On the variety $C_{\text{Sub}}(D)$

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Abstract. The variety of lattices generated by lattices of all convex sublattices of distributive lattices is investigated.

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0. Introduction.

Let $L$ be a lattice. Denote by $C_{\text{Sub}}(L)$ the lattice of all convex sublattices of $L$ (including the empty set $\emptyset$). For a variety $V$ of lattices, let $C_{\text{Sub}}(V)$ denote the variety of lattices generated by $\{C_{\text{Sub}}(L); L \in V\}$. In [4], it is shown that for any proper variety $V$ of lattices, the variety $C_{\text{Sub}}(V)$ is proper and that there are uncountably many varieties $C_{\text{Sub}}(V)$.

The aim of this paper is to obtain some information about the least nontrivial such variety, i.e. about $C_{\text{Sub}}(D)$, where $D$ denotes the variety of all distributive lattices. We shall show that this variety is locally finite. The meet $C_{\text{Sub}}(D)$ with the variety of all modular lattices will be described.

1. Preliminaries.

Any interval of a lattice $L$ is a convex sublattice of $L$. Denote by $\text{Int}(L)$ the lattice of all intervals of $L$ (including $\emptyset$). Clearly, $\text{Int}(L)$ is a sublattice of $C_{\text{Sub}}(L)$. The one-element sublattices of a lattice $L$ are just atoms of both $\text{Int}(L)$ and $C_{\text{Sub}}(L)$. If $I = [a, b]$ and $J = [c, d]$ are intervals of a lattice $L$, then we have in the lattice $C_{\text{Sub}}(L)$

$$I \lor J = [a \land c, b \lor d] \quad \text{and}$$
$$I \land J = I \cap J = [a \lor c, b \land d] \quad \text{or} \quad \emptyset \quad \text{if} \quad a \lor c \not\leq b \land d.$$

One can show (by induction) that, for any lattice term $p$ in $k$ variables and any $A_1, \ldots, A_k \in C_{\text{Sub}}(L)$ the following holds:

$$p(A_1, \ldots, A_k) = \bigcup \{p(I_1, \ldots, I_k); \quad I_j \subseteq A_j, I_j \in \text{Int}(L)\}.$$

Thus, for any variety $V$ of lattices, $\text{Int}(L) \in V$ iff $C_{\text{Sub}}(L) \in V$. Especially, the variety $C_{\text{Sub}}(V)$ is generated by $\{\text{Int}(L); L \in V\}$ (see [4]).
Let $L$ be a lattice and $A$ be a sublattice of the lattice $\text{Int}(L)$. If $A$ has the least element that is not $\emptyset$, then the meet of any pair of elements from $A$ is a non-empty interval of $L$ and, clearly, the mapping $h$ of $A$ into $L^* \times L$, where $L^*$ denotes the dual lattice of $L$, defined by

$$h([a, b]) = (a, b),$$

is an embedding of $A$ into $L^* \times L$.

**Lemma 1.** Let $V$ be a self-dual variety of lattices and $L \in V$ be a lattice. Then any dual ideal of $\text{Int}(L)$ generated by an atom of $\text{Int}(L)$ belongs to $V$.

**Proof:** Any dual ideal of $\text{Int}(L)$ generated by an atom of $\text{Int}(L)$ is a sublattice of $L^* \times L \in V$. \hfill $\Box$

2. Locally finite varieties.

In this section, let $V$ denote a locally finite (any finitely generated lattice in $V$ is finite) self-dual variety of lattices.

**Theorem 1.** The variety $\text{Csub}(V)$ is locally finite.

**Proof:** Let $d(n)$ denote the cardinality of the $V$-free lattice with $n$ generators. Let $A \in V$ and let $C$ be a sublattice of $\text{Int}(A)$ generated by $n$ elements. Then there exist atoms $a_1, \ldots, a_k$ of the lattice $\text{Int}(A)$, $k \leq n$, such that $C \subseteq \{\emptyset\} \cup \{a_1\} \cup \cdots \cup \{a_k\}$. By Lemma 1, $[a_i] \in V$ and the cardinality of $C \cap [a_i]$ is at most $d(n)$. Thus the cardinality of $C$ is at most $s(n) = 1 + n \cdot d(n)$. Since the variety $\text{Csub}(V)$ is generated by $\{\text{Int}(A); A \in V\}$ and for any $A \in V$ a sublattice of $\text{Int}(A)$ with $n$ generators has at most $s(n)$ elements, the variety $\text{Csub}(V)$ is locally finite (see [3]). \hfill $\Box$

**Lemma 2.** Let $L \in V$ be a lattice and let $A$ be a finite sublattice of the lattice $\text{Int}(L)$. Then $A$ is a sublattice of $\text{Int}(K)$ for some finite sublattice $K$ of $L$.

**Proof:** Denote $M_1 = \{x \in L; [x, y] \in A$ for some $y \in L\}$ and $M_2 = \{x \in L; [y, x] \in A$ for some $y \in L\}$. The sets $M_1$ and $M_2$ are finite, the sublattice $K$ of $L$ generated by $M_1 \cup M_2$ is finite and, clearly, $A$ is a sublattice of $\text{Int}(K)$. \hfill $\Box$

For a class $K$ of lattices, let $\text{H}(K)$, $\text{S}(K)$, and $\text{P}(K)$ denote the class of all homomorphic images, sublattices, and direct products of members of $K$, respectively. For a class $K$, the variety generated by $K$ is equal to $\text{HSP}(K)$.

**Theorem 2.** Let $A \in V$ be a finite lattice. Then $A \in \text{HSP}(\text{Int}(B))$ for some finite lattice $B \in V$. If $A$ is subdirectly irreducible, then $A \in \text{HS}(\text{Int}(B))$ for some finite lattice $B \in V$.

**Proof:** Since $A \in \text{HSP}(\{\text{Int}(L); L \in V\})$, there exist lattices $L_i \in V$, $i \in I$, a sublattice $C$ of the product of $\text{Int}(L_i)$, $i \in I$, and a homomorphism $f$ of $C$ onto $A$. We can assume that $C$ is finitely generated and so, by Theorem 1, $C$ is finite. Thus we may suppose that $I$ is finite. Let $\pi_i$ denote the $i$-th projection of the product of $\text{Int}(L_j)$, $j \in I$, onto $\text{Int}(L_i)$. For any $i \in I$, $\pi_i(C)$ is a finite sublattice of $\text{Int}(L_i)$ and, by Lemma 2, $\pi_i(C)$ is a sublattice of $\text{Int}(B_i)$ for some finite sublattice $B_i$ of $L_i$. We get that the lattice $A$ belongs to $\text{HSP}(\{\text{Int}(B_i); i \in I\})$. It is easy
to show that for any pair of lattices $A, B$, $A \subseteq B$ implies $\text{Int}(A) \subseteq \text{Int}(B)$; thus $\text{Int}(B_i)$, $i \in I$ are sublattices of $\text{Int}(B)$, where $B$ is the product of all $B_i$, $i \in I$; hence $A \in HSP(B)$. If $A$ is subdirectly irreducible, then, since congruence lattices of lattices are distributive, $A \in HSP(\text{Int}(B))$ (see [1]).

\textbf{Corollary 1.} Let $A \in \text{Csub}(V)$ be a finite subdirectly irreducible lattice. Then any dual ideal of $A$ generated by an atom of $A$ belongs to the variety $V$.

\textbf{Proof:} By Theorem 2, $A \in HSP(\text{Int}(B))$ for some finite lattice $B \in V$. Thus for any atom $a \in A$, the dual ideal $[a]$ of $A$ generated by $a$ is a homomorphic image of a sublattice of a dual ideal $[d]$ of $\text{Int}(A)$, $d \neq \emptyset$. By Lemma 1, $[d] \in V$ and so $[a] \in V$, too. \hfill $\square$

\textbf{3. The variety $C_{\text{sub}}(D)$.}

Let $D$ denote the class of all distributive lattices. The class $D$ is a self-dual locally finite variety. Any finite distributive lattice is a sublattice of a finite Boolean algebra. Now we can reformulate the results of Section 2 as follows.

\textbf{Theorem 3.} The following assertions hold:

1. The variety $C_{\text{sub}}(D)$ is locally finite.
2. Let $A \in C_{\text{sub}}(D)$ be a finite subdirectly irreducible lattice. Then
   (i) $A \in HSP(\text{Int}(B))$ for some finite Boolean algebra $B$;
   (ii) for any atom $a \in A$, the dual ideal $[a]$ is a distributive lattice.

Since any locally finite variety is generated by its finite members, we can immediately obtain

\textbf{Proposition 1.} $C_{\text{sub}}(D) = HSP(\{\text{Int}(B_n); n = 2, 3, \ldots \})$, where $B_n$ denotes the Boolean algebra with $n$ atoms.

Let us remark that, for any $n \geq 2$, the lattice $\text{Int}(B_n)$ is simple. Indeed, if $\alpha$ is a nontrivial congruence relation on $\text{Int}(B_n)$, then there exist intervals $I, J$ of $B_n$ such that $I \subseteq J, I \neq J$ and $I\alpha J$. Let $c$ be an element from $J \setminus I$. Then $([c, c] \cap I)\alpha([c, c] \cap J)$, i.e. $\emptyset\alpha[c, c]$. Let $c'$ be the complement of $c$. We can easily see that $[c', c']\alpha[0, 1]$ and that $([x, x] \cap [c', c'])\alpha([x, x] \cap [0, 1])$ for any $x \in B_n$. If $x \neq c'$, we get $\emptyset\alpha[x, x]$. If $c \notin \{0, 1\}$, then we have $\emptyset\alpha[0, 0], \emptyset\alpha[1, 1]$ and so $\emptyset\alpha[0, 1]$. Now assume that $c \in \{0, 1\}$. Let $b \in B_n \setminus \{0, 1\}$. Then $\emptyset\alpha[b, b]$ and $\emptyset\alpha[b', b']$; hence $\emptyset\alpha[0, 1]$.

An interesting problem is to describe the variety $C_{\text{sub}}(D) \cap M$, where $M$ denotes the variety of all modular lattices. We shall show that this variety contains all finite lattices $M_n$ having $n$ atoms and $n + 2$ elements. Since the lattice $M_{3,3}$ pictured in Fig. 1 belongs to any variety of modular lattices that is not a subvariety of the variety $HSP(\{M_n; n = 1, 2, \ldots \})$ (see [2]) and, by Theorem 3, $M_{3,3}$ does not belong to $C_{\text{sub}}(D)$, we can get the following result.

\textbf{Theorem 4.} $C_{\text{sub}}(D) \cap M = HSP(\{M_n; n = 1, 2, \ldots \})$.

To prove Theorem 4, it suffices to show that any lattice $M_n$ is a sublattice of a lattice $\text{Int}(B)$ for some finite Boolean algebra $B$. 

Lemma 3. For any natural number \( n \geq 2 \), there exist subsets \( A_i, B_i, i = 1, 2, \ldots, n \) of \( S = \{1, 2, \ldots, \frac{n}{2}(n+1)\} \) such that the following conditions hold:

1. if \( i \neq j \), then \( A_i \cap A_j = \emptyset \) and \( B_i \cup B_j = S \);
2. \( A_i \not\subseteq B_j \) if \( (i, j) = (n, 1) \) or \( (i, j) \neq (1, n) \) and \( i < j \).

Proof: By induction on \( n \). Let \( n = 2 \). Put \( A_1 = \{1\}, A_2 = \{2\}, B_1 = \{1, 3\}, B_2 = \{1, 2\} \). Now suppose that \( k \geq 2 \) and \( A_1', A_2', \ldots, A_k', B_1', \ldots, B_k' \) are subsets of \( T = \{1, 2, \ldots, \frac{k}{2}(k+1)\} \) satisfying the conditions (1) and (2). Denote \( s = \frac{k}{2}(k+1) \) and \( A_i = A_i' \cup \{s+i\} \) for \( i = 1, 2, \ldots, k \) and \( A_{k+1} = \{s+k+1\} \). Put \( B_1 = T \setminus \{s+k+1\} \) and for all \( i, 2 \leq i \leq k-1 \), \( B_i = B_i' \cup \{s+1, \ldots, s+k+1\} \), \( B_k = B_k' \cup \{s+2, \ldots, s+k+1\} \), and finally \( B_{k+1} = \{1, 2, \ldots, s+1\} \cup \{s+k+1\} \). One can easily verify that the sets \( A_i, B_i \) are subsets of \( \{1, 2, \ldots, \frac{k+1}{2}(k+2)\} \) satisfying the required conditions (1) and (2).

Proposition 2. For any natural number \( n \geq 2 \), the lattice \( M_n \) is a sublattice of \( \text{Int}(B) \) for some finite Boolean algebra \( B \).

Proof: Denote by \( B \) the Boolean algebra of all subsets of the set \( S = \{1, 2, \ldots, \frac{n}{2}(n+1)\} \). Let \( A_i, B_i \ (i = 1, \ldots, n) \) be subsets of \( S \) satisfying the conditions (1) and (2) of Lemma 3. Put \( I_i = [A_i, B_i], i = 1, \ldots, n \). Clearly, \( I_i \in \text{Int}(B) \) and for any pair \( i, j \) with \( i \neq j \), \( I_i \cap I_j = [A_i \cap A_j, B_i \cup B_j] = [\emptyset, S] \). Since for any pair \( i, j \) with \( i \neq j \), \( A_i \not\subseteq B_j \) or \( B_i \not\subseteq A_j \), we have \( A_i \cup A_j \not\subseteq B_i \cup B_j \); thus \( I_i \cup I_j = \emptyset \). We have showed that the intervals \( I_1, \ldots, I_n \) together with \( \emptyset \) and \( [\emptyset, S] \) form a sublattice of \( \text{Int}(B) \) isomorphic to \( M_n \).

Fig. 1: \( M_{3,3} \)

References


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