The category of uniform spaces
as a completion of the category of metric spaces

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Abstract. A criterion for the existence of an initial completion of a concrete category $K$ universal w.r.t. finite products and subobjects is presented. For $K = \text{metric spaces}$ and uniformly continuous maps this completion is the category of uniform spaces.

Keywords: universal completion, metric space, uniform space

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Introduction.

We investigate initial completions of concrete categories, i.e. initially complete categories $K^*$ such that $K$ is a full, concrete subcategory of $K^*$. Recall that $K^*$ is said to be a universal initial completion of $K$, see [H], provided that (a) $K$ is closed under initial sources in $K^*$ and (b) for each initially complete $L$ and each concrete functor $F : K \to L$ preserving initial sources there exists an extension to an initial-sources preserving functor $F^* : K^* \to L$, unique up-to a natural isomorphism. More in general, let $\Delta$ be a collection of sources in the base-category, then a $\Delta$-universal initial completion of $K$, see [E], is an initial completion $K^*$ such that

(a) $K$ is closed in $K^*$ under initial sources carried by $\Delta$-sources (shortly: initial $\Delta$-sources) and

(b) for each initially complete category $L$ and each concrete functor $F : K \to L$ preserving initial $\Delta$-sources there exists an extension to an initial-sources preserving functor $F^* : K^* \to L$, unique up-to a natural isomorphism.

A description of the $\Delta$-universal initial completion (as a category of “$\Delta$-complete sources”) has been presented in [E]. Unfortunately, the conclusion made in that paper that $K^*$ is always legitimate (i.e. lives in the universe $U$ of classes), formulated in Theorem 4, is false: For example, if $K$ is a discrete, large category, then the $\Delta$-universal completion is illegitimate, being codable by the collection of all subclasses of $K$. In the case $\Delta = \text{all sources}$ or $\Delta = \emptyset$, the legitimacy of the $\Delta$-universal completion is characterized in [AHS].

In the present paper we concentrate on the case

$\Delta_{fm} = \text{all finite products and monomorphisms}$. 
(Or, equivalently, all finite monosources.) We prove that each fibre-small, hereditary concrete category has a fibre-small (thus, legitimate) $\Delta_{fm}$-universal initial completion. For

$$K = \text{Met},$$

metric spaces and uniformly continuous maps

this completion is

$$K^* = \text{Unif},$$

uniform spaces and uniformly continuous maps.

In this special case the same result also holds for $\Delta = \text{all finite sources}$, or $\Delta = \text{all countable sources}$, but it does not hold for $\Delta = \text{all sources}$. In fact, a concrete description of the universal initial completion of $\text{Met}$ is not known.

The paper has been inspired by Zdeněk Frolík who asked us about a categorical motivation of uniform spaces from the point of view of metric spaces. This paper is, most unfortunately, the end of a long, fruitful, and happy collaboration of the authors: Jan Reiterman died precisely when it was completed.

I. $\Delta$-universal completion.

Recall from [A] that a construct (i.e. a concrete category over $\text{Set}$) is called hereditary provided that every subset of the underlying set of any object $K$ gives rise to an initial subobject of $K$. This can be generalized to concrete categories over a base category $X$ (i.e. pairs consisting of a category $K$ and a faithful, amnestic functor $| |: K \to X$) provided that a fixed factorization structure $(E, M)$ for $X$-morphisms is given:

**Definition.** A concrete category $K$ over an $(E, M)$-base category is called hereditary provided that given an object $K$ of $K$, every $M$-morphism $m: X \to |K|$ has an initial lift.

Recall that a concrete category is fibre-small provided that for each object $X$ of the underlying category the fibre $\{K \in K \mid |K| = X\}$ is a set.

**Theorem.** Let $X$ be an $E$-co-wellpowered $(E, M)$-category, and let $\Delta$ be a collection of sources containing all $M$-maps (considered as singleton sources). Then each hereditary, fibre-small concrete category over $X$ has a fibre-small $\Delta$-universal initial completion.

**Proof:** The $\Delta$-universal initial completion has been described in the proof of Theorem 4 of [E] (for the case of $\Gamma = \text{all sources}$ and $P = \text{forgetful functor of } K$) as the category of all $\Delta$-complete sources in $K$. It is our task to show that this category is fibre-small (thus, legitimate).

For each $\Delta$-complete source $\sigma = (X \xrightarrow{f_i} |S_i|)_{i \in I}$ and each $i$ we factorize $f_i = m_i \cdot e_i$ ($m_i \in M$, $e_i \in E$) and denote by $m_i: S'_i \to S_i$ the initial morphism induced by $m_i$ (which exists since $K$ is hereditary). From the fact that $\Delta$ contains $\{m_i\}$ it follows that $X \xrightarrow{e_i} |S'_i|$ is a member of the ($\Delta$-complete) source $\sigma$. Thus,

$$\sigma' = (X \xrightarrow{e_i} |S'_i|)_{i \in I}$$

is a subsource of $\sigma$. This subsource fully determines $\sigma$ (in other words, given two $\Delta$-complete sources $\sigma_1 \neq \sigma_2$, it follows that $\sigma'_1 \neq \sigma'_2$). In fact, from the $\Delta$-completeness
it follows that for each \( i \in I \) and each morphism \( h : S'_i \to S \) in \( K \) we have \( X \xrightarrow{hc_i} |S| \) in \( \sigma \); consequently, \( \sigma \) is precisely the source of all composites of the members of \( \sigma' \) with morphisms of \( K \). Thus, to prove the fibre-smallness, it is sufficient to observe that for each object \( X \) of \( X \), all the possible sources \( \sigma' \) form a set. In fact, since \( X \) is \( E \)-co-wellpowered, we have a set of representatives for all \( e_i \)'s, and for each such a representative \( e_i : X \to Y \) we have (since \( K \) is fibre-small) only a set of possible objects \( S'_i \) with \( |S'_i| = Y \). Thus, there exists a set \( A \) of representative structured maps \( e_i : X \to |S'_i| \), and the collection of all \( \Delta \)-complete sources with the domain \( X \) can, obviously, be coded by the set of all subsets of \( A \).

**Remark.** For constructs \( K \), i.e. concrete categories over \( \text{Set} \), the \( \Delta \)-universal initial completion \( K^* \) can be described as follows.

Objects on the underlying set \( X \) are all collections \( \sigma \) of \( K \)-objects \( K \) with the underlying sets \( |K| = X/\sim \) (where \( \sim \) is an equivalence relation on \( X \)) such that:

1. If \( K \in \sigma \) then \( K' \in \sigma \) whenever \( |K'| = X/\sim \) and the canonical map \( c : X \to |K'| \) of \( |K'| \) (assigning to each \( x \in X \) the equivalence class of \( x \)) factorizes as the canonical map of \( |K| \) followed by a \( K \)-morphism \( K \to K' \);

2. For each initial \( \Delta \)-source \( (S \xrightarrow{f_i} S_i)_{i \in I} \) in \( K \)

   \[
   \begin{array}{ccc}
   X & \xrightarrow{c} & |K| = X/\ker h \\
   |K_i| & \xleftarrow{h_i} & S_i \\
   \downarrow & & \downarrow f_i \\
   S & &
   \end{array}
   \]

and each map \( h : X \to |S| \) such that every \( f_i h \) factorizes as the canonical map of \( |K_i| \) for some \( K_i \in \sigma \) followed by a \( K \)-morphism \( h_i : K_i \to S_i \) it follows that \( \sigma \) contains the initial lift of the inclusion \( X/\ker h \hookrightarrow |S| \).

Morphisms from \( \sigma \) to \( \sigma' \) are maps \( f : |\sigma| \to |\sigma'| \) such that for each \( K' \in \sigma' \) (with the canonical map \( c : |\sigma'| \to |K'| \)) \( \sigma \) contains the initial lift of the inclusion \( |\sigma|/\sim \hookrightarrow |K| \), where \( \sim \) is the kernel equivalence of \( c.f. \).

**II. A completion of the category of metric spaces.**

**Proposition.** The category \( \text{Unif} \) is a \( \Delta_{fm} \)-universal initial completion of the category \( \text{Met} \).
Proof: We apply the general result above to the special case of $\Delta = \Delta_{fm}$ and $K = \text{Met}$. The objects of $\Delta_{fm}$-universal initial completion $K^*$ with the underlying set $X$ are collections $\sigma$ of metric spaces $K = (X/\sim, d_K)$, where $\sim$ is an equivalence on $X$, such that (1) and (2) are satisfied. Each $K$ induces a pseudometric $d_K^*$ on the set $X$ by $d_K^*(x,y) = d_K([x],[y])$ (where $x \mapsto [x]$ denotes the canonical map). The conditions (1) and (2) guarantee that the resulting set of pseudometrics on $X$ forms a uniformity (defined as a collection of pseudometrics). Conversely, for each uniformity on $X$ and each pseudometric $d$ of that uniformity we have a corresponding metric space on the set $X/\sim$ where $x \sim y$ means that $d(x,y) = 0$. The axioms of a uniformity guarantee that the resulting set of metric spaces is an object of $K^*$. The morphisms of $K^*$ correspond precisely to the uniformly continuous maps. Thus, $\text{Unif}$ is a $\Delta_{fm}$-universal initial completion of $\text{Met}$. $\square$

Remark. Since $\text{Met}$ is, obviously, closed under countable initial sources in $\text{Met}$, we can also say that for $\Delta = \text{all countable sources}$, $\text{Unif}$ is a $\Delta$-universal initial completion of $\text{Met}$. However, the universal completion ($\Delta = \text{all sources}$) is different:

Example of an initial source in $\text{Met}$ which is not initial in $\text{Unif}$.

Let $\delta$ be the discrete metric (with value 1) on the set $N$ of natural numbers. Let $F$ be a free ultrafilter on $N$ containing the set $E$ of even numbers. For each $F \in F$ let $\delta_F$ be the following metric on $N$:

$$\delta_F(x,y) = \begin{cases} 
\frac{1}{n} & \text{if } \{x,y\} = \{2n-1,2n\} \text{ with } 2n \in F \\
0 & \text{if } x = y \\
1 & \text{else}.
\end{cases}$$

As proved in [PRRS], the set of all $\delta_F$, $F \in F$, is a base of a uniformity $\sigma$ on $N$ which is an atom in the fibre of $N$ (i.e. the only strictly finer uniformity is that induced by $\delta$). Consequently, the following source

$$((N, \sigma) \rightarrow (N, \delta_F))_{F \in F}$$

is initial in $\text{Unif}$. We will show that, nevertheless,

$$((N, \delta) \rightarrow (N, \delta_F))_{F \in F}$$

is initial in $\text{Met}$. In fact, let $(X, \delta_0)$ be a metric space, and let $f : X \rightarrow N$ be a map such that $f : (X, \delta_0) \rightarrow (N, \delta_F)$ is uniformly continuous for each $F \in F$. We will prove that the image of $f$, considered as a uniform subspace of $(N, \sigma)$, is discrete — thus, $f : (X, \delta_0) \rightarrow (N, \delta)$ is uniformly continuous.

Suppose that image of $f$ is not discrete. Then it is isomorphic to $(N, \sigma)$. [In fact, every uniform subspace $A$ of $(N, \sigma)$ is isomorphic to $(N, \sigma)$ or is discrete, according to whether the set $\{2n \mid n \in N, 2n \in A, 2n + 1 \in A\}$ is a member of $F$ or not.] Thus, we can assume that $f$ is surjective. Since $\sigma$ is an atom, it follows that $f$ is a final morphism. However, $(N, \sigma)$ is not a quotient of a metric space, since it is not generated by a single pseudometric — this is a contradiction.
REFERENCES


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