Abstract. In this paper we introduce the concept of $\tau$-extending modules by $\tau$-rational submodules and study some properties of such modules. It is shown that the set of all $\tau$-rational left ideals of $R$ is a Gabriel filter. An $R$-module $M$ is called $\tau$-extending if every submodule of $M$ is $\tau$-rational in a direct summand of $M$. It is proved that $M$ is $\tau$-extending if and only if $M = \text{Rej}_M E(R/\tau(R)) \oplus N$, such that $N$ is a $\tau$-extending submodule of $M$. An example is given to show that the direct sum of $\tau$-extending modules need not be $\tau$-extending.

Keywords: torsion theory; $\tau$-rational submodules; $\tau$-closed submodules; $\tau$-extending modules

Classification: 16D10, 16D80 16D99

1. Introduction

Throughout this paper, $R$ is an associative ring with identity and $M$ is a unital left $R$-module. A subfunctor $\rho$ is called a preradical if it satisfies the following properties:

1. $\rho(M)$ is a submodule of an $R$-module $M$;
2. if $f : M \rightarrow N$ is an $R$-homomorphism, then $f(\rho(M)) \subseteq \rho(N)$ and $\rho(f) : f(M) \rightarrow f(N)$ is the restriction of $f$ to $\rho(M)$.

A preradical $\rho$ is idempotent if $\rho(\rho(M)) = \rho(M)$, and radical when $\rho(M/\rho(M)) = 0$ for all $M \in R$-Mod.

For a preradical $\rho$, let $T_\rho = \{N|\rho(N) = N\}$ and $F_\rho = \{N|\rho(N) = 0\}$. $T_\rho$ is called the torsion class of $\rho$ and $F_\rho$ the torsion free class of $\rho$. $\rho$ is called left exact if $\rho(N) = N \cap \rho(M)$ for every module $M$ and every submodule $N$ of $M$. A preradical $\rho$ is left exact if and only if $\rho$ is idempotent and $T_\rho$ is closed under submodules. A preradical $\rho$ is called cohereditary if $\rho(M/N) = (\rho(M) + N)/N$ for every module $M$ and every submodule $N$ of $M$. $\rho$ is cohereditary if and only if $\rho$ is radical and $F_\rho$ is closed under homomorphic images.

A pair $(T, F)$ of classes of modules is called a torsion theory if the following conditions hold:

(i) $\text{Hom}_R(A, B) = 0$ for every $A \in T$ and every $B \in F$;
(ii) $T$ and $F$ are maximal classes having property (i).
The modules in $T$ are called torsion modules of $\tau$ and the modules in $F$ are torsion-free of $\tau$. There is a 1-1 correspondence between torsion theories and idempotent radicals. In particular preradicals are connected to torsion theory as follows. If $\rho$ is an idempotent radical in $R$-Mod, then $(\mathbb{T}_\rho, \mathbb{F}_\rho)$ is a torsion theory, where $\mathbb{T}_\rho = \{M \in R$-Mod$| \rho(M) = M\}$ and $\mathbb{F}_\rho = \{M \in R$-Mod$| \rho(M) = 0\}$. Now for any torsion theory $(T, F)$, there is an associated idempotent radical $\tau_t$ (simply denoted by $\tau$), called the torsion radical associated to torsion theory $(T, F)$. Here for every module $N$, $\tau(N)$ will be the unique maximal submodule of $N$ such that $\tau(N) \in T$. Then $\tau$ is uniquely determined and $T$ is exactly the set $\{M| \tau(M) = M\}$ and $F = \{M| \tau(M) = 0\}$. Therefore we can denote this torsion theory by $\tau = (T, F)$, where $\tau$ is an idempotent radical associative to $(T, F)$. A torsion theory $\tau = (T, F)$ is called hereditary if $T$ is closed under submodules.

A torsion theory $\tau = (T, F)$ is hereditary if and only if $F$ is closed under injective hulls if and only if $t$ is a left exact radical. Thus there is a 1-1 correspondence between hereditary torsion theories and left exact radicals.

A module $M$ is extending if every submodule of $M$ is essential in a direct summand of $M$. In recent years, torsion-theoretic analogues of extending modules have been studied by many authors (see [4], [15], [5], [7], [16], [10], [8]).

In 2007, Charalambides and Clark [5] generalized extending modules to torsion theories. They defined that a module $M$ is $\tau$-extending if every $\tau$-dense, closed submodule of $M$ is a direct summand of $M$. In 2008, [15] the authors also studied $\tau$-CS (extending) modules under the name of type 2-$\tau$-extending modules. In [8] $s$-$t$-CS modules and CS modules were studied under the name of type 1 $\tau$-extending modules and type 2 $\tau$-extending modules respectively.

Following J. L. Gomez Pardo [10], a submodule $N$ of an $R$-module $M$ is called $\tau$-large in $M$ if, for $W \leq M$, $N \cap W \subseteq \tau(M)$ implies $W \subseteq \tau(M)$.

In [4] the authors say that $M$ is $\tau$-extending module if every submodule is $\tau$-large in a direct summand of $M$. They showed that every $\tau$-torsion module is $\tau$-extending and they also proved that a $\tau$-torsion free module is $\tau$-extending if and only if it is extending. In this paper, we generalize extending modules by using hereditary torsion theories. We say that a submodule $N$ of $M$ is $\tau$-rational in $M$ if $\text{Hom}(M/N, E(R/\tau(R))) = 0$, where $E(R/\tau(R))$ is the injective hull of $R/\tau(R)$. We say that an $R$-module $M$ is $\tau$-extending if for every submodule $X$, there exists a direct summand $D$ of $M$ such that $X$ is $\tau$-rational in $D$, i.e, $\text{Hom}(D/X, E(R/\tau(R))) = 0$. We prove that a module $M$ is $\tau$-extending if and only if every $\tau$-closed submodule of $M$ is a direct summand of $M$. We show that $M$ is $\tau$-extending if and only if $M = \text{Rej}_M E(R/\tau(R)) \oplus N$, and $N$ is a $\tau$-extending submodule of $M$. We also prove that the class of $\tau$-extending modules is closed under direct summands. It is proved that if $M$ is a $\tau$-extending module and $\text{Rej}_T(E(R/\tau(R))) = T$ for a module $T$, then $M \oplus T$ is $\tau$-extending. Moreover, we prove that $M = M_1 \oplus M_2$ is $\tau$-extending if and only if $M_i$ are $\tau$-extending and every $\tau$-closed submodule $K$ of $N_1 \oplus N_2$ with $K \cap N_1K \cap N_2 = 0$ is a direct summand of $M$, where $M_i = \text{Rej}_{M_i}(E(R/\tau(R))) \oplus N_i$. 
2. \(\tau\)-rational modules

Throughout this paper \(\tau\) is a hereditary preradical associative to a hereditary torsion theory.

**Definition 2.1.** We say that a submodule \(N\) of an \(R\)-module \(M\) is \(\tau\)-rational in \(M\), denoted by \(N \leq_{\tau\text{-r}} M\), if \(\text{Hom}(M/N, E(R/\tau(R))) = 0\). If \(M \in T_{\tau}\), then every submodule of \(M\) is \(\tau\)-rational.

Let \(\mathcal{U}\) be a class of modules. A module \(M\) is (finitely) cogenerated by \(\mathcal{U}\) (or \(\mathcal{U}\) (finitely) cogenerates \(M\)), in case there is an (a finite) indexed set \((U_\alpha)_{\alpha \in A}\) in \(\mathcal{U}\) and a monomorphism

\[
0 \longrightarrow M \longrightarrow \prod_A U_\alpha.
\]

An \(R\)-module \(RC\) is said to be a cogenerator if \(RC\) cogenerates every \(R\)-module.

It is recalled that a submodule \(N\) of \(M\) is dense in \(M\) if \(\text{Hom}_R(M/N, E(M)) = 0\), where \(E(M)\) denotes the injective envelope of \(M\).

**Lemma 2.2.** Let \(R/\tau(R)\) be an injective cogenerator in \(RM\) and \(N \leq_{\tau\text{-r}} M\). Then \(N\) is dense in \(M\).

**Proof:** As \(N \leq_{\tau\text{-r}} M\), then \(\text{Hom}_R(M/N, E(R/\tau(R))) = 0\). Since \(R/\tau(R)\) is an injective cogenerator, there is a set \(A\) for which \(E(M)\) can be embedded in \(\prod_A R/\tau(R)\).

Thus \(\prod_A \text{Hom}_R(M/N, R/\tau(R)) \simeq \text{Hom}_R(M/N, \prod_A R/\tau(R)) = 0\). It follows that \(\text{Hom}(M/N, E(M)) = 0\) and so \(N\) is dense in \(M\).

A ring \(R\) is called a left Kasch ring (or simply left Kasch) if every simple left module \(K\) embeds in \(RR\), equivalently if \(RR\) cogenerates \(K\). Every semisimple artinian ring is right and left Kasch, and a local ring \(R\) is left Kasch if and only if \(\text{Soc}_l(R) \neq 0\), because \(R\) has only one simple left module up to isomorphism.

**Corollary 2.3.** Consider the trivial torsion theory \(\tau = 0\). If \(R\) is a left Kasch ring and \(N \leq_{\tau\text{-r}} M\), for \(R\)-modules \(M\) and \(N\), then \(N\) is a dense submodule of \(M\).

**Proof:** This follows from the fact that \(RR\) is a left Kasch ring if and only if \(E(RR)\) is a cogenerator in \(RM\) and applying Lemma 2.2.

**Definition 2.4.** A non-empty set \(\mathcal{D}(R)\) of left ideals of \(R\) is called a filter radical if the following hold:

(i) for every \(I \in \mathcal{D}(R)\) and every \(a \in R\), we have \((I : a) \in \mathcal{D}(R)\), where \((I : a)\) is the ideal \(\{r \in R | ra \in I\}\);

(ii) for every \(J \in \mathcal{D}(R)\) and every left ideal \(I\) of \(R\) with \((I : a) \in \mathcal{D}(R)\) for each \(a \in J\), we have \(I \in \mathcal{D}(R)\).
Proposition 2.5. Let $\mathfrak{F}(R)$ be the set of all left ideals $I$ such that $_{R/I}$ is $\tau$-rational in $R$. Then $\mathfrak{F}(R)$ is a filter radical.

Proof: (i) Let $I \in \mathfrak{F}(R)$ and $a \in R$. Then $\text{Hom}_R(R/I, E(R/\tau(R))) = 0$ and by the injectivity of $E(R/\tau(R))$, $\text{Hom}_R((Ra + I)/I, E(R/\tau(R))) = 0$. As $R/(I : a) \cong (Ra + I)/I$ then we get $\text{Hom}_R(R/(I : a), E(R/\tau(R))) = 0$. Hence for every $I \in \mathfrak{F}(R)$ and $a \in R$, we get $(I : a) \in \mathfrak{F}(R)$.

(ii) Assume that $J \in \mathfrak{F}(R)$ and there exists a left ideal $I$ of $R$, such that $(I : a) \in \mathfrak{F}(R)$ for every $a \in J$, so that, $\text{Hom}_R(R/(I : a), E(R/\tau(R))) = 0$. If $f \in \text{Hom}_R(R/I, E(R/\tau(R)))$ then $(Ra + I)/I \cong R/(I : a) \subseteq \text{ker}(f)$ for every $a \in J$. Hence $\text{Hom}_R((Ra + I)/I, E(R/\tau(R))) = 0$ for every $a \in J$ and so $\text{Hom}_R((J + I)/I, E(R/\tau(R))) = 0$. Thus $f$ factors through $\overline{f} \in \text{Hom}_R(R/(I + J), E(R/\tau(R)))$. However $J \in \mathfrak{F}(R)$ implies $I + J \in \mathfrak{F}(R)$. Hence $\overline{f} = 0$ and so $f = 0$. This shows that $I \in \mathfrak{F}(R)$. □

Corollary 2.6. Let $I, J$ be left ideals of $R$. Then

(i) if $J \in \mathfrak{F}(R)$ and $J \subseteq I$, then $I \in \mathfrak{F}(R)$;

(ii) if $I, J \in \mathfrak{F}(R)$, then $I \cap J \in \mathfrak{F}(R)$;

(iii) if $I, J \in \mathfrak{F}(R)$, then $IJ \in \mathfrak{F}(R)$.

Proof: This follows by [3]. □

Lemma 2.7. If $RI$ is $\tau$-rational in $R$, then $(I + \tau(R))/\tau(R) \leq_{es} R/\tau(R)$.

Proof: Suppose that there exists a nonzero left ideal $L/\tau(R)$ of $R/\tau(R)$ such that $L/\tau(R) \cap (I + \tau(R))/\tau(R) = 0$. As $I \subseteq I + L$, by Corollary 2.6(i), we have $I + L \in \mathfrak{F}(R)$. Hence $\text{Hom}_R(R/(I + L), E(R/\tau(R))) = 0$.

Since $\text{Hom}_R(R/I, E(R/\tau(R))) = 0$, then $\text{Hom}_R((L + I)/I, E(R/\tau(R))) = 0$. Thus $\text{Hom}(L/(L \cap I), E(R/\tau(R))) = 0$, and since $I \cap L \subseteq \tau(R)$,

$$\text{Hom}(L/\tau(R), E(R/\tau(R))) = 0,$$

a contradiction. □

The following examples show that Lemma 2.7 need not be true, for $R$-modules.

Example 2.8. Consider the torsion theory $(0, R\mathcal{M})$ with associative radical $\tau = 0$, where $R = \mathbb{Z}$. Let $M = \mathbb{Z}_6$ and $N = 3\mathbb{Z}_6$, then $3\mathbb{Z}_6 \not\leq_{e} \mathbb{Z}_6$, however $N \leq_{\tau} M$ because $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}_6/3\mathbb{Z}_6, \mathbb{Q}) = 0$. □

Example 2.9. Consider the torsion theory $(R\mathcal{M}, 0)$ with associative radical $\tau = \text{id}$, where $R = \mathbb{Z}$. Then for every $R$-module $M$, we have $\text{Hom}_{\mathbb{Z}}(M/N, 0) = 0$, for every $\mathbb{Z}$-submodule $N$ of $M$, which implies that $N \leq_{\tau} M$. □

For each $M \in R\text{-Mod}$ we define

$$\delta_{\tau}(M) = \{x \in M | (0 : x) \text{ is a } \tau\text{-rational left ideal in } R\}.$$

Proposition 2.10. For an arbitrary ring $R$ and a left $R$-module $M$ the following assertions hold:

(1) $\delta_{\tau}(M)$ is a submodule of $M$;
(2) $\delta_{r}(M/\delta_{r}(M)) = 0$;
(3) for every $R$-homomorphism $f : M \rightarrow N$, $f(\delta_{r}(M)) \subseteq \delta_{r}(N)$;
(4) for every $K \leq M$ we have $\delta_{r}(K) = \delta_{r}(M) \cap K$.

**Proof:** (1) This is clear.
(2) Let $\overline{m} = m + \delta_{r}(M) \in M/\delta_{r}(M)$. Then $\overline{m} \in \delta_{r}(M/\delta_{r}(M))$ iff $(0 : \overline{m})$ is $\tau$-rational in $R$. As $(0 : \overline{m}) = \{r \in R | rm \in \delta_{r}(M)\}$ and $(0 : rm) = ((0 : m) : r)$, then $(0 : \overline{m}) = \{r \in R | ((0 : m) : r) \leq_{\tau_{r}} R\}$ is $\tau$-rational in $R$. Since the set of all $\tau$-rational left ideals of $R$ is a Gabriel filter, we get $(0 : m) \leq_{\tau_{r}} R$ and this shows that $m \in \delta_{r}(M)$, i.e; $\overline{m} = 0$.
(3) Let $m \in \delta_{r}(M)$. Then $(0 : m) \leq_{\tau_{r}} R$. As $(0 : m) \subseteq (0 : f(m))$ we get $(0 : f(m)) \leq_{\tau_{r}} R$.
(4) This is clear. \qed

**Corollary 2.11.** Let $(\mathbb{T}, \mathbb{F})$, where $\mathbb{T} = \{M | \delta_{r}(M) = M\}$, $\mathbb{F} = \{M \mid \delta_{r}(M) = 0\}$. Then $(\mathbb{T}, \mathbb{F})$ is a hereditary torsion theory.

**Proposition 2.12.** If $\delta_{r}(M/N) = M/N$ then $N$ is $\tau$-rational in $M$.

**Proof:** To the contrary assume that $\delta_{r}(M/N) = M/N$ but
\[
\text{Hom}(M/N, E(R/\tau(R))) \neq 0.
\]
Then we have $\text{Hom}((Rm + N)/N, E(R/\tau(R))) \neq 0$, for some $m \in M$. As $R/(N : m) \cong (Rm + N)/N$ for any $m \in M$, this gives
\[
\text{Hom}(R/(N : m), E(R/\tau(R))) \neq 0,
\]
a contradiction to the fact that $(N : m) \leq_{\tau_{r}} R$. \qed

**Corollary 2.13.** Let $N \leq_{\tau_{r}} M$ and $K$ a submodule of $M$. Then $N \cap K \leq_{\tau_{r}} K$.

**Corollary 2.14.** $\delta_{r}(M) = M$ iff $\text{Hom}_{R}(M, E(R/\tau(R))) = 0$.

**Proposition 2.15.** Let $M$ be an $R$-module and $N, L \leq M$. If $N \subseteq L \subseteq M$, then $N \leq_{\tau_{r}} L \leq_{\tau_{r}} M$ iff $N \leq_{\tau_{r}} M$.

**Proof:** If $N \leq_{\tau_{r}} M$, then obviously $N \leq_{\tau_{r}} L \leq_{\tau_{r}} M$.
Conversely, let $N \leq_{\tau_{r}} L \leq_{\tau_{r}} M$. Consider the exact sequence
\[
0 \rightarrow L/N \rightarrow M/N \rightarrow M/L \rightarrow 0.
\]
Then, since $\text{Hom}(\cdot, E(R/\tau(R)))$ is an exact functor we get the exact sequence
\[
0 \rightarrow \text{Hom}(M/N, E(R/\tau(R))) \rightarrow 0.
\]
Thus $\text{Hom}(M/N, E(R/\tau(R))) = 0$ and so $N \leq_{\tau_{r}} M$. \qed

**Corollary 2.16.** $N$ is $\tau$-rational in $M$ if and only if $C$ is $\tau$-rational in $M$, where $C/N = \tau(M/N)$.
Lemma 2.17. Let $N$ be a submodule of $M$ and suppose that every homomorphic image of $M$ has a non-zero $\tau$-torsion submodule. Then $N \leq_{\tau-c} M$.

Proof: On the contrary, assume $0 \neq f \in \text{Hom}(M/N, E(R/\tau(R)))$. Then $f$ factors through a monomorphism $0 \neq \overline{f} : M/\ker f \rightarrow E(R/\tau(R))$. As $\tau(M/\ker(f)) \neq 0$ we get $0 \neq \tau(M/\ker f) \subseteq \ker \overline{f}$, a contradiction. □

Corollary 2.18. If $M/N$ is a $\tau$-torsion module, then $N$ is $\tau$-rational in $M$.

The following example shows that the converse of Corollary 2.18 need not be true.

Example 2.19. Consider the trivial torsion theory $(0, R M)$ on $\mathbb{Z}M$. Then $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/4\mathbb{Z}, \mathbb{Q}) = 0$ while $\tau(\mathbb{Z}/4\mathbb{Z}) \neq \mathbb{Z}/4\mathbb{Z}$.

Definition 2.20. Let $M$ be a module and $K \leq M$. We say that $K$ is a $\tau$-closed submodule of $M$, denoted by $K \leq_{\tau-c} M$, if whenever for any submodule $L$ of $M$, $\text{Hom}(L/K, E(R/\tau(R))) = 0$ implies $K = L$. If $N$ is a submodule of $M$ such that $K \leq_{\tau-c} N$ and $N$ is $\tau$-closed in $M$ then $N$ is called a $\tau$-closure of $K$ in $M$. Note that $N \leq_{\tau-c} M$ if and only if for all $N < K \leq M$, $\text{Rej}_{K/N}(E(R/\tau(R))) = 0$.

Proposition 2.21. Let $N' \leq N \leq M$. Then the following are true:

1. If $N'$ is $\tau$-closed in $M$, then $N'$ is $\tau$-closed in $N$;
2. $\text{Rej}_M(E(R/\tau(R))) \leq_{\tau-c} M$ and $N \leq_{\tau-c} M$, moreover $\text{Rej}_M(E(R/\tau(R))) \subseteq N$. Besides, $\text{Rej}_M(E(R/\tau(R)))$ is the intersection of all $\tau$-closed submodules of $M$;
3. If $K \leq_{\tau-c} M$, then $M/K$ is a $\tau$-torsion free module. Clearly the converse is not true;
4. If $N'$ is $\tau$-closed in $N$ and $N$ is $\tau$-closed in $M$, then $N'$ is $\tau$-closed in $M$;
5. the class of $\tau$-closed submodules of $M$ is closed under intersections.

Proof: (1) This is clear.

(2) Clearly $\text{Rej}_M(E(R/\tau(R))) \leq_{\tau-c} M$. Now, on the contrary, assume that $N \leq_{\tau-c} M$ and $N \not\in \text{Rej}_M(E(R/\tau(R)))$. Then there is an $x \in \text{Rej}_M(E(R/\tau(R))) \setminus N$ and so $\text{Hom}((Rx + N)/N, E(R/\tau(R))) \simeq \text{Hom}(Rx/(Rx \cap N), E(R/\tau(R))) = 0$, a contradiction.

(3) Assume that $K \leq_{\tau-c} M$ and $\tau(M/K) = C/K \neq 0$. Then since $C/K \in T_\tau$ and $E(R/\tau(R)) \in F_\tau$, we get $\text{Hom}(C/K, E(R/\tau(R))) = 0$, a contradiction. Thus $C/K = 0$.

(4) If $\text{Hom}(L/N', E(R/\tau(R))) = 0$ for some $N' \leq L \leq M$, then we have $L \nsubseteq N$ and $N' \subseteq L \cap N$. Therefore, $\text{Hom}_R((L \cap N)/N', E(R/\tau(R))) = 0$. Since $N'$ is $\tau$-closed in $N$, we get $(L \cap N) = N'$. Hence $\text{Hom}((L + N)/N, E(R/\tau(R))) = 0$ and so $N = N + L$ because $N \leq_{\tau-c} M$. From $N = N + L$ we have $L \subseteq N$, a contradiction.
(5) Let \( N_i \leq_\tau c M \) for every \( i \in I \). Then \( \text{Rej}_{M/N_i}(E(R/\tau(R))) = 0 \), for every \( i \in I \). Thus we have \( \bigoplus_{i \in I} \text{Rej}_{M/N_i}(E(R/\tau(R))) = \text{Rej}_{\bigoplus_{i \in I} M/N_i}(E(R/\tau(R))) = 0 \). Since there is a monomorphism \( f : M/\bigcap_{i \in I} N_i \rightarrow \bigoplus_{i \in I} M/N_i \), the injectivity of \( E(R/\tau(R)) \) implies \( \text{Rej}_{M/\bigcap_{i \in I} N_i}(E(R/\tau(R))) = 0 \) and so \( \bigcap_{i \in I} N_i \leq_\tau c M \). \( \Box 

\textbf{Proposition 2.22.} For a module \( M \), every submodule \( N \) of \( M \) has a \( \tau \)-closure.

\textbf{Proof:} If \( \text{Hom}(M/N, E(R/\tau(R))) = 0 \), then there is nothing to prove. Hence suppose that \( \text{Hom}(M/N, E(R/\tau(R))) \neq 0 \), \( D/N = \text{Rej}_{M/N}(E(R/\tau(R))) \), for some submodule \( N \) of \( M \). Then \( \text{Hom}(D/N, E(R/\tau(R))) = 0 \) and

\[ \text{Hom}(D'/D, E(R/\tau(R))) \neq 0 \]

for every \( D < D' \leq M \). In this case \( D \) is a \( \tau \)-closure of \( N \) in \( M \). \( \Box 

\textbf{Example 2.23.} Consider the Goldie torsion theory, where \( \mathbb{T} = \{ M | \mathcal{Z}_2(M) = M \} \), \( \mathcal{F} = \{ M | \mathcal{Z}_2(M) = 0 \} \). It is not hard to see that the idempotent radical associated to Goldie torsion theory is \( \mathcal{Z}_2 \). If we take \( \mathbb{Z} \) as a \( \mathbb{Z} \)-module, then we can easily check that \( \mathcal{Z}_2(\mathbb{Z}) = 0 \) and \( E(\mathbb{Z}) = \mathbb{Q} \). Since \( \text{Hom}_\mathbb{Z}(n\mathbb{Z}, \mathbb{Q}) \neq 0 \), for every nonzero integer \( n \), the \( \mathcal{Z}_2 \)-closure of zero submodule is itself. Since for every nonzero integer \( m \) we have \( \text{Hom}_\mathbb{Z}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Q}) = 0 \), the \( \mathcal{Z}_2 \)-closure of every nonzero \( \mathbb{Z} \)-submodule is \( \mathbb{Z} \).

3. \( \tau \)-extending modules

In this section we introduce the concept of \( \tau \)-extending modules and give an example to show that the direct sum of \( \tau \)-extending modules may not be \( \tau \)-extending.

\textbf{Definition 3.1.} A module \( M \) is called \( \tau \)-extending if every submodule of \( M \) is \( \tau \)-rational in a direct summand of \( M \).

From Example 2.23, it follows that \( \mathbb{Z} \) is \( \mathcal{Z}_2 \)-extending module.

\textbf{Proposition 3.2.} A module \( M \) is \( \tau \)-extending if and only if every \( \tau \)-closed submodule of \( M \) is a direct summand of \( M \).

\textbf{Proof:} Suppose that \( M \) is \( \tau \)-extending and \( N \) a \( \tau \)-closed submodule of \( M \). By hypothesis, \( N \) is \( \tau \)-rational in a direct summand \( D \) of \( M \), so \( D = N \). Conversely, assume that every \( \tau \)-closed submodule of \( M \) is a direct summand of \( M \). Let \( N \) be a submodule of \( M \). Also, let \( \text{Rej}_{M/N}(E(R/\tau(R))) = C/N \). Since \( C \) is \( \tau \)-closed in \( M \), then by assumption \( C \) is a direct summand of \( M \). As \( N \) is \( \tau \)-rational in \( C \), \( M \) is \( \tau \)-extending. \( \Box 

\textbf{Lemma 3.3.} The class of \( \tau \)-extending modules is closed under direct summands.

\textbf{Proof:} Let \( M = M_1 \oplus M_2 \) and \( N_1 \leq_\tau c M_1 \). We show that \( N_1 \oplus M_2 \leq_\tau c M_1 \oplus M_2 \). Let there be a submodule \( K \) such that \( N_1 \oplus M_2 \leq K \leq M \) and \( \text{Hom}(K/(N_1 \oplus M_2), E(R/\tau(R))) = 0 \). By modularity \( K = M_2 \oplus (K \cap M_1) \) and so \( \text{Hom}((K \cap M_1)/N_1, E(R/\tau(R))) = 0 \). This gives \( K \cap M_1 = N_1 \), because
$N_1 \leq_{\tau c} M_1$. Thus $K = N_1 \oplus M_2$ and so $N_1 \oplus M_2 \leq_{\tau c} M$. As $M$ is $\tau$-extending, then $N_1 \oplus M_2 \oplus L = M$. Therefore $M_1 = N_1 \oplus (M_1 \cap (M_2 \oplus L))$, so $M_1$ is $\tau$-extending.

**Lemma 3.4.** Let $M$ be $\tau$-extending and $K$ a module for which $\text{Rej}_K(E(R/\tau(R))) = K$. Then $M \oplus K$ is $\tau$-extending.

**Proof:** We may assume that $\text{Rej}_M(E(R/\tau(R))) = 0$. Let $D$ be a $\tau$-closed submodule of $M \oplus K$. Then $\text{Rej}_{M \oplus K}(E(R/\tau(R))) = \text{Rej}_K(E(R/\tau(R))) = K$ and by Proposition 2.21, we get $K \subseteq D$. This shows that $D = (M \cap D) \oplus K$. If $\text{Hom}(L/(M \cap D), E(R/\tau(R))) = 0$, for some submodule $L$ of $M$ that contains $M \cap D$, then $\text{Hom}(L + K)/D, E(R/\tau(R))) = 0$. Since $D$ is $\tau$-closed in $M \oplus K$, we get $L + K = D$ and so $M \cap D = L$. This shows that $M \cap D$ is $\tau$-closed in $M$ and since $M$ is $\tau$-extending $M = (M \cap D) \oplus N$, for some $N \leq M$. Thus $M \oplus K = (M \cap D) \oplus N \oplus K = D \oplus N$, which implies that $D$ is a direct summand of $M \oplus K$.

**Corollary 3.5.** Let $M$ be a $\tau$-extending module and $K$ a $\tau$-torsion module. Then $M \oplus K$ is $\tau$-extending.

**Lemma 3.6.** The following statements are equivalent for a module $M$:

1. $M$ is $\tau$-extending;
2. $M = \text{Rej}_M(E(R/\tau(R))) \oplus N$, and $N$ is a $\tau$-extending submodule of $M$.

**Proof:** (i)$\implies$(ii). As $\text{Rej}_M(E(R/\tau(R))) \leq_{\tau c} M$ and $M$ is $\tau$-extending, we get $M = \text{Rej}_M(E(R/\tau(R))) \oplus N$, where $N$ is a $\tau$-extending submodule of $M$, by Lemma 3.3.

(ii)$\implies$(i). This follows by Lemma 3.4.

The following example shows that a direct sum of $\tau$-extending modules need not be $\tau$-extending.

**Example 3.7.** Let $R = \mathbb{Z}$ and $\tau = 0$. Then $M_1 = M_2 = \mathbb{Z}$ are $\tau$-extending, because $\text{Hom}_\mathbb{Z}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Q}) = 0$, for every nonzero ideal $m\mathbb{Z}$ of $\mathbb{Z}$. Next we show that $M_1 \oplus M_2$ is not $\tau$-extending. For, let $K$ be the $\mathbb{Z}$-submodule of $\mathbb{Z} \oplus \mathbb{Z}$ generated by $(2, 3)$, i.e, $K = \{(2n, 3n) | n \in \mathbb{Z}\}$. Then $f : \mathbb{Z} \oplus \mathbb{Z} \rightarrow \mathbb{Q}$ defined by $f(1, 0) = 1/2$, $f(0, 1) = -1/3$, is a $\mathbb{Z}$-homomorphism with $\text{ker}(f) = \{(m, n) | f(m, n) = f(m, 0) + f(0, n) = mf(1, 0) + nf(0, 1) = m/2 - n/3 = 0\}$. Therefore $K = \text{ker}(f)$ and so $\text{Hom}_\mathbb{Z}(\mathbb{Z} \oplus \mathbb{Z})/K, \mathbb{Q}) \neq 0$. This shows that $\mathbb{Z} \oplus \mathbb{Z}$ is not $\tau$-extending.

**Theorem 3.8.** Let $M_i$ ($i = 1, 2$) be $\tau$-extending modules and $N_i$, a submodule of $M_i$ such that $M_i = \text{Rej}_{M_i}(E(R/\tau(R))) \oplus N_i$ for each $i = 1, 2$. Then $M = M_1 \oplus M_2$ is $\tau$-extending if and only if every $\tau$-closed submodule $K$ of $N_1 \oplus N_2$ with $K \cap N_1 = K \cap N_2 = 0$ is a direct summand of $M$.

**Proof:** Assume that $M = M_1 \oplus M_2$ is $\tau$-extending. Then

$$\text{Rej}_{M_1 \oplus M_2}(E(R/\tau(R))) = \text{Rej}_{M_1}(E(R/\tau(R))) \oplus \text{Rej}_{M_2}(E(R/\tau(R)))$$
is a direct summand of $M$, by Lemma 3.6. Thus there exists $N \subseteq M_1 \oplus M_2$ such that

$$M_1 \oplus M_2 = \text{Rej}_{M_1}(E(R/\tau(R))) \oplus \text{Rej}_{M_2}(E(R/\tau(R))) \oplus N.$$ 

By modularity we get

$$M_i = \text{Rej}_{M_i}(E(R/\tau(R))) \oplus ((N \oplus \text{Rej}_{M_j}(E(R/\tau(R)))) \cap M_i)$$

for $i, j = 1, 2$ with $i \neq j$. Suppose that $N_i = (N \oplus \text{Rej}_{M_j}(E(R/\tau(R)))) \cap M_i$. Then $N = N_1 \oplus N_2$ is $\tau$-extending, and hence every $\tau$-closed submodule of $N$ is a direct summand of $N$ and so a direct summand of $M$.

Conversely, assume that for each $i = 1, 2$, the module $M_i = \text{Rej}_{M_i}(E(R/\tau(R))) \oplus N_i$ is $\tau$-extending, such that every $\tau$-closed submodule $K$ of $N_1 \oplus N_2$ with $K \cap N_1 = K \cap N_2 = 0$ is a direct summand of $M$. We will show that every $\tau$-closed submodule of $M_1 \oplus M_2$ is a direct summand of $M_1 \oplus M_2$. Let $K$ be a $\tau$-closed submodule of $M_1 \oplus M_2$ and $K \cap M_i = K_i$, for $i = 1, 2$.

Note that $\text{Hom}((M_i + K)/K, E(R/\tau(R))) = 0$ iff $M_i \subseteq K$, for $i = 1, 2$. Thus $\text{Hom}(M_i/K_i, E(R/\tau(R))) = 0$ iff $M_i = K_i$, for $i = 1, 2$. It follows that $K_i$ are $\tau$-closed submodule of $M_i$, for $i = 1, 2$ and by Proposition 2.21(2), $\text{Rej}_{M_i}(E(R/\tau(R))) \subseteq K_i$. Hence $K_i = \text{Rej}_{M_i}(E(R/\tau(R))) \oplus (K_i \cap N_i)$. It is not hard to see that $K_i \cap N_i$ is a $\tau$-closed submodule of $N_i$, for $i = 1, 2$.

Since $N_1$ and $N_2$ are $\tau$-extending, $N_i = (K_i \cap N_i) \oplus L_i$, for some $L_i \subseteq N_i$. This shows that $K = K_1 \oplus K_2 \oplus (K \cap (L_1 \oplus L_2))$. We can easily check that $K \cap (L_1 \oplus L_2)$ is a $\tau$-closed submodule of $N_1 \oplus N_2$ with $K \cap (L_1 \oplus L_2) \cap N_1 = 0$, for $i = 1, 2$. By assumption $K \cap (L_1 \oplus L_2)$ is a direct summand of $M$. Assume that $(K \cap (L_1 \oplus L_2)) \oplus S = M$. Then $L_1 \oplus L_2 = (K \cap (L_1 \oplus L_2)) \oplus ((L_1 \oplus L_2) \cap S)$. It follows that $M = K_1 \oplus K_2 \oplus (K \cap (L_1 \oplus L_2)) \oplus ((L_1 \oplus L_2) \cap S) = K \oplus ((L_1 \oplus L_2) \cap S)$, which implies that every $\tau$-closed submodule of $M$ is a direct summand of $M$. Thus $M$ is a $\tau$-extending module, by Proposition 3.2.

\begin{lemma}
Let $M_1$ be an $M_2$-injective module. Then $M = M_1 \oplus M_2$ is $\tau$-extending if and only if $M_1, M_2$ are $\tau$-extending.
\end{lemma}

\begin{proof}
Let $M_i$ be $\tau$-extending, for $i = 1, 2$. Then by Lemma 3.6, there exist $L_i \subseteq M_i$ such that $M_i = \text{Rej}_{M_i}(E(R/\tau(R))) \oplus L_i$. Applying Theorem 3.8, it suffices to show that every $\tau$-closed submodule $K$ of $L_1 \oplus L_2$, with $K \cap L_1 = K \cap L_2 = 0$, is a direct summand of $L_1 \oplus L_2$. As $M_1$ is an $M_2$-injective, then by [9, Lemma 7.5], there exists a submodule $L'$ of $L_1 \oplus L_2$ for which $K \subseteq L'$ and $L_1 \oplus L' = L_1 \oplus L_2$. As $L_2$ is $\tau$-extending and $L' \simeq L_2$, then by Proposition 3.2, $L'$ is $\tau$-extending. Hence $K$ is a direct summand of $L'$ and so a direct summand of $L_1 \oplus L_2$. By Proposition 3.2, $M_1 \oplus M_2$ is $\tau$-extending. \end{proof}
Corollary 3.10. Let $M = M_1 \oplus M_2$ be an injective module. Then $M$ is $\tau$-extending if and only if $M_1, M_2$ are $\tau$-extending modules.

References


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