Antiproximinal sets in the Banach space $c(X)$

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Abstract. If $X$ is a Banach space then the Banach space $c(X)$ of all $X$-valued convergent sequences contains a nonvoid bounded closed convex body $V$ such that no point in $C(X) \setminus V$ has a nearest point in $V$.

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The distance from an element $x$ of a normed space $X$ to a nonvoid subset $M$ of $X$ is defined by $d(x, M) = \inf\{\|x - y\| : y \in M\}$. An element $y \in M$ such that $\|x - y\| = d(x, M)$ is called a nearest point to $x$ in $M$ and the set of all nearest points to $x$ in $M$ is denoted by $P_M(x)$. The set $M$ is called proximinal if $P_M(x) \neq \emptyset$ for all $x \in X$, and antiproximinal if $P_M(x) = \emptyset$ for all $x \in X \setminus M$. (Observe that $P_M(y) = \{y\}$ for all $y \in M$.)

Let $X^*$ be the conjugate space to $X$ and let $M$ be a nonvoid convex subset of $X$. A functional $f \in X^*$ is said to support $M$ (at $x$) if there exists $x \in M$ such that $f(x) = \inf f(M)$ or $f(x) = \sup f(M)$. Obviously $f \in X^*$ supports the closed unit ball $B_X$ of $X$ if and only if there exists $x \in B_X$ such that $f(x) = \|f\|$. If $f \neq 0$ then every $x \in B_X$ verifying this equality must be of norm one, i.e. $\|x\| = 1$. We shall denote by $S(M)$ the set of all support functionals of the set $M$.

V. Klee [13] called a Banach space $X$ of type $N_1$ if it contains a nonvoid closed convex antiproximinal set and of type $N_2$ if it contains a nonvoid bounded closed convex antiproximinal set. A hyperplane $\{x \in X : f(x) = a\}$ with $f \in X^*$, $f \neq 0$, and $a \in \mathbb{R}$, is proximinal if $f \in S(B_X)$ and antiproximinal if $f \not\in S(B_X)$. Since, by James theorem, a Banach space $X$ is reflexive if and only if $S(B_X) = X^*$, it follows that a Banach space is of type $N_1$ if and only if it is non-reflexive.

The first example of a Banach space of type $N_2$ was exhibited by M. Edelstein and A.C. Thompson [9] — the Banach space $c_0$ contains a bounded symmetric closed antiproximinal convex body. By a convex body we mean a convex set with nonvoid interior. A bounded symmetric closed convex body is called a convex cell. In [4] it was shown that the space $c$ also contains an antiproximinal convex cell and this property is shared by any Banach space of continuous functions isomorphic to $c$ ([5]). The existence of antiproximinal convex cells in more general spaces of continuous functions was proved by V.P. Fonf [10] (see also [11]).

The aim of the present note is to prove the existence of an antiproximinal convex cell in the Banach space $c(X)$ of all $X$-valued convergent sequences, where $X$ is a
non-trivial Banach space. The proof is simpler than the proof in the scalar case given in [4]. The case of the space $c_0(X)$ was considered in [6]. The notation is standard and all spaces will be considered over $\mathbb{R}$.

Let $\omega$ be the first infinite ordinal. Then $\mathbb{N} = [1, \omega]$ and $[1, \omega]$ is a compact Hausdorff space with respect to the interval topology (called also ordinal topology). If $X \neq \{0\}$ is a Banach space then $c(X)$ can be identified with the Banach space $C([1, \omega], X)$ of all continuous functions from $[1, \omega]$ to $X$, equipped with the usual sup-norm. An element $x \in c(X)$ will be denoted by $x = (x(i) : 1 \leq i \leq \omega)$ and sometimes by $(x(\omega)|x(1), x(2), \ldots)$. The conjugate of $c(X)$ is the space $l^1(X^*) = l^1([1, \omega], X^*)$ of all sequences $f = (f_i : 1 \leq i \leq \omega)$ such that $\|f\| := \sum_{1 \leq i \leq \omega} \|f_i\| < \infty$, the duality between $c(X)$ and $l^1(X^*)$ being given by the formula

$$f(x) = \sum_{1 \leq i \leq \omega} f_i(x(i))$$

for $f \in l^1(X^*)$ and $x \in c(X)$. Again the alternate notation $(f_\omega|f_1, f_2, \ldots)$ will be used to designate an element of $l^1(X^*)$.

The main result of this paper is:

**Theorem 1.** The Banach space $c(X)$ contains a bounded closed antiproximinal convex body.

The proof will be based on the following characterization of antiproximinal sets.

**Lemma 2** ([9]). A nonvoid closed convex subset $M$ of a Banach space $X$ is antiproximinal if and only if

$$\mathcal{S}(M) \cap \mathcal{S}(B_X) = \{0\},$$

where $B_X$ denotes the closed unit ball of $X$.

The following lemma gives some information about the support functionals of the unit ball of $c(X)$. The characterization of support functionals of the unit ball of $C(T)$, for a compact Hausdorff space $T$, was given by S.I. Zukhovickij [19] in the scalar case and by V.L. Chakalov [1] for vector-valued functions. For characterization of support functionals of the unit balls in other concrete Banach spaces, see [7], [14] and [15].

**Lemma 3.** Let $B_c$ be the closed unit ball of $c(X)$ and let $f = (f_i : 1 \leq i \leq \omega)$, $f \neq 0$, be an element in $l^1(X^*)$.

(a) If $f = (f_i : 1 \leq i \leq \omega) \in \mathcal{S}(B_c) \setminus \{0\}$ and $x = (x(i) : 1 \leq i \leq \omega) \in B_c$ is such that $f(x) = \|f\|$, then $f_i(x(i)) = \|f_i\|$ for all $i \in [1, \omega]$ and $\|x(i)\| = 1$ for all $i \in [1, \omega]$ such that $f_i \neq 0$.

(b) Let $\mathbb{N} = [1, \omega]$ and let $\sigma_i : \mathbb{N} \to \mathbb{N}$, $i = 1, 2$, be two strictly increasing functions such that $\sigma_1(\mathbb{N}) \cap \sigma_2(\mathbb{N}) = \emptyset$. Let $h \in X^*$, $h \neq 0$, and $\alpha_j, \beta_j > 0$, $j \in \mathbb{N}$. 

Proof: (a) Let $f \in S(B_c) \setminus \{0\}$ and let $x \in B_c$ be such that $f(x) = \|f\|$. Since $f_i(x(i)) \leq \|f_i\| \cdot \|x(i)\|$, for all $i \in [1, \omega]$, it follows that

$$\sum_{1 \leq i \leq \omega} \|f_i\| = \|f\| = f(x) = \sum_{1 \leq i \leq \omega} f_i(x(i)) \leq \sum_{1 \leq i \leq \omega} \|f_i\| \cdot \|x(i)\| \leq \sum_{1 \leq i \leq \omega} \|f_i\|,$$

implying $f_i(x(i)) = \|f_i\|$, for all $i \in [1, \omega]$, and $\|x(i)\| = 1$ for all $i \in [1, \omega]$ such that $f_i \neq 0$.

(b) Let $h \in X^*$, $h \neq 0$, $\alpha_j$, $\beta_j$, $\sigma_1$, $\sigma_2$ and $f \in l^1(X^*)$ fulfill the hypotheses of the lemma and suppose, on the contrary, that there exists an element $x = (x(i) : 1 \leq i \leq \omega) \in B_c$ such that $f(x) = \|f\|$. Taking into account the first point of the lemma it follows that

$$\alpha_j \|h\| = \|f_{\sigma_1(j)}\| = \alpha_j h(x(\sigma_1(j))) = \sum_{1 \leq i \leq \omega} \|f_i\| \cdot \|x(i)\| \leq \sum_{1 \leq i \leq \omega} \|f_i\||x(i)\| = \|f\|,$$

and

$$\beta_j \|h\| = \|f_{\sigma_2(j)}\| = -\beta_j h(x(\sigma_2(j))) = -\beta_j \sum_{1 \leq i \leq \omega} \|f_i\| \cdot \|x(i)\| = \|f\|,$$

implying $h(x(\sigma_1(j))) = \|h\|$ and $h(x(\sigma_2(j))) = -\|h\|$, for all $j \in \mathbb{N}$. Since $\sigma_k(j) \rightarrow \omega$ for $j \rightarrow \omega$, $k = 1, 2$, and the functions $x$ and $h$ are continuous, the above equalities yield, for $j \rightarrow \omega$, the contradiction $h(x(\omega)) = \|h\| > 0$ and $h(x(\omega)) = -\|h\| < 0$. 

Other result we need for the proof of the Theorem 1 is the following one, emphasizing the behaviour of support functionals under linear isomorphisms. If $X$, $Y$ are Banach spaces and $A : X \rightarrow Y$ is an isomorphism then its conjugate $A^* : Y^* \rightarrow X^*$ is an isomorphism too and $(A^*)^{-1} = (A^{-1})^*$ ([8, Lemma VI 3.7]). The support functionals of a set $M \subseteq X$ and of the set $A(M) \subseteq Y$ are related as follows:

**Lemma 4** ([9, Lemma 1]). Let $X$, $Y$ be Banach spaces, $M$ a nonvoid closed convex subset of $X$ and $A : X \rightarrow Y$ an isomorphism. Then

\[ S(M) = A^*(S(A(M))). \]

More exactly

\[ g \in S(A(M)) \iff A^*g \in S(M). \]
Proof of Theorem 1: First we construct an isomorphism $A : c(X) \to c(X)$ in the following way. For an element $x = (x(i) : 1 \leq i \leq \omega) \in c(X)$ define $Ax : [1, \omega] \to X$ by

$$Ax(\omega) = x(\omega) + \sum_{1 \leq j < \omega} (-1)^j 2^{-j-2} x(2j - 1)$$

and

$$Ax(i) = x(i) + \sum_{1 \leq j \leq 2^i} (-1)^j 2^{-j-2} x(2j - 1) + 2^{-i-1} \sum_{1 \leq j < \omega} (-1)^j 2^{-j} x(2^i(2j - 1))$$

for $1 \leq i < \omega$. Since the series in the right hand sides of the equalities (5) and (6) are norm convergent and $X$ is a Banach space, it follows that the definition of $Ax$ makes sense. Since

$$||Ax(\omega) - Ax(i)|| \leq ||x(\omega) - x(i)|| + 2^{-i-1} \sum_{1 \leq j < \omega} 2^{-j} ||x|| = ||x(\omega) - x(i)|| + 2^{-i-1} ||x||,$$

and $\lim_{i \to \omega} x(i) = x(\omega)$, it follows that $\lim_{i \to \omega} Ax(i) = Ax(\omega)$, i.e. $Ax$ is an element of $c(X)$. Obviously the operator $A : c(X) \to c(X)$ is linear. By (5) and (6) we have

$$||Ax(\omega)|| \leq ||x|| + 2^{-2} ||x|| = (5/4) ||x||$$

and, respectively,

$$||Ax(i)|| \leq ||x|| + 2^{-2} ||x|| + 2^{-i-1} ||x|| \leq (3/2) ||x||$$

for $1 \leq i < \omega$, implying

$$||Ax|| \leq (3/2) ||x||,$$

for all $x \in c(X)$, which is equivalent to the continuity of the operator $A$.

Now let $x \in c(X)$, $x \neq 0$, and let $i_0 \in [1, \omega]$ be such that $||x(i_0)|| = ||x|| := \sup\{||x(i)|| : 1 \leq i \leq \omega\}$. If $i_0 = \omega$, then, by (5), $||Ax|| \geq ||Ax(\omega)|| \geq ||x(\omega)|| - 2^{-2} ||x|| = (3/4) ||x||$.

If $1 \leq i_0 < \omega$, then by (6)

$$||Ax|| \geq ||Ax(i_0)|| \geq ||x(i_0)|| - (2^{-2} + 2^{-i-1}) ||x|| \geq (1/2) ||x||.$$

It follows that

$$||Ax|| \geq (1/2) ||x||,$$

for all $x \in c(X)$, with $x \neq 0$.
for all \(x \in c(X)\). The inequalities (7) and (8) show that \(A\) is an isomorphism of \(c(X)\) onto \(c(X)\). Its conjugate \(A^*\) will be an isomorphism of \(l^1(X^*)\) onto \(l^1(X^*)\) acting by the formula

\[
A^* f(x) = f(Ax) = \sum_{1 \leq i \leq \omega} f_i(Ax(i)),
\]

for \(f \in l^1(X^*)\) and \(x \in c(X)\). Taking into account the formulae (5) and (6), defining the operator \(A\), one obtains

\[
f_\omega(Ax(\omega)) = f_\omega(x(\omega)) + \sum_{1 \leq j < \omega} (-1)^j 2^{-j-2} f_\omega(x(2j - 1))
\]

and

\[
f_i(Ax(i)) = f_i(x(i)) + \sum_{1 \leq j \leq 2^i} (-1)^j 2^{-j-2} f_i(x(2j - 1)) + 2^{-i-1} \sum_{1 \leq j < \omega} (-1)^j 2^{-j} f_i(x(2^i(2j - 1))).
\]

Let \(c_0(X)\) denote the Banach space of all \(X\)-valued sequences converging to zero. It follows that \(c_0(X) = \{x \in C([1, \omega], X) : x(\omega) = 0\}\). The spaces \(c(X)\) and \(c_0(X)\) are isomorphic, an isomorphism \(H : c(X) \rightarrow c_0(X)\) being given by the formula

\[
H(x) = (0|x(\omega), x(1) - x(\omega), x(2) - x(\omega), \ldots)
\]

for \(x = (x(\omega)|x(1), x(2), \ldots) \in c(X)\) (see [20, p.55]). Its conjugate \(H^*\) will be an isomorphism of \(c_0(X)^*\) onto \(c(X)^*\). The conjugate \(c_0(X)^*\) of \(c_0(X)\) can be identified with the space

\[
W := \{f \in l^1([1, \omega], X^*) : f = (f_i : 1 \leq i \leq \omega), f_\omega = 0\},
\]

or equivalently

\[
W = \{f \in l^1([1, \omega], X^*) : f = (0|f_1, f_2, \ldots)\},
\]

normed by \(\|f\| = \sum_{1 \leq i < \omega} \|f_i\|\). The duality between \(c_0(X)\) and \(W\) is given by the formula

\[
f(y) = \sum_{1 \leq i < \omega} f_i(y(i)),
\]

for \(f = (0|f_1, f_2, \ldots) \in W\) and \(y = (0|y(1), y(2), \ldots) \in c_0(X)\). Since for \(x = (x(\omega)|x(1), x(2), \ldots) \in c(X)\) and \(f = (0|f_1, f_2, \ldots) \in W\) we have

\[
H^* f(x) = f(Hx) = f((0|x(\omega), x(1) - x(\omega), x(2) - x(\omega), \ldots))
\]
it follows that
\begin{equation}
H^* f = (f_1 - \sum_{2 \leq j \omega} f_j |f_2, f_3, \ldots). \tag{15}
\end{equation}

Denote by $B_c$ and $B_{c_0}$ the closed unit balls of $c(X)$ and $c_0(X)$ respectively, and put
\begin{equation}
V = (HA)^{-1}(B_{c_0}). \tag{16}
\end{equation}

Since $A$ and $H$ are isomorphisms, it follows that $V$ is a bounded symmetric closed convex body in $c(X)$. We shall show that the set $V$ is antiproximinal in $c(X)$. To this end, by Lemma 2, it suffices to show that
\begin{equation}
S(V) \cap S(B_c) = \{0\}. \tag{17}
\end{equation}

Since, by (16), $B_{c_0} = HA(V)$ we have
\begin{equation}
S(B_{c_0}) = S(HA(V)). \tag{18}
\end{equation}

By Lemma 4, $S(V) = \{(HA)^* f : f \in S(HA(V))\}$ and therefore
\begin{equation}
S(V) = \{(HA)^* f : f \in S(B_{c_0})\}. \tag{19}
\end{equation}

It follows that the relation (17) will be a consequence of the implication
\begin{equation}
f \in S(B_{c_0}) \setminus \{0\} \Rightarrow (HA)^* f \notin S(B_c). \tag{20}
\end{equation}

In order to prove (20) observe that $f = (0|f_1, f_2, \ldots) \in c_0(X)^*$, $f \neq 0$, supports the unit ball $B_{c_0}$ of $c_0(X)$ if and only if there exists $n \in [1, \omega]$ such that $f_i = 0$ for $i > n$ and $f_i \in S(B_X)$, for $1 \leq i \leq n$, where $B_X$ denotes the closed unit ball of the space $X$.

Now let $f = (0|f_1, \ldots, f_n, 0, \ldots)$, $f_n \neq 0$, be a support functional of $B_{c_0}$ and let us show that $(HA)^* f \notin S(B_c)$.

First suppose $n = 1$, i.e. $f = (0|f_1, 0, \ldots)$ with $f_1 \in S(B_X)$, $f_1 \neq 0$. By (15), $H^* f = (f_1|0, \ldots)$ so that, denoting $g = A^* H^* f = (HA)^* f$, formula (10) gives
\begin{equation}
g(x) = f_1(x(\omega)) + \sum_{1 \leq j < \omega} (-1)^j 2^{-j-2} f_1(x(2j - 1)) \tag{10}
\end{equation}
for all $x \in c(X)$. For $j = 2k$ and $j = 2k - 1$, $1 \leq k < \omega$, one obtains $g_{4k-1} = 2^{-2k-2} f_1$ and $g_{4k-3} = -2^{2k-3} f_1$, respectively, so that, by Lemma 3(b), $g \notin S(B_c)$.

If $n \geq 2$ then
\begin{equation}
h := H^* f = (f_1 - \sum_{2 \leq i \leq n} f_i |f_2, \ldots, f_n, 0, \ldots). \tag{15}
\end{equation}

Taking into account formula (11) it follows that $g = A^* h$ verifies
\begin{equation}
g_{2n-1(4k-3)} = -2^{-2k+1-n} f_n \quad \text{and} \quad g_{2n-1(4k-1)} = 2^{-2k-n} f_n \quad \text{for all } k \in [1, \omega[. \tag{16}
\end{equation}

Appealing again to Lemma 3(b) it follows that $g = A^* H^* f \notin S(B_c)$.

Theorem 1 is completely proved. \qed
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References


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