THE DIVISION METHOD FOR SUBSPECTRA OF Self-Adjoint Differential Vector-Operators

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Abstract. We discuss the division method for subspectra which appears to be one of the key approaches in the study of spectral properties of self-adjoint differential vector-operators, that is operators generated as a direct sum of self-adjoint extensions on an Everitt-Markus-Zettl multi-interval system. In the current work we show how the division method may be applied to obtain the ordered spectral representation and Fourier-like decompositions for self-adjoint differential vector-operators, after which we obtain the analytical decompositions for the measurable (relative to a spectral parameter) generalized eigenfunctions of a self-adjoint differential vector-operator.

1. Introduction

Problem Overview. We begin with a physical example of a Schrödinger vector-operator. Gesztesy and Kirsch [6] in particular considered a Schrödinger operator generated by the Hamiltonian

\[ H = -\frac{d^2}{dx^2} + (s^2 - \frac{1}{4}) \frac{1}{\cos^2 x}, \quad s > 0. \] (1.1)

Since the potential in Hamiltonian has a countable number of singularities on a discreet set \( X \) in \( \mathbb{R} \), leading to spoiling of the local integrability, it is impossible to apply the standard methods of the theory of ordinary differential operators. In order to proceed and build a self-adjoint extension of a minimal operator generated by (1.1) on \( \mathbb{R} \setminus X \), one may take self-adjoint extensions \( T_i \), generated by the same Hamiltonian (1.1) in the coordinate spaces \( L^2\left(-\frac{\pi}{2} + i\pi, \frac{\pi}{2} + i\pi\right) \), \( i \in \mathbb{Z} \), and then consider the direct sum operator \( \oplus_{i \in \mathbb{Z}} T_i \) in the space \( \oplus_{i \in \mathbb{Z}} L^2\left(-\frac{\pi}{2} + i\pi, \frac{\pi}{2} + i\pi\right) \).

This direct sum operator appears to be one of the possible self-adjoint extensions of the minimal operator considered on \( \mathbb{R} \setminus X \). Moreover in the case \( s \geq 1 \) the minimal
on $\mathbb{R} \setminus X$ operator appears to be essentially self-adjoint and its only self-adjoint extension is a direct sum operator.

This physical example gave birth to the theory of general differential direct sum operators, or in the text below vector-operators. Beginning from 1992, the theory of differential vector-operators has been investigated in connection with their non-spectral properties in a Hilbert space ([1], [2, 3] and in complete locally convex spaces [4, 5]. The interest in such a theory is explained by its numerous applications in theoretical physics and pure mathematics. Thus, physical applications may be found in a single or a multi-particle quantum mechanics, especially in problems where a quantum system is split into a number of disconnected subsystems under the influence of a potential. For applications in quantum mechanics see also the respective references in [3].

As it was shown in the fundamental works [2] and [3], a differential vector-operator is an object which resembles an ordinary differential operator by its general properties, but in fact it has much more complicated structure.

Although the bigger part of studies concerned only non-spectral properties of differential vector-operators, there has been some development of their spectral theory recently. Some results describing position of spectra of Schrödinger vector-operators were presented in 1985 in [6] and the most recent results for general quasi-differential vector-operators belong to Sobhy El-Sayed Ibrahim [7, 8].

The internal spectral structure of abstract self-adjoint vector-operators was first investigated in [9], for which see also [10, 11]. The structure of coordinate operators as differential operators played the key role in [12] where the unitary transformation making the ordered representation was described in terms of generalized eigenfunctions of a differential vector-operator. These generalized eigenfunctions appear to be only measurable relative to the spectral parameter, therefore it is an essential problem to obtain their decomposition over some set of analytical kernels. This problem is positively solved by Theorem 2.4 of the current work.

Mathematical background. Basic concepts of quasi-differential operators are well described in [2, 3]. A good reference for operators with real coefficients is the book of Naimark [13].

Let $\Omega$ be a finite or a countable set of indices. On $\Omega$, we have a multi-interval differential Everitt-Markus-Zettl system $\{I_i, \tau_i\}_{i \in \Omega}$, where $I_i$ are arbitrary intervals of the real line and $\tau_i$ are formally self-adjoint differential expressions of a finite order. This EMZ system generates a family of Hilbert spaces $\{L^2(I_i) = L^2_i\}_{i \in \Omega}$ and families of minimal $\{T_{\min,i} \}_{i \in \Omega}$ and maximal $\{T_{\max,i} \}_{i \in \Omega}$ differential operators. Consider a respective family $\{T_i \}_{i \in \Omega}$ of self-adjoint extensions. Further, we introduce a system Hilbert space $L^2 = \oplus_{i \in \Omega} L^2_i$, consisting of the vectors $f = \oplus_{i \in \Omega} f_i$ such that $f_i \in L^2_i$ and

$$\|f\|^2 = \sum_{i \in \Omega} \|f_i\|^2 = \sum_{i \in \Omega} \int_{I_i} |f_i|^2 \, dx < \infty.$$  

In the space $L^2$ consider the operator $T : D(T) \subseteq L^2 \to L^2$, defined on the domain

$$D(T) = \{f \subseteq L^2 : \sum_{i \in \Omega} \|T_i f_i\|^2_i < \infty\}$$

by $T f = \oplus_{i \in \Omega} T_i f_i$. 
The operator $T$ is called a differential vector-operator generated by the self-adjoint extensions $T_i$, or a self-adjoint differential vector-operator. If $\Omega$ is infinite, the vector-operator $T$ is called infinite. The operators $T_i$ are called coordinate operators.

The abstract preliminaries for this work may be found, for instance, in books \[14, 15\].

Fix $i \in \Omega$. For each $T_i$ there exists a unique resolution of the identity $E_i^\lambda$ and a unitary operator $U_i$, making the isometrically isomorphic mapping of the Hilbert space $L_i^2$ onto the space $L^2(M_i, \mu_i)$, where the operator $T_i$ is represented as a multiplication operator. Below, we remind the structure of the mapping $U_i$.

We call $\phi \in L_i^2$ a cyclic vector if for each $z \in L_i^2$ there exists a Borel function $f$, such that $z = f(T_i)\phi$. Generally, there is no a cyclic vector in $L_i^2$ but there is a collection $\{\phi^k\}$ of them in $L_i^2$, such that $L_i^2 = \oplus_k L_i^2(\phi^k)$, where $L_i^2(\phi^k)$ are $T_i$-invariant subspaces in $L_i^2$ generated by the cyclic vectors $\phi^k$. That is $L_i^2(\phi^k) = \{f(T_i)\phi^k\}$, for a varying Borel function $f$, such that $\phi^k \in D(f(T_i))$.

A vector $\phi \in L_i^2$ is called maximal relative to the operator $T_i$, if each measure $(E^i(\cdot), x, x)_i$, $x \in L_i^0$, is absolutely continuous with respect to the measure $(E^i(\cdot), \phi, \phi)_i$.

For each Hilbert space $L_i^2$, there exist a unique (up to unitary equivalence) decomposition $L_i^2 = \oplus_k L_i^2(\varphi^k_1)$, where $\varphi^k_1$ is maximal in $L_i^2$ relative to $T_i$, and a decreasing set of multiplicity sets $e^k_1$, where $e^k_1$ is the whole line, such that $\oplus_e L_i^2(\varphi^k_1)$ is equivalent with $\oplus_k L_i^2(e^k_1, \mu_i)$, where the measure of the ordered representation is defined as $\mu_i = (E^i(\cdot)\varphi^k_1, \varphi^k_1)_i$. A spectral representation of $T_i$ in $\oplus_k L_i^2(e^k_1, \mu_i)$ is called the ordered representation and it is unique, up to a unitary equivalence. Two operators are called equivalent, if they create the same ordered representation of their spaces.

For $i \in \Omega$, we introduce a sliced union of sets $M_i$ (see also preliminaries) as a set $M_i$, containing all $M_j$ on different copies of $\bigcup_{i \in \Omega} M_i$. The sets $M_i$ do not intersect in $M$, but they can superpose, i.e. two sets $M_i$ and $M_j$ superpose, if their projections in the set $\bigcup_{i \in \Omega} M_i$ intersect.

For $z_i \in L_i^2$, $i \in \Omega$, define $\widehat{z_i} = \{0, \ldots, 0, z_i, 0, \ldots, 0\} \in L^2$, where $z_i$ is on the $i$-th place.

For each $i \in \Omega$, let $\delta(T_i)$ denote the subspectrum of the operator $T_i$, i.e. $\delta(T_i) = \sigma_{pp}(T_i) \cup \sigma_{cont}(T_i)$, where $\sigma_{pp}(T_i)$ is the set of eigenvalues which may be open and $\sigma_{cont}(T_i)$ is the continuous spectrum with a removed set of spectral measure zero. $\sigma_{cont}(T_i)$ may be also open. Note that $\delta(T_i) = \sigma(T_i)$. For instance, the subspectrum of an operator having the complete system of eigenfunctions with eigenvalues being the rational numbers of $[0, 1]$ equals to $\mathbb{Q} \cap [0, 1]$; the subspectrum of an operator having the continuous spectrum $[0,1]$ is assumed to equal to $\mathbb{Q}$ without loss of generality. The notion of the subspectrum arises quite naturally. Indeed, let we are given a self-adjoint operator $A$ with a simple spectrum $\sigma(A) = [a,b]$. Choosing any point $\xi \in \sigma(A)$ we can obtain $\sigma(A) = [a, \xi] \cup [\xi, b]$. If we are interested in obtaining a formula $A_1 \oplus A_2 = A$, where $\sigma(A_1) = [a, \xi]$ and $\sigma(A_2) = [\xi, b]$, we have to suppose that $\xi \notin \sigma_{pp}(A)$. But if we pass to subspectra, we will not need to care about inessential points appearing as limit points.

Consider a projecting mapping $P : M \to \bigcup_{i \in \Omega} M_i$ such that $P(\delta(T_i)) = \delta(T_i)$. Let $\Omega = \bigcup_{k=1}^K A_k$, $A_k \cap A_s = \emptyset$ for $k \neq s$ and $A_k = \{s \in \Omega : \forall s, l \in A_k, s \neq l, P(\delta(T_s)) \cap P(\delta(T_l)) = B_{sl}\}$.
where \( \|E'(B_u)\varphi_t\|^2 = 0 \) for any cyclic \( \varphi_t \in L^2_t, t = s,l \). From all such divisions of \( \Omega \) we choose and fix the one, which contains the minimal number of \( A_k \). In case when all the coordinate spectra \( \sigma(T_i) \) are simple, we define the number \( \Lambda = \min\{K\} \) as the spectral index of the vector-operator \( T \).

The following two lemmas were proved in [9].

**Lemma 1.1.** The identity resolution \( \{E_{\lambda}\} \) of the vector-operator \( T \) equals to the direct sum of the coordinate identity resolutions \( \{E_{\lambda}^i\} \), that is \( \{E_{\lambda}\} = \oplus_{i \in \Omega}\{E_{\lambda}^i\} \)

**Lemma 1.2.** Let each \( T_i \) have a cyclic vector \( a_i \) in \( L^2 \). Then the vector-operator \( T \) has minimum \( \Lambda \) cyclic vectors \( \{a_k\}_{k=1}^\Lambda \), having the form \( a_k = \sum_{i \in A_k} \tilde{a}_i \).

In the next section we will see what a spectral multiplicity of a vector-operator is. Nevertheless, this notation is intuitively clear. Running ahead, let us present here two examples, which will show the difference between the spectral index and the spectral multiplicity of the vector-operator \( T \).

**Example 1.** We have a three-interval EMZ system \( \{I_i, \tau_i, 1\}_{i=1}^3 \) (a kinetic energy, a mirror kinetic energy, an impulse):

\[
I_1 = [0, +\infty), \quad \tau_1 = -\left(\frac{d}{dt}\right)^2, \\
D(T_1) = \{f \in D(T_{max,1}) : f(0) + kf'(0) = 0, -\infty < k \leq \infty\}; \\
I_2 = [0, +\infty), \quad \tau_2 = \left(\frac{d}{dt}\right)^2, \\
D(T_2) = \{f \in D(T_{max,2}) : f(0) + sf'(0) = 0, -\infty < s \leq \infty\}; \\
I_3 = [0,1], \quad \tau_3 = \frac{1}{t} \frac{d}{dt}, \\
D(T_3) = \{f \in D(T_{max,3}) : f(0) = e^{i\alpha} f(1), \alpha \in [0,2\pi]\}.
\]

(a) If \( k, s \in (-\infty,0) \cup (+\infty) \) then

\[
\delta(T_1) = (0, +\infty), \quad \delta(T_2) = (-\infty,0), \quad \delta(T_3) = \bigcup_{n=-\infty}^{\infty} (2\pi n - \alpha).
\]

For this system we have: \( \{1,2,3\} = \bigcup_{k=1}^2 A_k \) and \( A_1 = \{1,2\}, A_2 = \{3\} \). Thus, here the spectral index does not coincide with the spectral multiplicity (which is 1) and equals 2.

(b) The case \( 0 < k, s < +\infty \) leads to

\[
\delta(T_1) = \left\{-\frac{1}{k^2}\right\} \cup (0, +\infty), \quad \delta(T_2) = (-\infty,0) \cup \left\{\frac{1}{s^2}\right\}, \quad \delta(T_3) = \bigcup_{n=-\infty}^{\infty} (2\pi n - \alpha).
\]

If \( \alpha \notin \left[ \bigcup_{n=-\infty}^{\infty} (2\pi n + \frac{1}{k^2}) \right] \bigcup \left[ \bigcup_{n=-\infty}^{\infty} (2\pi n - \frac{1}{s^2}) \right] \), we have \( A_1 = \{1\}, A_2 = \{2\}, A_3 = \{3\} \). That is \( \Lambda = 3 \) but \( \oplus_{i=1}^3 T_i \) has a simple spectrum.

**Example 2.** Let us have a vector-operator \( \oplus_{i=1}^3 T_i \) with

\[
\delta(T_1) = \bigcup_{n \in \mathbb{Z}, n \geq 0} n, \quad \delta(T_2) = \bigcup_{n \in \mathbb{Z}, n \leq 0} n, \quad \delta(T_3) = \bigcup_{n \in \mathbb{Z}, n \neq 0} n.
\]

The spectral index is 3 but spectral multiplicity is 2.
Thus, the unitary representation of the space \( L^2 \) with the simple coordinate spectra \( \sigma(T_i) \) is called \textit{distorted} if its spectral index does not equal its spectral multiplicity.

Generally it is not possible to build a spectral representation for a distorted vector-operator without applying the division method, but in some cases (Example 1) Theorem 1.2 may lead to the construction of a spectral representation even for some distorted vector-operators. If we want to obtain an ordered spectral representation for any self-adjoint vector-operator, only implementation of the division method can achieve this.

2. THE DIVISION METHOD FOR SUBSPECTRA (DMS)

Below we present the three theorems (2.1, 2.2 and 2.3) without their complete proofs. Only the structural parts of the proofs essential for the demonstration of the DMS are presented. The complete proof of Theorem 2.1 may be found in [11, 10] and refer to [12] for the proofs of Theorems 2.2 and 2.3.

**Theorem 2.1.** If \( \theta_i \) and \( \{e_n^i\}_{n=1}^{m_i} \) are measures and multiplicity sets of ordered representations for coordinate operators \( T_i, i \in \Omega \), then there exist processes \( Pr_1 \) and \( Pr_2 \), such that the measure

\[
\theta = Pr_1(\{\theta_i\}_{i \in \Omega})
\]

is the measure of an ordered representation and the sets

\[
s_n = Pr_2(\{e_n^i\}_{i \in \Omega; k = 1 \ldots m_i})
\]

are the canonical multiplicity sets of the ordered representation of the operator \( T \). Thus, the unitary representation of the space \( L^2 \) on the space \( \oplus_n L^2(s_n, \theta) \) is the ordered representation and it is unique up to unitary equivalence.

**Proof.** We divide the proof into units for convenience. Parts (A) and (B) represent the DMS.

(A) Let \( a_i \) be maximal vectors relative to the operators \( T_i \) in \( L^2 \). We want to find a maximal vector relative to the vector-operator \( T \). We know, that the vector \( \oplus_{i \in \Omega} a_i \) does not give a single measure, if a set \( P(\delta(T_i)) \cap P(\delta(T_j)) \) has a non-zero spectral measure for \( i \neq j \). Consider restrictions \( T_i|_{L^2(a_i)} = T'_i \). Since all the operators \( T'_i \) have single cyclic vectors \( a_i \), we can divide \( \Omega \) into \( A_k, k = 1 \ldots \Lambda \) and apply Lemma 1.2 for the operator \( \oplus_{i \in \Omega} T'_i \). Then we have derive \( \Lambda \) vectors \( a^k = \oplus_{j \in A_k} a_j \), which are maximal in the respective spaces \( L^2(a^k) = \oplus_{j \in A_k} L^2_j(a_j) \).

(B) Let now \( 1 < \Lambda < \infty \). Define \( T^k = \oplus_{j \in A_k} T'_j \). For any two operators \( T^k \) and \( T^s \), \( k \neq s \), let us introduce the sets \( \delta_{k,s} = P(\delta(T^k)) \cap P(\delta(T^s)) \) and \( \delta_k = P(\delta(T^k)) \setminus \delta_{k,s} \). There exist unitary representations

\[
U^k : L^2(a^k) \to L^2(\mathbb{R}, \mu_{a^k}).
\]

Consider measures \( \mu_k \) and \( \mu_{k,s} \), defined as

\[
\mu_{k,s}(e) = \mu_{a^k}(e \cap \delta_{k,s})
\]

and \( \mu_k(e) = \mu_{a^k}(e \cap \delta_k) \), for any measurable set \( e \). For any operator \( T^k \) (with respect to \( T^s \)), measures \( \mu_k \) and \( \mu_{k,s} \) are mutually singular and \( \mu_k + \mu_{k,s} = \mu_{a^k} \); therefore,

\[
L^2(\mathbb{R}, \mu_{a^k}) = L^2(\mathbb{R}, \mu_k) \oplus L^2(\mathbb{R}, \mu_{k,s}).
\]
This means that (according to our designations):
\[ U^{k-1} : L^2(\mathbb{R}, \mu_{\mathbf{k}}) \rightarrow L^2(\mathbf{a}_k^k) \oplus L^2(\mathbf{a}_{k,s}^k) \]
and
\[ \mathbf{a}_k^k = \mathbf{a}_k^k \oplus \mathbf{a}_{k,s}^k, \]
where \( \mathbf{a}_k^k \) and \( \mathbf{a}_{k,s}^k \) form the measures \( \mu_k \) and \( \mu_{k,s} \) respectively. Define also as \( \max\{w, \tilde{w}\} \) the vector, which is maximal of the two vectors in the brackets (Note that this designation is valid only for vectors, considered on the same set. In order not to complicate the investigation we assume here that any two vectors are comparable in this sense. In order to achieve this, it is enough to decompose each coordinate operator \( T_i \) into the direct sum \( T_i^{pp} \oplus T_i^{cont} \), where the operators have respectively pure point and continuous spectra. Then after redesignation we obtain the equivalent vector-operator to the initial vector-operator \( \oplus T_i \)).

Consider first two operators \( T^1 \) and \( T^2 \). It is clear, that the vector
\[ \mathbf{a}_1^{1\oplus 2} = \mathbf{a}_1^1 \oplus \mathbf{a}_2^2 \oplus \max\{\mathbf{a}_1^1, \mathbf{a}_2^2\} \]
is maximal in \( L^2(\mathbf{a}_1^1) \oplus L^2(\mathbf{a}_2^2) \). Note that \( \mathbf{a}_1^1 \) and \( \mathbf{a}_2^2 \) and they both may equal zero. The maximal vector in \( L^2(\mathbf{a}_1^1) \oplus L^2(\mathbf{a}_2^2) \oplus L^2(\mathbf{a}_3^3) \) will have the form:
\[ \mathbf{a}_1^{1\oplus 2\oplus 3} = \mathbf{a}_1^{1\oplus 2} \oplus \mathbf{a}_2^2 \oplus \max\{\mathbf{a}_{1,2,3}^{1\oplus 2}, \mathbf{a}_3^3\}, \]
where \( \mathbf{a}_{1,2,3}^{1\oplus 2} \) is the narrowed vector \( \mathbf{a}_1^{1\oplus 2} \), corresponding to the set which is free from the superposition with \( \delta(T_3) \), as shown in (2.1).

Continuing this process, we obtain a maximal vector in the main space \( L^2 \):
\[ \mathbf{a}_1^{1\oplus \cdots \oplus \Lambda} = \mathbf{a}_{1,\Lambda}^{1\oplus \cdots \oplus \Lambda-1} \oplus \mathbf{a}_\Lambda^\Lambda \oplus \max\{\mathbf{a}_{1,\Lambda}^{1\oplus \cdots \oplus \Lambda-1, \Lambda}, \mathbf{a}_\Lambda^\Lambda, \mathbf{a}_\Lambda^{1\oplus \cdots \oplus \Lambda-1}\}. \]

Let \( \Lambda = \infty \). We obtain \( \mathbf{a}_1^{1\oplus \cdots \oplus \Lambda} \) as a vector which satisfies the following equality:
\[ \|\bigoplus_{\Omega \in \Omega} E^i(\cdot) a_1^{1\oplus \cdots \oplus \Lambda} \|^2 = \lim_{L \to \infty} \|\bigoplus_{\Omega \in \Omega} E^i(\cdot) a_1^{1\oplus \cdots \oplus \Lambda} \| \|^2, \]
(2.3)
since the limit on the right side exists.

\( \text{(C)} \) The next step is to build the measure of the ordered representation for the vector-operator. From Lemma 1.1 and the reasonings above, it follows that such a measure will be
\[ \theta(\cdot) = \left(\bigoplus_{\Omega \in \Omega} E^i(\cdot) a_1^{1\oplus \cdots \oplus \Lambda}, a_1^{1\oplus \cdots \oplus \Lambda}\right). \]

\( \text{(D)} \) The canonical multiplicity sets \( s_n \) of the vector-operator have the form:
\[ s_n = \left[ \bigcup_i P(e_i^t) \right] \bigcup \bigcup_{m_i \geq n} \left[ \bigcap_{m_i \geq n} P\left(e_{m_i}^t \setminus e_{m_i+1}^t\right) \right]. \]

(2.4)

**Example 3.** Let us have a vector-operator \( T = \bigoplus_{i=1}^5 T_i \), generated by five coordinate operators with simple continuous spectra:
\[ \delta(T_1) = (0, 2), \quad \delta(T_2) = (1, 3), \quad \delta(T_3) = (2, 4), \quad \delta(T_4) = (3, 6), \quad \delta(T_5) = (0, 4). \]
Divide \( \Omega = \{1, 2, 3, 4, 5\} \) into \( A_k \):
\[ A_1 = \{1, 3\}, \quad A_2 = \{2, 4\}, \quad A_3 = \{5\}. \]
The spectral index \( \Lambda \) is 3 and we pass to the three operators
\[ T^1 = T_1 \oplus T_3, \quad T^2 = T_2 \oplus T_4, \quad T^3 = T_5, \]
with maximal elements respectively:
\[ a^1 = a_1 \oplus a_3, \quad a^2 = a_2 \oplus a_4, \quad a^3 = a_5. \]
Consider first two sub-vector-operators \( T^1 \) and \( T^2 \). Find elements \( a^1_1, a^1_2, a^2_2, a^2_1 \).
Since the spectra are continuous, we may assume that
\[ \max \{a^1_1, a^2_1\} = a^1_1. \]
We derive a maximal vector \( a^{1 \oplus 2} \) in the space \( \oplus_{i=1}^{4} L^2_i \):
\[ a^{1 \oplus 2} = a^1_1 \oplus a^1_2 \oplus a^2_2 = a^1 \oplus a^2_2. \]
Apply the DMS to the vectors \( a^{1 \oplus 2} \) and \( a^3 \). We obtain the vectors
\[ a^1_{1 \oplus 2}, a^1_{1 \oplus 2}, a^3_{3 \oplus 2} = a^3, \quad a^3_{3 \oplus 2} = 0. \]
Eventually, we find the maximal element in \( \oplus_{i=1}^{5} L^2_i = \oplus_{k=1}^{3} L^2(\mathcal{A}^k) \):
\[ a^{1 \oplus 2 \oplus 3} = a^1_{1 \oplus 2} \oplus \max \{a^1_{1 \oplus 2}, a^3\} \oplus 0 = a^1_{1 \oplus 2} \oplus a^1_{1 \oplus 2} \oplus 0 = a^1 \oplus a^2_2 \oplus 0. \]
It is easy to see that the multiplicity sets for the initial vector-operator are: \( s_1 = \mathbb{R} \), \( s_2 = (0, 4) \), \( s_3 = (1, 4) \).

Let us return to Examples 1 and 2. For the distorted vector-operator \( T_1 \oplus T_2 \oplus T_3 \) from Example 1, a spectral measure will be constructed on the maximal vector \( a^{1 \oplus 2 \oplus 3} \). The multiplicity sets \( s_i, i \geq 2 \) have measures zero. For the vector-operator from Example 2 two spectral measures are constructed on \( a^{1 \oplus 2 \oplus 3} \) and
\[ \min \{a^1_{1 \oplus 2}, a^2_{2 \oplus 1}\} \oplus \min \{a^2_{2 \oplus 3}, a^3_{3 \oplus 2}\} \oplus \min \{a^3_{3 \oplus 2}, a^1_{1 \oplus 3}\}, \]
where the sense of the minimums is clear. The multiplicity set \( s_2 \) will be
\[ [P(\delta(T_1)) \cap P(\delta(T_2)) \cup [P(\delta(T_2)) \cap P(\delta(T_3))] \cup [P(\delta(T_3)) \cap P(\delta(T_1))]. \]
Now the term 'distorted vector-operator' is clearly explained by the form of the cyclic vectors for such an operator.

Let \( I = \bigvee_{i \in \Omega} I_i \) denote the sliced union of intervals \( I_i \). Similarly, \( I^k = \bigvee_{j \in A_i} I_j \).
If \( x_i \) are variables on \( I_i \), then \( \forall x_i \) will designate a variable either on \( I \) or \( I^k \) depending on the context. This notation shows, that a vector-function
\[ z = \{z_1(x_1), \ldots, z_n(x_n), \ldots\} \]
on \( I \) or \( I^k \) may be written as \( z(\forall x_i) \). In particular, we may also write \( z(\forall x_i) \) instead of \( z = \bigoplus_{i \in \Omega} z_i \).

Let us introduce the space \( \bigoplus_{i \in \Omega} L^\infty(I^*_i) \). Here, \( z(\forall x_i) \in \bigoplus_{i \in \Omega} L^\infty(I^*_i) \) means that
\[ \sup_{i \in \Omega} \{\text{ess sup}_{x_i \in I^*_i} |z_i(x_i)|\} < \infty, \]
where for each \( i \), families \( \{I^*_i\}^\infty_{n=1} \) represent compact subintervals of \( I_i \), such that
\[ \bigcup_{n=1}^\infty I^*_i = I_i. \] In [3] Lemma 2.1, it was shown that \( \bigoplus_{i \in \Omega} L^\infty(I^*_i) = (\bigoplus_{i \in \Omega} L^1(I^*_i))^* \), where the space of Lebesgue-integrable vector-functions \( \bigoplus_{i \in \Omega} L^1(I^*_i) \) is defined analogously to \( L^1 \).

We also need to introduce a symbolic integral \( \int_{\bigvee J} f(\forall x) d(\forall x) \) defined by:
\[ \int_{\bigvee J} f(\forall x) d(\forall x) = \bigoplus_{i \in \Omega} \int_{J_i} f_i(x_i) \, dx_i, \]
where \( f(\forall x) \) is understood to be measurable relative to \( d(\forall x) \), if \( f_i(x_i) \) are measurable relative to Lebesgue measures \( dx_i \).
Theorem 2.2. Let $T$ be a self-adjoint vector-operator, generated by an EMZ system $\{I_i, \tau_i\}_{i \in \Omega}$. Let $U$ be an ordered representation of the space $L^2 = \oplus_{i \in \Omega} L^2(I_i)$ relative to $T$ with the measure $\theta$ and the multiplicity sets $s_k$, $k = 1, m$. Then there exist kernels $\Theta_k(\forall x, \lambda)$, measurable relative to $d(\forall x) \times \theta$, such that $\Theta_k(\forall x, \lambda) = 0$ for $\lambda \in \mathbb{R} \setminus s_k$ and $\Theta_k(\forall x, \lambda) = 0$ for each fixed $\lambda$. Moreover for any bounded Borel set $\Delta$,

$$
\int_{\Delta} |\Theta_k(\forall x, \lambda)|^2 \, d\theta(\lambda) \in \oplus_{i \in \Omega} L^\infty(I_i) \quad \forall n \in \mathbb{N}.
$$

(2.5)

$$
(Uw)^k(\lambda) = \lim_{n \to \infty} \int_{I_n} w(\forall x) \Theta_k(\forall x, \lambda) \, d(\forall x), \quad w \in L^2,
$$

(2.6)

where the limit exists in $L^2(s_k, \theta)$. The kernels $\{\Theta_k(\forall x, \lambda)\}_{k=1}^n$, $n \leq m$, are linearly independent as vector-functions of the first variable almost everywhere relative to the measure $\theta$ on $s_n$.

Proof. We again need the DMS to prove this theorem. Fix $i$. If $\theta_i$ and $\{e_j^i\}_{j=1}^{m_i}$ are respectively the measure and the multiplicity sets of an ordered representation for $T_i$, then there exists the decomposition $L^i = \oplus_{p=1}^{m_i} I^p(e_i, \theta_i)$, which implies $T_i = \oplus_{p=1}^{m_i} T_i^p$ and $L^i(e_i, \theta_i)$ are $T_i$-invariant. For vector-operator $(\oplus_{i \in \Omega} \oplus_{p=1}^{m_i} T_i^p) \to$ redesignate $\to \oplus T_s$, $s = \{i, p\} \in \Omega$, we may write $\Omega = \bigcup_{k=1}^A A_k$.

Let us separate the proof into units for convenience.

(A) For each $T_j$, $j \in A_k$ and $k = 1, A$, there exists a single cyclic vector $a_j \in L^2$ and [15] XII.3, Lemma 9 and XIII.5, Theorem 1, a function $W_j(x_j, \lambda)$ defined on $I \times e_j$ (note, that for a fixed $i \in \Omega$, $I_j = I_i$ for all $p = 1, m_i$) and measurable relative to $d(x_j) \times \mu_{a_j}$, such that $W_j(x_j, \lambda) = 0$, $\lambda \in \mathbb{R} \setminus e_j$ and for any bounded $\Delta \subset e_j$:

$$
\int_{\Delta} |W_j(x_j, \lambda)|^2 \, d\mu_{a_j}(\lambda) \in L^\infty(I_n^p), \quad n \in \mathbb{N}.
$$

Also

$$
(E^i(\Delta) F_j(T_j) a_j) (x_j) = \int_{\Delta} W_j(x_j, \lambda) F_j(\lambda) \, d\mu_{a_j}(\lambda),
$$

(2.7)

for any $F_j \in L^2(e_j, \mu_{a_j})$. On $I^k = \bigcup_{j \in A_k} I_j$, we construct the vector-function

$$
W^k(\forall x, \lambda) = \{W_1(x_1, \lambda), \ldots, W_n(x_n, \lambda), \ldots\},
$$

which is obviously measurable relative to $d(\forall x) \times \sum \mu_{a_j}$. Separate arguments show that this vector-function is a correctly constructed generalized eigenfunction and thus satisfies the statement of the theorem within each $A_k$.

Note that since for all $p = 1, m_i$ there exists the equality $(\tau_i - \lambda)W_i^p = 0$ (see [15] XIII.5, Theorem 1), it is obvious that $(\oplus_{j \in A_k} \tau_j - \lambda)W^k = 0$, where $\tau_j = \tau_i$ for a fixed $i$ and all $p = 1, m_i$. If $P(\delta(T_i)) \cap P(\delta(T_j))$ has zero spectral measures for all $i, j \in \Omega$, then $A_k : \Omega_1 = \bigcup_{k=1}^{A_k} A_k$ may be constructed such that $A_k$ contains of indices $\{i, k\}$, $i \in \Omega$, $k = 1, \max_i \{m_i\}$.

(B) Consider the set of indices $\Omega_2 = \{j \in \Omega : j = \{i, 1\}, i \in \Omega\}$. Construct $A_k : \Omega_2 = \bigcup_{k=1}^{A_k} A_k$. Apply the reasonings used in (A), considering everywhere $\Omega_2$ instead of $\Omega_1$. Hence, for each $A_k$ and we find a vector-function $W^k(\forall x, \lambda)$ which is the solution of the equation $((\oplus_{j \in A_k} \tau_j - \lambda)y = 0$. Consider $W^k_1$ and $W^k_s$ for $s \neq k$.

For $a^k$ there exists the decomposition $a^k = a^k_k \oplus a^k_s$ (see the proof of Theorem 2.1). This fact induces the decomposition for $W^k_1$: $W^k_1 = W^k_{1,k} \oplus W^k_{1,k,s}$. It is clear that being the restrictions of $W^k_1$, the vector-functions $W^k_1$ and $W^k_{1,k,s}$ are also the solutions of the equation $((\oplus_{j \in A_k} \tau_j - \lambda)y = 0$. They, along with $a^k_k$ and $a^k_s$
define unitary transformations \( U^k_{\alpha;k} \) and \( U^k_{\beta;k} \), such that: \( U^0_k : L^2(a^k_0) \to L^2(\mathbb{R}, \mu_k) \) and \( U^k_{\alpha;k} : L^2(a^k_{\alpha;k}) \to L^2(\mathbb{R}, \mu_k) \) (see the proof of Theorem 2.1). This implies, that the decomposition \( W^k = W^k_{1;k} \oplus W^k_{2;k} \) is correct.

Define as \( \max\{W^k_{1,k,s}, W^k_{2,s,k}\} \) the vector-function, which corresponds to the vector \( \max\{a^k_{\alpha,k}, a^k_{\beta,k}\} \), respectively \( \min\{W^k_{1,k,s}, W^k_{2,s,k}\} \) as the vector-function which corresponds to that \( a^k_{\alpha,k} \) or \( a^k_{\beta,k} \), which is not maximal of the two.

(C) Without loss of generality, suppose that \( k = 1 \) and \( s = 2 \). From the reasonings presented in Part (A) of this proof, it follows that

\[
\Theta_1^{1,2} = W_{1,1}^1 \oplus W_{1,2}^2 \oplus \max\{W_{1,1,2}^1, W_{1,2,1}^1\}
\]

is correctly constructed vector-function satisfying the statement of the theorem for the case \( T = [\mathbb{E}_{j \in A_1} T_j] \oplus [\mathbb{E}_{j \in A_2} T_j] \). Apply the above described process to \( \Theta_1^{1,2} \) and \( W_1^3 \) to obtain the correctly constructed vector-function:

\[
\Theta_1^{1,2,3} = \Theta_1^{1,2} \oplus W_1^3 \oplus \max\{\Theta_1^{1,2,3}, W_1^3, W_2^3, W_3^3, W_1^3, W_2^3, W_3^3\}.
\]

Continuing this process, we finally obtain:

\[
\Theta_1(\mathbb{x}, \mathbf{\lambda}) = \Theta_1^{1,\cdots,\Lambda_2} = \Theta_1^{1,\cdots,\Lambda_2-1} \oplus W_1^{\Lambda_2} \oplus \max\{\Theta_1^{1,\cdots,\Lambda_2-1}, W_1^{\Lambda_2}, W_2^{\Lambda_2}, W_3^{\Lambda_2}, \ldots\},
\]

where in the case of \( \Lambda_2 = \infty \), \( \Theta_1^{1,\cdots,\Lambda_2} \) is the function which satisfies (analogously to (2.3)):

\[
\|\left[\sum_{i=1}^{\infty} \Theta_1^{1,\cdots,\Lambda_2} d\theta(\mathbf{\lambda})\right]\|_2 = \lim_{L \to \infty} \left\|\sum_{j=1}^{L} E_j(\Delta) \right\|_2 \int_{\Delta} \Theta_1^{1,\cdots,\Lambda_2} d\theta_L(\mathbf{\lambda}),
\]

for any bounded Borel set \( \Delta \), where

\[
\theta_L(\cdot) = \left[\sum_{j=1}^{L} E_j(\cdot)\right] a^{1,\cdots,\Lambda_2} + a^{1,\cdots,\Lambda_2}
\]

is the measure of the ordered representation of the space \( \oplus_{j=1}^{L} L^2_j \). The limit on the right side exists and in fact it appears that

\[
\int_{\Delta} \Theta_1^{1,\cdots,\Lambda_2} d\theta_L(\mathbf{\lambda}) \to \int_{\Delta} \Theta_1^{1,\cdots,\Lambda_2} d\theta(\mathbf{\lambda}),
\]

as \( L \to \infty \).

(D) Define \( \Omega_3 = \{j \in \Omega_1 : j = \{i, 2\}, i \in \Omega\} \). Construct \( A_k : \Omega_3 = \cup_{k=1}^{A_1} A_k \). Apply processes (B) and (C) of this proof, substituting everywhere \( \Omega_3 \) instead of \( \Omega_2 \). We obtain a vector-function \( \Theta_2^{1,\cdots,\Lambda_3} \), which is defined on the set \( \cup_i P(e_i^k) \). But, as we know (see (2.4)), the set \( s_2 \) also includes the sets where there are non-empty superpositions of \( \delta(T_i) \). Therefore, designating

\[
\Theta_1^2 = \Theta_2^{1,\cdots,\Lambda_3}, \quad \Theta_3^2 = \min\{W_{1,1,2}^1, W_{1,2,1}^1\},
\]

we may again use the process (C) to build the vector-function \( \Theta_2(\mathbb{x}, \mathbf{\lambda}) \) defined on \( s_2 \) and \( \Theta_2(\mathbb{x}, \mathbf{\lambda}) = 0 \) for \( \lambda \in \mathbb{R} \setminus s_2 \). Using processes (B), (C), (D) and formula (2.4), we finally obtain \( \Theta_m(\mathbb{x}, \mathbf{\lambda}) \).

(E) The linear independence is proved by separate arguments.
**Theorem 2.3** (Eigenfunction expansions). For any \( w \in L^2 \), there exists a decomposition

\[
    w = \sum_{k=1}^{m} \lim_{n \to \infty} \int_{-n}^{+n} (Uw)^k(\lambda) \Theta_k(\forall x_i, \lambda) \, d\theta(\lambda),
\]

Since the kernels from Theorem 2.2 are only measurable relative to \( \lambda \), the following theorem is important:

**Theorem 2.4.** Each kernel \( \Theta_k(\forall x_i, \lambda) \), \( k = 1, \ldots, m \), may be decomposed as

\[
    \Theta_k(\forall x_i, \lambda) = \sum_{s=1}^{M_k} \gamma_{sk}(\lambda)\sigma_{sk}(\forall x_i, \lambda),
\]

where the \( M_k \) are finite for each \( k \) and \( \sigma_{sk}(\forall x_i, \lambda) \) depend analytically on \( \lambda \).

**Proof.** We separate the proof in parts which will correspond to the analogous parts of the proof of Theorem 2.2.

\( (A^*) \) Each kernel \( W_j(x_j, \lambda) \) from the part (A) of the proof of Theorem 2.2 may be decomposed:

\[
    W_j(\cdot, \lambda) = \sum_{s=1}^{n_j} \alpha_{js}(\lambda)\sigma_{js}(\cdot, \lambda),
\]

where \( \alpha_{js} \) are supposed to equal zero on \( \mathbb{R} \setminus e_j \), see \cite[p. 1351]{15}. Supplemen
ting the defining systems with zeros where necessary, we obtain:

\[
    W_k^e(\forall x_j, \lambda) = \oplus_{j \in A_k} W_j(x_j, \lambda)
    = \oplus_{j \in A_k} \sum_{s=1}^{n_j} \alpha_{js}(\lambda)\sigma_{js}(x_j, \lambda)
    = \sum_{q=1}^{N_k} \oplus_{j \in A_k} \alpha_{jq}(\lambda)\sigma_{jq}(x_j, \lambda)
    = \sum_{q=1}^{N_k} \alpha^k_q(\lambda)\sigma_q^k(\forall x_j, \lambda),
\]

where

\[
    N_k = \max_{j \in A_k} n_j, \quad \alpha^k_q(\lambda) = \sum_{j \in A_k} \alpha_{jq}(\lambda), \quad \sigma_q^k(\forall x_j, \lambda) = \oplus_{j \in A_k} \sigma_{jq}(x_j, \lambda).
\]

Since \( e_j \) and \( e_k \) do not intersect almost everywhere for \( j, k \in \Omega_2, j \neq k \), the series \( \sum_{j \in A_k} \alpha_{jq}(\lambda) \) converges almost everywhere on \( \cup_{j \in \Omega_2} P(e_j) \).

\( (B^*) \) and \( (C^*) \) Now pass to the part (B). There we obtained the decompositions

\[
    W^k_1 = W^k_{1,k} + W^k_{1,k,s} \quad \text{and} \quad W^s_1 = W^s_{1,s} + W^s_{1,s,k}.
\]

Let us totally order the set \( \{T^j\}_{j=1}^{\Lambda_2} \) saying that \( T^k \preceq T^s \) if \( \max\{W^k_{1,k,s}, W^s_{1,s,k}\} = W^k_{1,k,s} \). At that, \( T^k \approx T^s \) if and only if \( T^k \preceq T^s \) and \( T^s \preceq T^k \). According to this, we build \( \oplus_{j=1}^{\Lambda_2} T^j \), where \( T^j \preceq T^{j+1} \), \( j = 1, \Lambda_2 - 1 \) if \( \Lambda_2 \geq 2 \). The obtained vector-operator is obviously equivalent to the initial vector-operator (comprising unordered operators). Note that

\[
    W^k_1(\forall x_i, \lambda) = \sum_{q=1}^{N_k} \alpha^k_q(\lambda)\sigma_q^k(\forall x_j, \lambda)
\]
and analogously
\[ W_1^s(\forall x_i, \lambda) = \sum_{p=1}^{N_s} \alpha_{1p}^s(\lambda)\sigma_{1p}(\forall x_j, \lambda). \]

All the above leads to the following equality:
\[
\Theta_1^{1\oplus 2} = W_1^1 \oplus W_1^2 \oplus \max\{W_1^{1,1,2}, W_1^{2,1,2}\} = W_1^1 \oplus W_1^2 \\
= \left( \sum_{q=1}^{N_1} \alpha_{1q}^1(\lambda)\sigma_{1q}(\forall x, \lambda) \right) \oplus \left( \sum_{p=1}^{N_2} \alpha_{1p}^2(\lambda)\chi_{\delta_2}(\lambda)\sigma_{1p}^2(\forall x, \lambda) \right) \\
= \sum_{s=1}^{N^{1\oplus 2}} \alpha_{1s}^{1\oplus 2}(\lambda)\sigma_{1s}^{1\oplus 2}(\forall x, \lambda),
\]

where \( N^{1\oplus 2} = \max\{N_1, N_2\} \); \( \alpha_{1s}^{1\oplus 2}(\lambda) = \alpha_{1s}^1(\lambda) + \alpha_{1s}^2(\lambda)\chi_{\delta_2}(\lambda) \),
\[ \sigma_{1s}^{1\oplus 2}(\forall x, \lambda) = \sigma_{1s}^1(\forall x, \lambda) \oplus (\sigma_{1s}^2(\forall x, \lambda)\chi_{\delta_2}(\lambda)), \]

\( s = \overline{1, N^{1\oplus 2}} \).

Continuing this process until the finite \( \Lambda_2 \), we obtain:
\[
\Theta_1(\forall x, \lambda) = \Theta_1^{1\oplus \cdots \oplus \Lambda_2} = \sum_{s=1}^{N^{1\oplus \cdots \oplus \Lambda_2}} \alpha_{1s}^{1\oplus \cdots \oplus \Lambda_2}(\lambda)\sigma_{1s}^{1\oplus \cdots \oplus \Lambda_2}(\forall x, \lambda), \tag{2.10}
\]

where \( N^{1\oplus \cdots \oplus \Lambda_2} = \max\{N_1, N_2, \ldots, N_{\Lambda_2}\} \) and for \( s = \overline{1, N^{1\oplus \cdots \oplus \Lambda_2}} \):
\[
\alpha_{1s}^{1\oplus \cdots \oplus \Lambda_2}(\lambda) = \alpha_{1s}^1(\lambda) + \sum_{i=2}^{\Lambda_2} \alpha_{1s}^i(\lambda)\chi_{\delta_i}(\lambda); \\
\sigma_{1s}^{1\oplus \cdots \oplus \Lambda_2}(\forall x, \lambda) = \sigma_{1s}^1(\forall x, \lambda) \oplus (\oplus_{i=2}^{\Lambda_2} \sigma_{1s}^i(\forall x, \lambda)\chi_{\delta_i}(\lambda)). \tag{2.11}
\]

In the case of infinite \( \Lambda_2 \), \( N^{1\oplus \cdots \oplus \Lambda_2} \) is clearly finite. The series in the right side of \( (2.11) \) pointwise converges, since it consists of items defined on non-intersecting sets. \( \sigma_{1s}^{1\oplus \cdots \oplus \Lambda_2}(\forall x, \lambda) \) is defined by induction.

\((D^*)\) Borrowing the designations from \((D)\) and using processes described in \((A^*)\) and \((C^*)\), we shall come to the decomposition of \( \Theta_2^{1\oplus \cdots \oplus \Lambda_3} \):
\[
\Theta_2^{1\oplus \cdots \oplus \Lambda_3} = \sum_{s=1}^{N^{1\oplus \cdots \oplus \Lambda_3}} \alpha_{2s}^{1\oplus \cdots \oplus \Lambda_3}(\lambda)\sigma_{2s}^{1\oplus \cdots \oplus \Lambda_3}(\forall x, \lambda).
\]

To obtain \( \Theta_2(\forall x, \lambda) \), as in \((D)\), we repeat part \((C^*)\) for
\[
\Theta_2^1 = \Theta_2^{1\oplus \cdots \oplus \Lambda_3}, \quad \Theta_2^2 = W_2^{2,1,2}, \quad \ldots, \quad \Theta_2^{\Lambda_2+1} = W_1^{\Lambda_2,1,\Lambda_2,1\oplus \cdots \oplus \Lambda_2-1}.
\]

Finally, the same way we obtain decompositions for all \( \Theta_k(\forall x, \lambda), \ k = \overline{1, m} \), which will have the form \( (2.9) \).

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References


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