Multipliers of Hankel transformable generalized functions

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Abstract. Let \( \mathcal{H}_\mu \) be the Zemanian space of Hankel transformable functions, and let \( \mathcal{H}'_\mu \) be its dual space. In this paper \( \mathcal{H}_\mu \) is shown to be nuclear, hence Schwartz, Montel and reflexive. The space \( \mathcal{O} \), also introduced by Zemanian, is completely characterized as the set of multipliers of \( \mathcal{H}_\mu \) and of \( \mathcal{H}'_\mu \). Certain topologies are considered on \( \mathcal{O} \), and continuity properties of the multiplication operation with respect to those topologies are discussed.

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1. Introduction.

Let \( \mu \in \mathbb{R} \). The space \( \mathcal{H}_\mu \), introduced by A. H. Zemanian [7], consists of all those infinitely differentiable functions \( \phi = \phi(x) \) defined on \( I = [0, \infty[ \) such that the quantities

\[
\lambda_{m,k}^\mu(\phi) = \sup_{x \in I} |x^m(x^{-1}D)^k x^{-\mu - 1/2}\phi(x)| \quad (m, k \in \mathbb{N})
\]

are finite. Endowed with the topology generated by the family of seminorms \( \{\lambda_{m,k}^\mu\}_{(m,k) \in \mathbb{N} \times \mathbb{N}} \), \( \mathcal{H}_\mu \) is a Fréchet space.

We note that this topology of \( \mathcal{H}_\mu \) can be also defined by means of the seminorms

\[
\tau_{m,k}^\mu(\phi) = \sup_{x \in I} |(1 + x^2)^m(x^{-1}D)^k x^{-\mu - 1/2}\phi(x)| \quad (m, k \in \mathbb{N}, \phi \in \mathcal{H}_\mu).
\]

The vector space \( \mathcal{O} \) of all those \( \theta \in C^\infty(I) \) such that for every \( k \in \mathbb{N} \) there exist \( n_k \in \mathbb{N}, A_k > 0 \) satisfying

\[
|(x^{-1}D)^k \theta(x)| \leq A_k (1 + x^2)^{n_k} \quad (x \in I)
\]

was shown in [7] to be a space of multipliers for \( \mathcal{H}_\mu \). Here we prove that \( \mathcal{O} \) is precisely the space of multipliers of \( \mathcal{H}_\mu \) (Section 2) and of \( \mathcal{H}'_\mu \) (Section 4). In characterizing \( \mathcal{O} \) as the space of multipliers for \( \mathcal{H}'_\mu \) we use the reflexivity of \( \mathcal{H}_\mu \), which derives from the fact, previously established in that section, that \( \mathcal{H}_\mu \) is nuclear.

Sections 3 and 5 mainly deal with the problem of topologizing \( \mathcal{O} \). We show that this can be done in such a way that the bilinear maps \( (\theta, \vartheta) \mapsto \theta \vartheta \) from \( \mathcal{O} \times \mathcal{O} \) into \( \mathcal{O} \), \( (\theta, \phi) \mapsto \theta \phi \) from \( \mathcal{O} \times \mathcal{H}_\mu \) into \( \mathcal{H}_\mu \), and \( (\theta, T) \mapsto \theta T \) from \( \mathcal{O} \times \mathcal{H}'_\mu \) into \( \mathcal{H}'_\mu \), are separately continuous (Section 3) or even hypocontinuous with respect to bounded subsets (Section 5).
We note that most of the properties established here for $\mathcal{H}_\mu$, $\mathcal{H}_\mu'$, and $\mathcal{O}$ are similar to the corresponding ones for the Schwartz space $\mathcal{S}$, its dual $\mathcal{S}'$ (the space of tempered distributions), and their space of multipliers $\mathcal{O}_M$. A difference between $\mathcal{O}$ and $\mathcal{O}_M$ should be pointed out, however: $\mathcal{O}$ is not a normal space of distributions (see the remark following Proposition 3.5).

2. Multipliers of $\mathcal{H}_\mu$.

A function $\theta = \theta(x)$ defined on $I$ is said to be a multiplier for $\mathcal{H}_\mu$ if the map $\phi \mapsto \theta \phi$ is continuous from $\mathcal{H}_\mu$ into $\mathcal{H}_\mu$. Our purpose in this section is to characterize the space of multipliers of $\mathcal{H}_\mu$. This will be done in Theorem 2.3; some preliminary results are needed.

Lemma 2.2 below provides certain useful examples of functions in $\mathcal{H}_\mu$. The following particular case of Peetre’s Inequality (see, e.g., [1, Lemma 5.2]) is helpful in constructing such functions.

Lemma 2.1. For every $\xi, \eta \in \mathbb{R}$, there holds:

\[
\frac{1 + \xi^2}{1 + \eta^2} \leq 2 \left(1 + |\xi - \eta|^2\right).
\]

Lemma 2.2. Let $\alpha \in \mathcal{D}(I)$ be such that $0 \leq \alpha \leq 1$, $\text{supp} \alpha = [1/2, 3/2]$ and $\alpha(1) = 1$. Also, let $\{x_j\}_{j \in \mathbb{N}}$ be a sequence of real numbers satisfying $x_0 > 1$ and $x_{j+1} > x_j + 1$. Define

\[
(2.1) \quad \phi(x) = x^{\mu+1/2} \sum_{j=0}^{\infty} \frac{\alpha(x - x_j + 1)}{(1 + x_j^2)^j} \quad (x \in I).
\]

Then $\phi \in \mathcal{H}_\mu$.

**Proof:** It should be noted that the sum on the right-hand side of (2.1) is finite, because the functions $\alpha(x - x_j + 1)$ have pairwise disjoint supports. In fact, if $m, k \in \mathbb{N}$ and $x_j - 1/2 \leq x \leq x_j + 1/2$, we may write:

\[
(1 + x^2)^m(x^{-1}D)^kx^{-\mu-1/2}\phi(x) = \frac{(1 + x_j^2)^m(x^{-1}D)^k\alpha(x - x_j + 1)}{(1 + x_j^2)^j - m}.
\]

Lemma 2.1 guarantees that $\tau_{m,k}^\mu(\phi) < +\infty$, thus showing that $\phi \in \mathcal{H}_\mu$, as asserted. $\square$

We are now in a position to characterize the multipliers of $\mathcal{H}_\mu$.

**Theorem 2.3.** Any one of the following statements is equivalent to the other two:

(i) The function $\theta = \theta(x)$ belongs to $C^\infty(I)$, and for every $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$ such that

\[
(1 + x^2)^{-n_k}(x^{-1}D)^k\theta(x)
\]
is bounded on \( I \).

(ii) The product \( \theta \phi \) lies in \( \mathcal{H}_\mu \) whenever \( \phi \in \mathcal{H}_\mu \), and the map \( \phi \mapsto \theta \phi \) is a continuous endomorphism of \( \mathcal{H}_\mu \).

(iii) The function \( \theta \) is infinitely differentiable on \( I \), for every \( k \in \mathbb{N} \) and every \( \phi \in \mathcal{H}_\mu \) the function \( \phi(x) (x^{-1}D)^k \theta(x) \) belongs to \( \mathcal{H}_\mu \), and the map \( \phi(x) \mapsto \phi(x) (x^{-1}D)^k \theta(x) \) is a continuous endomorphism of \( \mathcal{H}_\mu \).

**Proof:** That (i) implies (ii) has already been proved by Zemanian ([7, p. 134]).

To show that (ii) implies (iii), let us consider the function \( \phi \in \mathcal{H}_\mu \) defined by

\[
\phi(x) = x^{\mu+1/2} e^{-x^2}.
\]

(2.2)

According to (ii),

\[
\psi(x) = x^{\mu+1/2} \theta(x) e^{-x^2}
\]

(2.3)

lies in \( \mathcal{H}_\mu \), so that

\[
\theta(x) = x^{\mu+1/2} \psi(x) e^{-x^2}
\]

(2.4)

is infinitely differentiable on \( I \). At this point, it suffices to show that \( (x^{-1}D)^k \theta(x) \) is a multiplier of \( \mathcal{H}_\mu \) whenever \( \theta \) is. But this can be easily established by induction on \( k \), taking into account the formula

\[
\phi(x) (x^{-1}D) \theta(x) = \frac{x^{\mu+1/2} (x^{-1}D) x^{-\mu-1/2} \theta(x) \phi(x) - \theta(x) x^{\mu+1/2} (x^{-1}D) x^{-\mu-1/2} \phi(x)}{x^{\mu+1/2} (x^{-1}D)^k x^{-\mu-1/2} \phi(x)}
\]

along with the fact that if \( \phi \) is in \( \mathcal{H}_\mu \) then so is

\[
x^{\mu+1/2} (x^{-1}D)^k x^{-\mu-1/2} \phi(x).
\]

Finally, let \( \theta(x) \) satisfy (iii). Since (2.2) belongs to \( \mathcal{H}_\mu \), so does (2.3). Then \( \theta(x) \) can be represented by (2.4), and, in particular, the limit \( \lim_{x \to 0^+} \theta(x) \) exists.

According to (iii), each \( (x^{-1}D)^k \theta(x) \) is a multiplier of \( \mathcal{H}_\mu \), and we conclude that \( \lim_{x \to 0^+} (x^{-1}D)^k \theta(x) \) exists for all \( k \in \mathbb{N} \).

Arguing by contradiction, let us assume that (i) is false. Then there exist \( k \in \mathbb{N} \) and a sequence \( \{x_j\}_{j \in \mathbb{N}} \) of real numbers, which, by what has been just proved, may be chosen so that \( x_0 > 1 \) and \( x_{j+1} > x_j + 1 \), such that:

\[
| (x^{-1}D)^k \theta(x) |_{x=x_j} > (1 + x_j^2)^j.
\]

The function \( \phi \in \mathcal{H}_\mu \) constructed by means of \( \{x_j\}_{j \in \mathbb{N}} \) as in Lemma 2.2 plainly satisfies

\[
| x_j^{-\mu-1/2} \phi(x) (x^{-1}D)^k \theta(x) |_{x=x_j} | > \alpha(1) = 1 \quad (j \in \mathbb{N}),
\]

contradicting (iii).

\[\Box\]
3. Topology and properties of the space of multipliers.

Following [7], we denote by $\mathcal{O}$ the linear space of all those $\theta \in C^\infty(I)$ such that for every $k \in \mathbb{N}$ there exist $n_k \in \mathbb{N}$, $A_k > 0$ satisfying

$$|(x^{-1}D)^k \theta(x)| \leq A_k (1 + x^2)^{n_k} \quad (x \in I).$$

The equivalence between the conditions (i) and (ii) in Theorem 2.3 above characterizes $\mathcal{O}$ as the space of multipliers of $H_\mu$, with independence of the value of the real parameter $\mu$. However, once $\mu$ has been fixed, the condition (iii) suggests to introduce on $\mathcal{O}$ the (separating) family of seminorms

$$\Gamma_\mu = \{\gamma^\mu_{\phi,k} : \phi \in H_\mu, \; k \in \mathbb{N}\},$$

where

$$\gamma^\mu_{\phi,k} (\theta) = \sup_{x \in I} |x^{-\mu-1/2} \phi(x)(x^{-1}D)^k \theta(x)|.$$

Since the map $\phi(x) \mapsto x^{\nu-\mu} \phi(x) = \varphi(x)$ establishes an isomorphism between $H_\mu$ and $H_{\nu}$ for any $\mu, \nu \in \mathbb{R}$, the equality $\gamma^\mu_{\phi,k}(\theta) = \gamma^\nu_{\varphi,k}(\theta)$ holds whenever $k \in \mathbb{N}$ and $\theta \in \mathcal{O}$. Therefore, all families $\Gamma_\mu$ ($\mu \in \mathbb{R}$) define one and the same topology on $\mathcal{O}$. In the sequel, unless otherwise stated, it will always be assumed that $\mathcal{O}$ is endowed with this topology, and $\mu$ will be any real number.

Remarks. (i) If $\theta \in C^\infty(I)$ is such that $\gamma^\mu_{\phi,k}(\theta) < +\infty$ for every $\phi \in H_\mu$ and $k \in \mathbb{N}$, then $\theta \in \mathcal{O}$. In fact, fix $\phi \in H_\mu$, $m, k \in \mathbb{N}$ and for $0 \leq p \leq k$ define $\phi_p \in H_\mu$ by

$$\phi_p(x) = (1 + x^2)^m x^{\mu+1/2}(x^{-1}D)^{k-p} x^{-\mu-1/2} \phi(x) \quad (x \in I).$$

Since

$$(1 + x^2)^m (x^{-1}D)^k x^{-\mu-1/2}(\theta \phi)(x) = \sum_{p=0}^k \binom{k}{p} x^{-\mu-1/2} \phi_p(x)(x^{-1}D)^p \theta(x) \quad (x \in I),$$

necessarily

$$(3.1) \quad \tau^\mu_{m,k}(\theta \phi) \leq \sum_{p=0}^k \binom{k}{p} \gamma^\mu_{\phi_p,p}(\theta).$$

In general

$$\tau^\mu_{m,k}(\phi(x)\left(\frac{1}{x}D\right)^k \theta(x)) \leq \sum_{p=0}^k \binom{k}{p} \gamma^\mu_{\phi_p,p+n}(\theta), \quad (n \in \mathbb{N}).$$
Our assertion now follows as in the proof that (iii) implies (i) in Theorem 2.3.

(ii) The topology of \( \mathcal{O} \) may be also generated by means of the family of seminorms \( \{ \gamma_{m,k}^\mu : (m, k) \in \mathbb{N} \times \mathbb{N}, \phi \in \mathcal{H}_\mu \} \), where

\[
\gamma_{m,k}^\mu(\theta) = \tau_{m,k}^\mu(\theta\phi) \quad (m, k \in \mathbb{N}, \phi \in \mathcal{H}_\mu).
\]

Certainly, let \( k \in \mathbb{N} \) and, for every \( \phi \in \mathcal{H}_\mu \) and every \( p \in \mathbb{N} \) with \( 0 \leq p \leq k \), define \( \phi_p \in \mathcal{H}_\mu \) by

\[
\phi_p(x) = x^{\mu+1/2}(x^{-1}D)^p x^{-\mu-1/2}\phi(x) \quad (x \in I).
\]

If \( \phi \in \mathcal{H}_\mu \) and \( \theta \in \mathcal{O} \), the equality

\[
x^{-\mu-1/2}\phi(x)(x^{-1}D)^k\theta(x) = \sum_{p=0}^{k} (-1)^p \binom{k}{p} (x^{-1}D)^{k-p} x^{-\mu-1/2}(\theta\phi_p)(x) \quad (x \in I)
\]

then shows that

\[
\gamma_{\phi,k}^\mu(\theta) \leq \sum_{p=0}^{k} \binom{k}{p} \gamma_{0,k-p;\phi_p}(\theta).
\]

Along with (3.1), this estimate proves our assertion.

**Proposition 3.1.** The identity map \( \mathcal{O} \hookrightarrow \mathcal{E}(I) \) is continuous.

**Proof:** It is enough to observe that

\[
D^k\theta(x) = \frac{1}{x^{-\mu-1/2}\phi(x)} \sum_{p=0}^{k} C_p x^{\alpha(p)} x^{-\mu-1/2}\phi(x)(x^{-1}D)^{\beta(p)}\theta(x) \quad (x \in I)
\]

for every \( k \in \mathbb{N} \) and every \( \theta \in \mathcal{O} \), where \( \phi(x) = x^{\mu+1/2}e^{-x^2} \quad (x \in I) \) belongs to \( \mathcal{H}_\mu \), \( C_p > 0 \ (0 \leq p \leq k) \) are suitable constants, and \( \alpha(p) \leq k, \beta(p) \leq k \ (0 \leq p \leq k) \) denote nonnegative integers, with \( C_k = 1 \) and \( \alpha(k) = \beta(k) = k \). \( \square \)

**Proposition 3.2.** The linear topological space \( \mathcal{O} \) is locally convex, Hausdorff, nonmetrizable, and complete.

**Proof:** The only property that needs to be checked out is completeness.

Let \( \{\theta_i\}_{i \in J} \) be a Cauchy net in \( \mathcal{O} \). Since \( \mathcal{O} \) injects continuously into \( \mathcal{E}(I) \) (Proposition 3.1), \( \{\theta_i\}_{i \in J} \) is also a Cauchy net in \( \mathcal{E}(I) \). \( \mathcal{E}(I) \) being complete, \( \{\theta_i\}_{i \in J} \) converges to some \( \theta \in \mathcal{E}(I) \) in \( \mathcal{E}(I) \). We must show that \( \theta \in \mathcal{O} \) and that \( \{\theta_i\}_{i \in J} \) converges to \( \theta \) in the topology of \( \mathcal{O} \).

Fix \( \phi \in \mathcal{H}_\mu \), \( k \in \mathbb{N} \), \( \varepsilon > 0 \). By hypothesis, there exists \( \iota_0 = \iota_0(\phi, k, \varepsilon) \in J \) such that

\[
(3.2) \quad \gamma_{\phi,k}^\mu(\theta_{\iota} - \theta_{\iota'}) < \varepsilon \quad (\iota, \iota' \geq \iota_0).
\]
Let us consider \( x \in I, \eta > 0. \) Since \( \{\theta_i\}_{i \in J} \) converges to \( \theta \) in \( E(I) \), there holds
\[
(3.3) \quad \left| x^{-\mu-1/2} \phi(x) (x^{-1} D)^k (\theta - \theta_i')(x) \right| < \eta
\]
for some \( i' = i'(\phi, x, \eta) \geq t_0 \). The combination of (3.2) and (3.3) yields
\[
\left| x^{-\mu-1/2} \phi(x) (x^{-1} D)^k (\theta - \theta_i)(x) \right| < \varepsilon + \eta \quad (i \geq t_0),
\]
and from the arbitrariness of \( x \) and \( \eta \), we infer that
\[
\gamma_{\phi,k}^\mu(\theta - \theta_i) \leq \varepsilon \quad (i \geq t_0).
\]
With the inequality
\[
\gamma_{\phi,k}^\mu(\theta) \leq \gamma_{\phi,k}^\mu(\theta - \theta_i) + \gamma_{\phi,k}^\mu(\theta_i) \quad (i \geq t_0)
\]
we finally prove that \( \theta \in O \) and \( \{\theta_i\}_{i \in J} \) converges to \( \theta \) in \( O \).

The next Proposition 3.3 collects several continuity properties of certain operators on \( O \).

**Proposition 3.3.** The following holds:
(i) The bilinear map
\[
O \times O \to O \\
(\theta, \vartheta) \mapsto \theta \vartheta
\]
is separately continuous.
(ii) If \( R(x) = P(x)/Q(x) \), where \( P(x) \) and \( Q(x) \) are polynomials and \( Q \) does not vanish in \([0, \infty[\), then the map \( \theta(x) \mapsto R(x^2)\theta(x) \) is continuous from \( O \) to \( O \).
(iii) For every \( k \in \mathbb{N} \), the map \( \theta(x) \mapsto (x^{-1} D)^k \theta(x) \) is continuous from \( O \) to \( O \).

**Proof:** Let \( \theta \in O \), \( k \in \mathbb{N} \), and for \( 0 \leq p \leq k \) let \( n_p \in \mathbb{N} \), \( A_p > 0 \) be such that
\[
\left| (x^{-1} D)^p \theta(x) \right| \leq A_p (1 + x^2)^{n_p} \quad (x \in I).
\]
If \( \phi \in \mathcal{H}_\mu \), set
\[
\phi_p(x) = (1 + x^2)^{n_p} \phi(x) \quad (x \in I).
\]
Note that \( \phi_p \in \mathcal{H}_\mu \). The formula
\[
x^{-\mu-1/2} \phi(x) (x^{-1} D)^k (\theta \vartheta)(x) =
= \sum_{p=0}^k \binom{k}{p} x^{-\mu-1/2} \phi_p(x) (x^{-1} D)^p \theta(x) (1 + x^2)^{n_p} (x^{-1} D)^{k-p} \vartheta(x),
\]
valid for all \( x \in I \), leads to the inequality
\[
\gamma_{\phi,k}^\mu(\theta \vartheta) \leq \sum_{p=0}^k \binom{k}{p} A_p \gamma_{\phi_p,k-p}^\mu(\vartheta),
\]
which proves (i).

Assertion (ii) may be immediately deduced from (i) and from Lemma 5.3.1 in [7], whereas (iii) derives from the relationship
\[
\gamma_{\phi,k}^\mu((x^{-1} D)^k \theta(x)) = \gamma_{\phi,k+p}^\mu(\theta).
\]
\[\square\]
Proposition 3.4. The bilinear map

\[ \mathcal{O} \times \mathcal{H}_\mu \rightarrow \mathcal{H}_\mu \]

\[ (\theta, \phi) \mapsto \theta \phi \]

is separately continuous.

Proof: See Theorem 2.3 and part (i) of the remark preceding Proposition 3.1. □

Proposition 3.5. The map \( \varphi(x) \mapsto x^{-\mu-1/2} \varphi(x) \) is continuous from \( \mathcal{H}_\mu \) into \( \mathcal{O} \).

Proof: There holds:

\[ \gamma_{\phi,k}^\mu(x^{-\mu-1/2} \varphi(x)) \leq \sup_{x \in I} |x^{-\mu-1/2} \varphi(x)| \lambda_{0,k}^\mu(\varphi) \quad (\varphi, \phi \in \mathcal{H}_\mu, \ k \in \mathbb{N}). \]

□

Remark. We claim that the test space \( \mathcal{D}(I) \) is not dense in \( x^{-\mu-1/2} \mathcal{H}_\mu \) with respect to the topology of \( \mathcal{O} \). Admitting for the moment the veracity of this assertion, it follows from Proposition 3.5 that \( \mathcal{D}(I) \) is not dense in \( \mathcal{O} \), which prevents \( \mathcal{O} \) from being a normal space of distributions. This differs from the case of Schwartz multipliers (cf. [1, Theorem 4.7]).

To prove the claim, take \( \varphi \in \mathcal{H}_\mu \) and assume (to reach a contradiction) that \( \{x^{-\mu-1/2} \alpha_{\iota}(x)\}_{\iota \in J} \) is a net in \( \mathcal{D}(I) \), converging to \( x^{-\mu-1/2} \varphi(x) \) in the topology of \( \mathcal{O} \). Given \( k \in \mathbb{N} \), \( \varepsilon > 0 \), there exists \( \iota_0 = \iota_0(k, \varepsilon) \in J \), with

\[ |e^{-x^2} (x^{-1} D)^k x^{-\mu-1/2} (\alpha_{\iota_0} - \varphi)(x) | < \varepsilon/e \quad (x \in I). \]

For \( x \in ]0, 1[ \), we may write:

\[ |(x^{-1} D)^k x^{-\mu-1/2} (\alpha_{\iota_0} - \varphi)(x) | \leq e |e^{-x^2} (x^{-1} D)^k x^{-\mu-1/2} (\alpha_{\iota_0} - \varphi)(x) | < \varepsilon. \]

Therefore, to every \( k \in \mathbb{N} \) and every \( n = 1, 2, 3, \ldots \) there corresponds \( \iota_n \in J \), \( x_n \in ]0, 1/n[ \), such that

\[ |(x^{-1} D)^k x^{-\mu-1/2} \varphi(x)|_{x=x_n} | \leq |(x^{-1} D)^k x^{-\mu-1/2} (\alpha_{\iota_0} - \varphi)(x)|_{x=x_n} | + |(x^{-1} D)^k x^{-\mu-1/2} \alpha_{\iota_n}(x)|_{x=x_n} | < 1/n, \]

whence

\[ \lim_{n \to \infty} (x^{-1} D)^k x^{-\mu-1/2} \varphi(x)|_{x=x_n} = 0. \]

However, the particularizations \( \varphi(x) = x^{\mu+1/2} e^{-x^2} \) and \( k = 0 \) lead to

\[ \lim_{x \to 0^+} (x^{-1} D)^k x^{-\mu-1/2} \varphi(x) = 1, \]

thus yielding a contradiction, as expected.
Proposition 3.6. Set $\mu \geq -1/2$. Given $\theta \in \mathcal{O}$, the function $x^{\mu+1/2}\theta(x)$ defines an element in $\mathcal{H}_\mu'$ by the formula

\begin{equation}
\langle x^{\mu+1/2}\theta(x), \phi(x) \rangle = \int_0^\infty x^{\mu+1/2}\theta(x)\phi(x) \, dx \quad (\phi \in \mathcal{H}_\mu),
\end{equation}

and the map $\theta(x) \mapsto x^{\mu+1/2}\theta(x)$ is continuous from $\mathcal{O}$ into $\mathcal{H}_\mu'$.

PROOF: Take $\theta \in \mathcal{O}$, $\phi \in \mathcal{H}_\mu$, and choose $r \in \mathbb{N}$, $A_r > 0$ satisfying

$|\theta(x)| \leq A_r (1 + x^2)^r \quad (x \in I)$.

Also, let $s \in \mathbb{N}$, $s > \mu + 1$, be such that

$C^\mu_s = \int_0^\infty \frac{x^{2\mu+1}}{(1 + x^2)^s} \, dx < +\infty$.

Upon multiplying and dividing the integrand in (3.4) by $x^{-\mu-1/2}(1 + x^2)^s$ we find that:

$|\langle x^{\mu+1/2}\theta(x), \phi(x) \rangle| \leq A_r C^\mu_s \tau^\mu_{\tau+s,0}(\phi)$,

and that:

$|\langle x^{\mu+1/2}\theta(x), \phi(x) \rangle| \leq C^\mu_s \gamma^\mu_{\psi,0}(\theta)$,

where $\psi(x) = (1 + x^2)^s \phi(x) \in \mathcal{H}_\mu$. \Box

4. Multipliers of $\mathcal{H}_\mu'$.

Next we aim to characterize $\mathcal{O}$ as the space of multipliers of $\mathcal{H}_\mu'$ ($\mu \in \mathbb{R}$). The reflexivity of $\mathcal{H}_\mu$ will be needed for that purpose. In Proposition 4.2 we prove that $\mathcal{H}_\mu$ is nuclear ([4, Definition III.50.1]), a property stronger than reflexivity; to this end, the following is useful.

Lemma 4.1. Let $m, k \in \mathbb{N}$, and let $\phi \in \mathcal{H}_\mu$. There holds:

$$\sum_{k=0}^m \sup_{x \in I} |(1 + x^2)^m (x^{-1}D)^k x^{-\mu-1/2} \phi(x)| \leq$$

$$\leq (m + 1) \sum_{k=0}^{m+1} \int_0^\infty |(1 + t^2)^{m+1} (t^{-1}D)^k t^{-\mu-1/2} \phi(t)| \, dt.$$

PROOF: In fact, we have:

$$(1 + x^2)^m (x^{-1}D)^k x^{-\mu-1/2} \phi(x) = - \int_x^\infty \left( (1 + t^2)^m (t^{-1}D)^k t^{-\mu-1/2} \phi(t) \right) \, dt$$

$$= - \int_x^\infty 2mt(1 + t^2)^{m-1} (t^{-1}D)^k t^{-\mu-1/2} \phi(t) \, dt$$

$$- \int_x^\infty t(1 + t^2)^m (t^{-1}D)^{k+1} t^{-\mu-1/2} \phi(t) \, dt \quad (x \in I).$$
Since $2t \leq 1 + t^2$ ($t \in I$), it follows that

$$
|(1 + x^2)^m (x^{-1} D)^k x^{-\mu - 1/2} \phi(x)| \leq m \int_0^\infty |(1 + t^2)^m (t^{-1} D)^k t^{-\mu - 1/2} \phi(t)| \, dt
$$

+ $\int_0^\infty |(1 + t^2)^{m+1} (t^{-1} D)^{k+1} t^{-\mu - 1/2} \phi(t)| \, dt$ ($x \in I$),

whence the lemma. □

**Proposition 4.2.** The space $H_\mu$ is nuclear.

**Proof:** Let $m, k \in \mathbb{N}$, and let $\phi \in H_\mu$. For $t \in I$ and $0 \leq k \leq m + 2$, define $u_{t,k} \in H'_\mu$ by the formula:

$$
\langle u_{t,k}, \phi \rangle = (1 + t^2)^{m+2} (t^{-1} D)^k t^{-\mu - 1/2} \phi(t) \quad (\phi \in H_\mu),
$$

and consider

$$
V = \{ \phi \in H_\mu : \sum_{k=0}^{m+2} \sup_{t \in I} |(1 + t^2)^{m+2} (t^{-1} D)^k t^{-\mu - 1/2} \phi(t)| < 1 \}.
$$

Note that $V$ is a neighborhood of the origin in $H_\mu$, and that each $u_{t,k}$ ($t \in I$, $0 \leq k \leq m + 2$) belongs to $V^\circ$, the polar set of $V$. Thus, a positive Radon measure $\mu$ may be defined on $V^\circ$ by the equation:

$$
\langle \mu, \varphi \rangle = \int_{V^\circ} \varphi \, d\mu = (m + 1) \sum_{k=0}^{m+2} \int_0^\infty \varphi(u_{t,k})(1 + t^2)^{-1} \, dt \quad (\varphi \in C(V^\circ)).
$$

Lemma 4.1 now implies:

$$
\sum_{k=0}^{m} \sup_{x \in I} |(1 + x^2)^m (x^{-1} D)^k x^{-\mu - 1/2} \phi(x)| \leq (m + 1) \sum_{k=0}^{m+2} \int_0^\infty |(1 + t^2)^{m+1} (t^{-1} D)^{k+1} t^{-\mu - 1/2} \phi(t)| \, dt
$$

$$
= (m + 1) \sum_{k=0}^{m+2} |\langle u_{t,k}, \phi \rangle| (1 + t^2)^{-1} \, dt
$$

$$
= \int_{V^\circ} |\langle u, \phi \rangle| \, d\mu(u) \quad (\phi \in H_\mu).
$$

Since the sets

$$
V(m, \varepsilon) = \{ \phi \in H_\mu : \sum_{k=0}^{m} \sup_{x \in I} |(1 + x^2)^m (x^{-1} D)^k x^{-\mu - 1/2} \phi(x)| < \varepsilon \} \quad (m \in \mathbb{N}, \varepsilon > 0)
$$

form a basis of neighborhoods of the origin in $H'_\mu$, the nuclearity of this space follows from [3, Proposition 4.1.5]. □

Once that Proposition 4.2 has been established, a number of consequences may be deduced by applying general properties of nuclear spaces.
Corollary 4.3. The space $\mathcal{H}'_\mu$ is nuclear with respect to its strong topology.

Proof: See [4, Proposition III.50.6]. □

Corollary 4.4. $\mathcal{H}_\mu$ (with its usual topology) and $\mathcal{H}'_\mu$ (with the strong topology) are Schwartz spaces.

Proof: This is derived from [5, Proposition 3.2.5]. □

Corollary 4.5. The space $\mathcal{H}_\mu$ is Montel, hence reflexive.

Proof: Fréchet-Schwartz spaces are Montel ([2, Corollary to Proposition 3.15.4]), and Montel spaces are reflexive ([2, Corollary to Proposition 3.9.1]). □

We turn to the study of the multipliers of $\mathcal{H}'_\mu$.

Definition 4.6. For $\theta \in O$ and $T \in \mathcal{H}'_\mu$, $\theta T$ is defined by transposition:

$$\langle \theta T, \varphi \rangle = \langle T, \theta \varphi \rangle \quad (\varphi \in \mathcal{H}_\mu).$$

Proposition 3.4 implies that $\theta T \in \mathcal{H}'_\mu$ and that each map $T \mapsto \theta T$ is continuous from $\mathcal{H}'_\mu$ to $\mathcal{H}'_\mu$. By applying the universal property of initial topologies, we also find that the map $\theta \mapsto \theta T$ is continuous from $O$ into $\mathcal{H}'_\mu$ if the latter is equipped with its weak* topology. We are thus led to the following.

Proposition 4.7. The bilinear map

$$O \times \mathcal{H}'_\mu \rightarrow \mathcal{H}'_\mu$$

$$(\theta, T) \mapsto \theta T$$

is separately continuous when $\mathcal{H}'_\mu$ is endowed with its weak* topology.

Given $a > 0$ and $\mu \in \mathbb{R}$, $B_{\mu,a}$ (see [6]) is the subspace of $\mathcal{H}_\mu$ formed by all those functions $\psi = \psi(x)$ infinitely differentiable on $I$ such that $\psi(x) = 0$ ($x \geq a$), for which the quantities

$$\lambda_k(\psi) = \sup_{x \in I} |(x^{-1}D)^k x^{-\mu - 1/2} \psi(x)| \quad (k \in \mathbb{N})$$

are finite. When equipped with the topology generated by the family of seminorms $\{\lambda_k\}_{k \in \mathbb{N}}$, $B_{\mu,a}$ becomes a Fréchet space. It is easy to see that $B_{\mu,a} \subset B_{\mu,b}$ if $0 < a < b$, and that $B_{\mu,a}$ inherits from $B_{\mu,b}$ its own topology. These facts allow us to define $B_{\mu} = \bigcup_{a > 0} B_{\mu,a}$ as the inductive limit of the family $\{B_{\mu,a}\}_{a > 0}$. The space $B_{\mu}$ turns out to be dense in $\mathcal{H}_\mu$.

Definition 4.8. Let $\theta \in C^\infty(I)$ be such that $(x^{-1}D)^k \theta(x)$ is bounded in a neighborhood of zero for every $k \in \mathbb{N}$. If $T \in \mathcal{H}'_\mu$ then $T$ lies in $B'_\mu$, the dual space of $B_\mu$, and $\theta T \in B'_\mu$ may be consistently defined by the formula

$$\langle \theta T, \psi \rangle = \langle T, \theta \psi \rangle \quad (\psi \in B_\mu).$$

We are now ready to prove that the space of multipliers of $\mathcal{H}'_\mu$ is precisely $O$:
Theorem 4.9. Assume that $\theta \in C^\infty(I)$ is such that each $(x^{-1}D)^k \theta(x)$ $(k \in \mathbb{N})$ is bounded in a neighborhood of zero. If, for every $T \in \mathcal{H}_\mu^\prime$, the functional $\theta T \in \mathcal{B}_\mu^\prime$ (given by Definition 4.8) can be (a fortiori uniquely) extended up to $\mathcal{H}_\mu$ as a member of $\mathcal{H}_\mu^\prime$ in such a way that the map $\theta \mapsto \theta T$ is continuous from $\mathcal{H}_\mu^\prime$ into itself, then $\theta \in \mathcal{O}$.

Proof: Let $\phi \in \mathcal{H}_\mu$. Our hypotheses imply that the linear functional $T \mapsto \langle \theta T, \phi \rangle$ is continuous on $\mathcal{H}_\mu^\prime$. By the reflexivity of $\mathcal{H}_\mu$ (Corollary 4.5), there exists $\varphi \in \mathcal{H}_\mu$ satisfying

$$\langle \theta T, \phi \rangle = \langle T, \varphi \rangle \quad (T \in \mathcal{H}_\mu^\prime).$$

In particular:

$$\langle \theta \phi, \psi \rangle = \langle \theta \psi, \phi \rangle = \langle \psi, \varphi \rangle = \langle \varphi, \psi \rangle \quad (\psi \in \mathcal{B}_\mu).$$

Thus, $\theta \phi = \varphi \in \mathcal{H}_\mu$. Since the space of multipliers of $\mathcal{H}_\mu$ is $\mathcal{O}$ (Theorem 2.3), we conclude that $\theta \in \mathcal{O}$. \hfill \Box

5. Another topology on $\mathcal{O}$.

Let $\mu$ be any real number, and let $\mathcal{B}_\mu$ denote the family of all bounded subsets of $\mathcal{H}_\mu$. Throughout this section we shall assume that $\mathcal{O}$ is endowed with the topology generated by the family of seminorms

$$\gamma_{B,k}^\mu = \sup \{ \gamma_{\phi,k}^\mu : \phi \in B \} \quad (B \in \mathcal{B}_\mu, k \in \mathbb{N}).$$

Remark. Clearly, the topology just defined on $\mathcal{O}$ is finer than that introduced in Section 3. As before, any two spaces $\mathcal{H}_\mu$ and $\mathcal{H}_\nu$ being isomorphic, this topology does not depend on the parameter $\mu$.

Proposition 5.1. The topological vector space $\mathcal{O}$ is locally convex, Hausdorff, nonmetrizable, and complete.

Proof: Again, the only property to be checked out is completeness.

Let $\{\theta_i\}_{i \in J}$ be a Cauchy net in $\mathcal{O}$. Since $\{\theta_i\}_{i \in J}$ is also Cauchy with respect to the topology considered on $\mathcal{O}$ in Section 3 above (see the preceding remark), there exists $\theta \in \mathcal{O}$ such that $\{\theta_i\}_{i \in J}$ converges to $\theta$ in that topology.

Take $B \in \mathcal{B}_\mu$, $k \in \mathbb{N}$, $\varepsilon > 0$. By hypothesis, there exists $\iota_0 = \iota_0(B, k, \varepsilon) \in J$ such that

$$\gamma_{B,k}^\mu(\theta_i - \theta_{i'}) < \varepsilon/2 \quad (i, i' \geq \iota_0).$$

Moreover, as just observed, to every $\phi \in B$ there corresponds $\iota' = \iota'(\phi, k, \varepsilon) \geq \iota_0$ satisfying

$$\gamma_{\phi,k}^\mu(\theta_{i'} - \theta) < \varepsilon/2.$$

A combination of the last two inequalities shows that

$$\gamma_{B,k}^\mu(\theta_i - \theta) < \varepsilon \quad (i \geq \iota_0).$$

Therefore, $\{\theta_i\}_{i \in J}$ converges to $\theta$ in $\mathcal{O}$. \hfill \Box
**Proposition 5.2.** The bilinear map

\[ \mathcal{O} \times \mathcal{H}_\mu \to \mathcal{H}_\mu \]

\[ (\theta, \phi) \mapsto \theta \phi \]

is hypocontinuous.

**Proof:** That (5.2) is separately continuous follows from Proposition 3.4 and from the remark preceding Proposition 5.1 above.

Since \( \mathcal{H}_\mu \) is a Fréchet space, the uniform boundedness principle guarantees the hypocontinuity with respect to the bounded subsets of \( \mathcal{O} \). On the other hand, take \( m, k \in \mathbb{N} \), and for every \( \phi \in \mathcal{H}_\mu \) and every \( p \in \mathbb{N} \), \( 0 \leq p \leq k \), define \( \phi_p \in \mathcal{H}_\mu \) by

\[ \phi_p(x) = (1 + x^2)^m x^{\mu+1/2} (x^{-1} D)^k p x^{-\mu-1/2} \phi(x) \quad (x \in I). \]

Leibniz’s rule shows that the map \( \phi \mapsto \phi_p \) is continuous from \( \mathcal{H}_\mu \) into \( \mathcal{H}_\mu \). Denoting by \( B_p \in \mathfrak{B}_\mu \) the image of \( B \in \mathfrak{B}_\mu \) through this map, it can be proved, as in the part (i) of the remark preceding Proposition 3.1 that

\[ \tau_{\mu,m,k}^\mu (\theta \phi) \leq \sum_{p=0}^k \binom{k}{p} \gamma_{B_p,\mu}^\mu (\theta) \quad (\theta \in \mathcal{O}, \phi \in B). \]

Thus, (5.2) is \( \mathfrak{B}_\mu \)-hypocontinuous. \( \square \)

It should be observed that the topology generated on \( \mathcal{O} \) by the seminorms (5.1) is compatible with the family

\[ \gamma_{m,k;B}^\mu (\theta) = \sup \{ \tau_{m,k}^\mu (\theta \phi) : \phi \in B \} \quad (m, k \in \mathbb{N}, B \in \mathfrak{B}_\mu). \]

In fact, let \( k \in \mathbb{N} \). For every \( p \in \mathbb{N} \) with \( 0 \leq p \leq k \), the map \( \phi \mapsto \phi_p \), defined from \( \mathcal{H}_\mu \) into \( \mathcal{H}_\mu \) by the formula

\[ \phi_p(x) = x^{\mu+1/2} (x^{-1} D)^p x^{-\mu-1/2} \phi(x) \quad (x \in I), \]

is continuous; as before, we denote by \( B_p \in \mathfrak{B}_\mu \) the image of \( B \in \mathfrak{B}_\mu \) through this map. Now, the argument in the part (ii) of the remark preceding Proposition 3.1 shows that

\[ \gamma_{B,k}^\mu (\theta) \leq \sum_{p=0}^k \binom{k}{p} \gamma_{0,k-p;B_p}^\mu (\theta) \quad (B \in \mathfrak{B}_\mu, k \in \mathbb{N}, \theta \in \mathcal{O}). \]

Along with (5.3), this estimate proves our assertion.
Proposition 5.3. The bilinear map
\[ O \times \mathcal{H}'_{\mu} \rightarrow \mathcal{H}'_{\mu} \]
\[ (\theta, T) \mapsto \theta T \]
is separately continuous when \( \mathcal{H}'_{\mu} \) is endowed either with its weak* or with its strong topology.

Proof: The continuity in the second variable follows from [4, Propositions II.19.5 and II.35.8]. On the other hand, let \( T \in \mathcal{H}'_{\mu}, \theta \in O, B \in \mathfrak{B}_{\mu}. \) There exist \( r \in \mathbb{N} \) and a constant \( C > 0 \) such that
\[ |\langle T, \varphi \rangle| \leq C \max_{0 \leq m, k \leq r} \tau_{m,k}^{\mu}(\varphi) \quad (\varphi \in \mathcal{H}_{\mu}), \]
Hence
\[ |\langle \theta T, \phi \rangle| = |\langle T, \theta \phi \rangle| \leq C \max_{0 \leq m, k \leq r} \tau_{m,k}^{\mu}(\theta \phi) \quad (\phi \in B), \]
which leads to the inequality
\[ \sup\{ |\langle \theta T, \phi \rangle| : \phi \in B \} \leq C \max_{0 \leq m, k \leq r} \gamma_{m,k;B}^{\mu}(\theta). \]

Proposition 5.4. The bilinear map
\[ O \times O \rightarrow O \]
\[ (\theta, \vartheta) \mapsto \theta \vartheta \]
is hypocontinuous.

Proof: Let \( \mathfrak{B} \) denote the family of all bounded subsets of \( O. \) If \( A \in \mathfrak{B} \) and \( B \in \mathfrak{B}_{\mu}, \) a fortiori \( AB \in \mathfrak{B}_{\mu} \) (Proposition 5.2 and [2, Proposition 4.7.2]). Fix \( m, k \in \mathbb{N}, \theta \in A, \vartheta \in O, \phi \in B; \) then
\[ \gamma_{m,k;B}^{\mu}(\theta \vartheta) \leq \gamma_{m,k;AB}^{\mu}(\vartheta). \]

References

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