A note on intermediate
differentiability of Lipschitz functions

L. Zajíček

Abstract. Let $f$ be a Lipschitz function on a superreflexive Banach space $X$. We prove that then the set of points of $X$ at which $f$ has no intermediate derivative is not only a first category set (which was proved by M. Fabian and D. Preiss for much more general spaces $X$), but it is even $\sigma$-porous in a rather strong sense. In fact, we prove the result even for a stronger notion of uniform intermediate derivative which was defined by J.R. Giles and S. Sciffer.

Keywords: Lipschitz function, intermediate derivative, $\sigma$-porous set, superreflexive Banach space

Classification: Primary 46G05; Secondary 58C20

1. Introduction

In this note we show that a theorem of [2] implies a new result on intermediate differentiability of Lipschitz functions.

Let $X$ be a real Banach space. The open ball with center $c$ and radius $r$ is denoted by $B(c, r)$. If $f$ is a Lipschitz function, then the Lipschitz constant of $f$ is denoted by $\text{Lip}(f)$.

If $f$ is a real function on $X$ and $x, v \in X$, then we consider the upper and lower (one-sided) directional derivatives

$$\overline{f}(x, v) = \limsup_{t \to 0^+} \frac{f(x + tv) - f(x)}{t} \quad \text{and} \quad \underline{f}(x, v) = \liminf_{t \to 0^+} \frac{f(x + tv) - f(x)}{t}.$$ 

Following [3] we say that $x^* \in X^*$ is an intermediate derivative of a function $f : X \to \mathbb{R}$ at a point $x \in X$ if

$$\underline{f}(x, v) \leq (v, x^*) \leq \overline{f}(x, v) \quad \text{for every} \quad v \in X.$$ 

Of course, if $f$ has at $x$ the Gâteaux derivative, then it has also the (unique) intermediate derivative. Therefore Aronszajn's differentiability theorem ([1]) implies that every (locally) Lipschitz function on a separable Banach space has an intermediate derivative at all points except a set $E$ which is null in Aronszajn’s sense.

M. Fabian and D. Preiss [3] proved the following theorem.

Supported by CEZ J13/98113200007, GAČR 201/97/1161 and GAUK 190/1996.
Theorem FP. Suppose that a Banach space $Y$ contains a dense continuous linear image of an Asplund space and that $X$ is a subspace of $Y$. Then every locally Lipschitz function defined on an open subset $\Omega$ of $X$ is intermediate differentiable at every point of $\Omega \setminus A$, where $A$ is a first category set.

J.R. Giles and S. Sciffer [4] considered the following stronger notion of uniform intermediate differentiability.

**Definition 1.** A real function $f$ defined on an open subset $\Omega$ of a Banach space $X$ is said to be uniformly intermediate differentiable at $x \in \Omega$ if there exists (a “uniform intermediate derivative”) $x^* \in X^*$ and a sequence $t_n \downarrow 0$ such that

$$\lim_{n \to \infty} \frac{f(x + t_n v) - f(x)}{t_n} = (v, x^*)$$

for each direction $v \in X$ with $\|v\| = 1$.

The following result is proved in [4] using the Preiss deep differentiability theorem of [5].

**Theorem GS.** Let $X$ be an Asplund space. Then every locally Lipschitz function defined on an open subset $\Omega$ of $X$ is uniformly intermediate differentiable at every point of $\Omega \setminus A$, where $A$ is a first category set.

To formulate the result of the present note, we need the following definition (cf. [8], p.327).

**Definition 2.** Let $P$ be a metric space and $M \subset P$. We say that

(i) $M$ is globally very porous if there exists $c > 0$ such that for every open ball $B(a, r)$ there exists an open ball $B(b, cr) \subset B(a, r) \setminus M$ and

(ii) $M$ is $\sigma$-globally very porous if it is a countable union of globally very porous sets.

**Remark 1.** Every globally very porous set is clearly nowhere dense and thus every $\sigma$-globally very porous set is of the first category. It is not difficult to prove that in each Banach space there exists a first category set which is not $\sigma$-globally very porous. (For the more difficult result concerning the weaker notion of a $\sigma$-porous set see [10].)

Now we can formulate our result.

**Theorem.** Let $X$ be a superreflexive Banach space. Then every locally Lipschitz function $f$ defined on an open subset $\Omega$ of $X$ is uniformly intermediate differentiable at every point of $\Omega \setminus A$, where $A$ is a $\sigma$-globally very porous set.

By Remark 1, our Theorem is, in the case of a superreflexive $X$, an improvement of Theorem GS.

A result analogous to Theorem for the weaker notion of (non-uniform) intermediate differentiability is proved in [7] in the case of a separable Banach space $X$. 
Intermediate differentiability of Lipschitz functions

In this case the set $A$ can be taken to be “$\sigma$-directionally porous”. Note that the notions of smallness “$\sigma$-globally very porous” and “$\sigma$-directionally porous” are incomparable in infinite-dimensional spaces.

We will need also the notion of a very porous set which is clearly weaker than this of a globally very porous set.

**Definition 3.** Let $P$ be a metric space, $M \subset P$ and $x \in P$. We say that

(i) $M$ is very porous at $x$ if there exist numbers $\delta > 0, \eta > 0$ such that, for each $0 < \rho < \delta$, there exists a ball $B(y, \omega) \subset B(x, \rho) \setminus M$ with $\omega \geq \eta \rho$,

(ii) $M$ is very porous if it is very porous at each of its points and

(iii) $M$ is $\sigma$-very porous if it is a countable union of very porous sets.

The basic ingredient of the proof of our Theorem is the following result of [2]. In the terminology of [2], it says that the pair of Banach spaces $(X, \mathbb{R})$ has the “uniform approximation by affine property (UAAP)” if $X$ is superreflexive. (Moreover, it is proved in [2] that $(X, \mathbb{R})$ has (UAAP) iff $X$ is superreflexive.)

**Theorem BJLPS.** Let $X$ be a superreflexive Banach space. Then for each $\varepsilon > 0$ there exists $c = c(\varepsilon) > 0$ such that for every ball $B(x, \rho)$ in $X$ and every Lipschitz function $f : B(x, \rho) \mapsto \mathbb{R}$ there exist a ball $B(y, \tilde{\rho}) \subset B(x, \rho)$ and an affine function $a : X \mapsto \mathbb{R}$ such that $\tilde{\rho} \geq c \rho$ and

$$|f(z) - a(z)| \leq \varepsilon \tilde{\rho} \text{Lip}(f) \quad \text{for each} \quad z \in B(y, \tilde{\rho}).$$

We will use also the following relatively easy fact (see [11], Lemma E).

**Proposition Z.** Let $X$ be a Banach space and $M \subset X$. Then $M$ is $\sigma$-globally very porous iff it is $\sigma$-very porous.

2. Proof of Theorem

Let $G_n$ be the union of all balls $B(c, r) \subset \Omega$ such that $r < 1/n$ and there exists an affine function $a$ on $X$ for which $|f(z) - a(z)| \leq r/n$ whenever $z \in B(c, 2r)$. Put $P_n = \Omega \setminus G_n$ and $A = \bigcup_{n=1}^{\infty} P_n$. It is sufficient to prove that

(1) each $P_n$ is $\sigma$ - globally porous and

(2) $f$ has a uniform intermediate derivative at each point of $\Omega \setminus A = \bigcap_{n=1}^{\infty} G_n$.

First we will prove (1). By Proposition Z, it is sufficient to prove that each $P_n$ is very porous at each point $x \in \Omega$. To this end choose $n, x$ and find $\delta > 0, K > 0$ such that $B(x, \delta) \subset \Omega, \delta < 1/n$ and $f$ is Lipschitz with constant $K$ on $B(x, \delta)$. Now find $c = c(1/2nK)$ by Theorem BJLPS and consider an arbitrary $0 < \rho < \delta$. 

By the choice of \( c \) there exists a ball \( B(y, \tilde{\rho}) \subset B(x, \rho) \) and an affine function \( a \) on \( X \) such that \( \tilde{\rho} \geq c\rho \) and

\[
|f(z) - a(z)| \leq \frac{1}{2nK\tilde{\rho}} = \frac{\tilde{\rho}}{2n}
\]

for each \( z \in B(y, \tilde{\rho}) \).

Therefore \( B(y, \tilde{\rho}/2) \subset G_n \) and we see that \( P_n \) is very porous at \( x \) (with \( \eta = c/2 \)).

To prove (2), suppose that \( z \in \bigcap_{n=1}^\infty G_n \) is given. Then there exist sequences \((B(c_n, r_n))\) of balls and \((a_n)\) of affine functions on \( X \) such that \( 0 < r_n < 1/n, \ z \in B(c_n, r_n) \)

(3) \[
|f(y) - a_n(y)| < r_n/n \text{ for each } y \in B(c_n, 2r_n).
\]

Let \( a_n(t) = q_n + x_n^*(t) \), where \( q_n \in R \) and \( x_n^* \) is a linear function on \( X \). If \( v \in X, \|v\| = 1 \), then (3) implies

(4) \[
\left| \frac{f(z + r_nv) - f(z)}{r_n} - (v, x_n^*) \right| = \left| \frac{f(z + r_nv) - f(z)}{r_n} - \frac{a_n(z + r_nv) - a_n(z)}{r_n} \right| < \frac{2}{n}.
\]

Since \( f \) is locally Lipschitz, there exist \( L > 0 \) and \( n_0 \in N \) such that \( |(v, x_n^*)| < L + 2/n \) whenever \( n \geq n_0 \) and \( \|v\| = 1 \). Therefore \( (x_n^*)_{n=n_0}^\infty \) is a norm bounded sequence in \( X^* \). By the Eberlein-Smulyan theorem we can choose a subsequence \((x_{n_k})_{k=1}^\infty \) and \( x^* \in X^* \) such that

(5) \[
x_{n_k}^* \to x^* \text{ in the } w^* \text{ - topology}.
\]

Put \( t_k := r_{n_k} \). Then (4) and (5) clearly imply that

\[
\lim_{k \to \infty} \frac{f(z + t_kv) - f(z)}{t_k} = (v, x^*)
\]

for each \( v \in X, \|v\| = 1 \), which completes the proof.

**Acknowledgment.** In [11] a characterization of \( \sigma \)-globally very porous sets based on a modification of the Banach-Mazur game is given. In the original version of the present paper, this characterization (similarly as in [9]) was used. The author thanks the anonymous referee who suggested the more direct argument (based on the definition of \( G_n \)) which is used in the present version.

**References**


Intermediate differentiability of Lipschitz functions


Department of Mathematical Analysis, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 75 Praha 8, Czech Republic
E-mail: Zajicek@karlin.mff.cuni.cz

(Received December 17, 1998, revised June 26, 1999)