'(g_{n+1})(z)
\]

we see that \(g_{n+1}(T(y_n)) = T'(g_{n+1})(y_n) \geq \frac{\epsilon}{2}\) must hold for all \(n\) sufficiently large (because \(2^{-n} T'(g_{n+1})(z) \to 0\), which contradicts with our hypothesis (4).

(5) ⇒ (1) Let \((x_n)\) be a weak null sequence in \(E\) consisting of pairwise disjoint terms, and let \((f_n)\) be a weak* null sequence in \(F'\) consisting of pairwise disjoint terms, it follows from Remark(1) of [6] that \(\langle |x_n| \rangle\) is weakly null in \(E\), and from lemma 2.2 that \(\langle |f_n| \rangle\) is weak* null in \(F'\). So by our hypothesis (5), \(|f_n| T(|x_n|) \to 0\). Now, from the inequality \(|f_n(T(x_n))| \leq |f_n| T(|x_n|)\) for each \(n\), we deduce that \(f_n(T(x_n)) \to 0\), and this completes the proof. \(\square\)

The domination property for almost weak* Dunford-Pettis operators can be derived from Theorem 2.3.

**Corollary 2.4.** Let \(E\) and \(F\) be two Banach lattices such that \(F\) is \(\sigma\)-Dedekind complete. If \(S\) and \(T\) are two positive operators from \(E\) into \(F\) such that \(0 \leq S \leq T\) and \(T\) is an almost weak* Dunford-Pettis, then \(S\) is also almost weak* Dunford-Pettis.

**Proof:** Let \((x_n)\) be a weakly null sequence in \(E^+\) and \((f_n)\) be a weak* null sequence in \((F')^+\). According to (5) of Theorem 2.3, it suffices to show that \(f_n(S(x_n)) \to 0\). Since \(T\) is almost weak* Dunford-Pettis, then Theorem 2.3 implies that \(f_n(T(x_n)) \to 0\). Now, by the inequality \(0 \leq f_n(S(x_n)) \leq f_n(T(x_n))\) for each \(n\), we conclude that \(f_n(S(x_n)) \to 0\). \(\square\)
As a consequence of Theorem 2.3 and Theorem 3.2 of Chen [2], other characterizations of Banach lattices with the wDP* property are given in the following Corollary.

**Corollary 2.5.** Let \( E \) be a \( \sigma \)-Dedekind complete Banach lattice. Then, the following assertions are equivalent.

1. \( E \) has the wDP* property.
2. The solid hull of every relatively weakly compact set in \( E \) is almost limited.
3. The identity operator \( \text{Id}_E : E \to E \) is almost weak* Dunford-Pettis.
4. For every disjoint weakly null sequence \( (x_n) \subset E \), and every disjoint weak* null sequence \( (f_n) \subset E' \) it follows that \( f_n(x_n) \to 0 \).
5. For every disjoint weakly null sequence \( (x_n) \subset E^+ \), and every disjoint weak* null sequence \( (f_n) \subset (E')^+ \) it follows that \( f_n(x_n) \to 0 \).
6. For every disjoint weakly null sequence \( (x_n) \subset E^+ \), and every weak* null sequence \( (f_n) \subset E' \) it follows that \( f_n(x_n) \to 0 \).
7. For every disjoint weakly null sequence \( (x_n) \subset E^+ \), and every weak* null sequence \( (f_n) \subset (E')^+ \) it follows that \( f_n(x_n) \to 0 \).
8. For every weakly null sequence \( (x_n) \subset E^+ \), and every weak* null sequence \( (f_n) \subset (E')^+ \) it follows that \( f_n(x_n) \to 0 \).

**Proof:** (3) ⇔ (4) Obvious.
(3) ⇔ (5) ⇔ (6) ⇔ (7) ⇔ (8) follows from Theorem 2.3.
(1) ⇔ (2) ⇔ (4) follows from Theorem 3.2 of [2].

The proof of the next theorem is based on the following proposition.

**Proposition 2.6.** Let \( E, F \) and \( G \) be three Banach lattices such that \( G \) has the DP* property. Then, each operator \( T : E \to F \) that admits a factorization through the Banach lattice \( G \) is almost weak* Dunford-Pettis.

**Proof:** Let \( P : E \to G \) and \( Q : G \to F \) be two operators such that \( T = Q \circ P \). Let \( (x_n) \) be a disjoint weakly null sequence in \( E \) and let \( (f_n) \) be a disjoint weak* null sequence in \( F' \). It is clear that \( P(x_n) \overset{w}{\to} 0 \) in \( G \) and \( Q'(f_n) \overset{w^*}{\to} 0 \) in \( G' \). As \( G \) has the DP* property, then
\[
f_n(Tx_n) = f_n(Q \circ P(x_n)) = (Q'f_n)(P(x_n)) \to 0.
\]
This proves that \( T \) is almost weak* Dunford-Pettis.

Note that every almost Dunford-Pettis operator is almost weak* Dunford-Pettis, but the converse is not true in general. In fact, \( \text{Id}_{\ell^\infty} : \ell^\infty \to \ell^\infty \) is almost weak* Dunford-Pettis operator because \( \ell^\infty \) has the wDP* property, but it fails to be almost Dunford-Pettis because \( \ell^\infty \) does not have the positive Schur property.

Now, we characterize Banach lattices such that each positive almost weak* Dunford-Pettis operator is almost Dunford-Pettis.

**Theorem 2.7.** Let \( E \) and \( F \) be two Banach lattices such that \( F \) is \( \sigma \)-Dedekind complete. Then the following assertions are equivalent.
(1) Each positive almost weak* Dunford-Pettis operator $T : E \to F$ is almost Dunford-Pettis.

(2) One of the following assertions is valid:
   (a) $E$ has the positive Schur property,
   (b) the norm of $F$ is order continuous.

Proof: (1) $\Rightarrow$ (2) Assume by way of contradiction that $E$ does not have the positive Schur property and the norm of $F$ is not order continuous. We have to construct a positive almost weak* Dunford-Pettis operator which is not almost Dunford-Pettis. As $E$ does not have the positive Schur property, then there exists a disjoint weakly null sequence $(x_n)$ in $E^+$ which is not norm null. By choosing a subsequence we may suppose that there is $\varepsilon > 0$ with $\|x_n\| > \varepsilon > 0$ for all $n$. From the equality $\|x_n\| = \sup \{ f(x_n) : f \in (E')^+, \|f\| = 1 \}$, there exists a sequence $(f_n) \subset (E')^+$ such that $\|f_n\| = 1$ and $f_n(x_n) \geq \varepsilon$ holds for all $n$. Now, consider the operator $R : E \to \ell^\infty$ defined by

$$R(x) = (f_n(x))_{n=1}^\infty$$

On the other hand, since the norm of $F$ is not order continuous, it follows from Theorem 4.51 of [1] that $\ell^\infty$ is lattice embeddable in $F$, i.e., there exists a lattice homomorphism $S : \ell^\infty \to F$ and there exist two positive constants $M$ and $m$ satisfying

$$m \|((\lambda_k)_k)\|_\infty \leq \|S((\lambda_k)_k)\|_F \leq M \|((\lambda_k)_k)\|_\infty$$

for all $(\lambda_k)_k \in \ell^\infty$. Put $T = S \circ R$, and note by Proposition 2.6 that $T$ is a positive almost weak* Dunford-Pettis operator because $\ell^\infty$ has DP* property. However, for the disjoint weakly null sequence $(x_n) \subset E^+$, we have

$$\|T(x_n)\| = \|S((f_k(x_n))_k)\| \geq m \|(f_k(x_n))_k\|_\infty \geq mf_n(x_n) \geq m\varepsilon$$

for every $n$. This shows that $T$ is not almost Dunford-Pettis, and we are done.

(a) $\Rightarrow$ (1) In this case, each operator $T : E \to F$ is almost Dunford-Pettis.

(b) $\Rightarrow$ (1) Let $(x_n) \subset E$ be a positive disjoint weakly null sequence. We shall show that $\|T(x_n)\| \to 0$. By Corollary 2.6 of [3], it suffices to prove that $\|T(x_n)\| \to 0$ and $f_n(T(x_n)) \to 0$ for every disjoint and norm bounded sequence $(f_n) \subset (F')^+$. Let $f \in (F')^+$ and by Theorem 1.23 of [1] there exists some $g \in [-f,f]$ with $f(Tx_n) = g(Tx_n)$. Since $x_n \wto 0$ then $f(Tx_n) = g(Tx_n) = (T'g)(x_n) \to 0$, thus $\|T(x_n)\| \to 0$. On the other hand, let $(f_n) \subset (F')^+$ be a disjoint and norm bounded sequence. As the norm of $F$ is order continuous, then by Corollary 2.4.3 of [5] $f_n \wto 0$. Now, since $T$ is positive almost weak* Dunford-Pettis then, $f_n(T(x_n)) \to 0$. This completes the proof. \qed

Remark 1. The assumption that $F$ is $\sigma$-Dedekind complete is essential in Theorem 2.7. In fact, if we consider $E = \ell^\infty$ and $F = c$, the Banach lattice of all convergent sequences, it is clear that $F = c$ is not $\sigma$-Dedekind complete, and it follows from the proof of Proposition 1 of [7] and Theorem 5.99 of [1] that each
operator from $\ell^\infty$ into $c$ is Dunford-Pettis (and hence is almost Dunford-Pettis). But $\ell^\infty$ does not have the positive Schur property and the norm of $c$ is not order continuous.

As consequences of Theorem 2.7, we have the following characterization.

**Corollary 2.8.** Let $E$ be a $\sigma$-Dedekind complete Banach lattice. Then the following assertions are equivalent.

1. Each positive almost weak* Dunford-Pettis operator $T : E \to E$ is almost Dunford-Pettis.
2. The norm of $E$ is order continuous.

**Proof:** The result follows from Theorem 2.7 by noting that if $E$ has the positive Schur property then the norm of $E$ is order continuous. \(\Box\)

Now, from Corollary 2.8 and Theorem 4.9 (Nakano) of [1], we obtain the following result, which is just Proposition 3.3 of [2].

**Corollary 2.9.** Let $E$ be a Banach lattice. Then $E$ has the positive Schur property if and only if $E$ has the wDP* property and its norm is order continuous.

**Proof:** The “only if” part is trivial.

For the “if” part, since $E$ has wDP* property, then $Id_E : E \to E$ is almost weak* Dunford-Pettis operator. As the norm of $E$ is order continuous, it follows from Theorem 4.9 (Nakano) of [1] that $E$ is $\sigma$-Dedekind complete, and by Corollary 2.8 we have that $Id_E : E \to E$ is almost Dunford-Pettis. This proves that $E$ has the positive Schur property. \(\Box\)

**References**


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