Last zero time or maximum time of the winding number of Brownian motions

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Abstract

In this paper we consider the winding number, $\theta(s)$, of planar Brownian motion and study asymptotic behavior of the process of the maximum time, the time when $\theta(s)$ attains the maximum in the interval $0 \leq s \leq t$. We find the limit law of its logarithm with a suitable normalization factor and the upper growth rate of the maximum time process itself. We also show that the process of the last zero time of $\theta(s)$ in $[0,t]$ has the same law as the maximum time process.

Keywords: Brownian motion; winding number; Last zero time; Maximum time.

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1 Introduction and Main results

In this paper we seek for an analogue of the arcsine law of the linear Brownian motion for the argument of a complex Brownian motion $\{W(t) = W_1(t) + iW_2(t) : t \geq 0\}$ started at $W(0) = (1,0)$. Skew-product representation tells us that there exist two independent linear Brownian motions $\{B(t) : t \geq 0\}$ and $\{\hat{B}(t) : t \geq 0\}$ such that

$$W(t) = \exp(\hat{B}(H(t))) + iB(H(t))$$

for all $t \geq 0$, (1.1)

where

$$H(t) = \int_0^t ds \frac{dW(s)}{|W(s)|^2} = \inf\{u \geq 0 : \int_0^u \exp(2\hat{B}(s))ds > t\},$$

which entails that $B$ is independent of $|W|$ and hence of $H$, while $\log |W|$ is time change of $\hat{B}$ (cf. e.g., [5], Theorem 7.26).

We let $\theta(t) = B(H(t))$ so that $\theta(t) = \arg W(t)$, which we call the winding number. Without loss of generality we suppose $\theta(0) = 0$. The well-known result of Spitzer [9] states the convergence of $2\theta(t)/\log t$ in law:

$$\lim_{t \to \infty} P\left(\frac{2\theta(t)}{\log t} \leq a\right) = \frac{1}{\pi} \int_{-\infty}^a \frac{dx}{1 + x^2}.$$  

It is shown in [1] that for any increasing function $f : (0, \infty) \to (0, \infty)$

$$\limsup_{t \to \infty} \frac{\theta(t)}{f(t)} = 0 \text{ or } \infty \quad \text{a.s.}$$

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according as the integral \( \int_1^\infty \frac{f(t)}{t} dt \) converges or diverges and

\[
\liminf_{t \to \infty} \frac{1}{f(t)} \sup\{\theta(s), 1 \leq s \leq t\} = 0 \text{ or } \infty \text{ a.s.}
\]

according as the integral \( \int_1^\infty \frac{f(t)}{t \log t} dt \) diverges or converges; moreover, it is shown that the square root of the random time \( H_t \) is subjected to the same growth law as of \( \theta \) in (1.2) and the \( \liminf \) behavior of \( H_t \) is also given. Another proof of (1.2) is given in [8]. Also, it is shown in [7]

\[
\liminf_{t \to \infty} \log \log \log \frac{t}{\log t} \sup\{|\theta(s)|, 1 \leq s \leq t\} = \pi \frac{1}{4} \text{ a.s.}
\]

Before advancing our result we recall the two arcsine laws whose analogues are studied in this paper. Let \( \{B(t) : t \geq 0\} \) be a standard linear Brownian motion started at zero and denote by \( Z_t \) the time when the maximum of \( B_s \) in the interval \( 0 \leq s \leq t \) is attained. Then, the process \( Z_t \) and the process \( \sup\{s \in [0, t] : B(s) = 0\} \), the last zero of Brownian motion in the time interval \( [0, t] \), are subject to the same law, and according to Lévy’s arcsine law the scaled variable \( Z_t / t \) is subject to the arcsin law. (cf. e.g., [5] Theorem 5.26 and 5.28)

In order to state the results of this paper we set

\[
V(a) = \frac{4}{\pi^2} \int_0^1 \int_{0 \leq y \leq ax} \frac{dx}{1 + x^2} \frac{dy}{1 + y^2}.
\]

(1.3)

We also define a random variable \( M_t \in [0, t] \) by

\[
\theta(M_t) = \max_{s \in [0, t]} \theta(s),
\]

the time when \( \theta(s) \) attains the maximum in the interval \( 0 \leq s \leq t \), and a random variable \( L_t \) by

\[
L_t = \sup\{s \in [0, t] : \theta(s) = 0\},
\]

the last zero of \( \theta(s) \) in \( [0, t] \). According to Theorem 2.11 of [5] a linear Brownian motion attains its maximum at a single point on each finite interval with probability one. In view of the representation \( \theta(t) = B(H(t)) \), it therefore follows that the maximiser \( M_t \) is uniquely determined for all \( t \) with probability one.

**Theorem 1.1.** (a) For every \( 0 < a < 1 \)

\[
\lim_{t \to \infty} P\left( \frac{\log M_t}{\log t} \leq a \right) = V\left( \frac{a}{1 - a} \right).
\]

(b) It holds that

\[
\{L_t : t \geq 0\} = \{M_t : t \geq 0\}.
\]

**Theorem 1.2.** Let \( \alpha(t) \) be a positive function that is non-increasing, tends to zero as \( t \to \infty \) and satisfies

\[
2\alpha(t^e) \geq \alpha(t),
\]

(1.4)

and put

\[
I\{\alpha\} = \int_1^\infty \frac{\alpha(t)|\log \alpha(t)|}{t \log t} dt.
\]
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Then, with probability one

$$\liminf_{t \to \infty} \frac{M_t}{\rho(t)} = \infty \text{ or } 0$$

according as the integral $I\{\alpha\}$ converges or diverges.

It may be worth noting that the distribution function $V(a/(1-a))$ ($0 \leq a \leq 1$) is expressed as

$$V\left(\frac{a}{1-a}\right) = \int_0^a \frac{1}{2u-1} \log \frac{u}{1-u} du.$$ 

Indeed,

$$V'(c) = \int_0^{\infty} \frac{xdx}{(1+x^2)(1+c^2x^2)} = \frac{\log c}{c^2-1} \quad (c \neq 1),$$

where

$$\frac{d}{da} V\left(\frac{a}{1-a}\right) = \frac{1}{(1-a)^2} V'\left(\frac{a}{1-a}\right) \quad (a \neq \frac{1}{2}),$$

and we find the density asserted above.

2 Proofs

2.1 Proof of Theorem 1.1

Let $\{N(t) : t \geq 0\}$ be the maximum process of a winding number $\{\theta(t) : t \geq 0\}$, i.e. the process defined by

$$N(t) = \max_{s \in [0,t]} \theta(s).$$

Lemma 2.1. If $a > 0$, then $P(N(t) > a) = 2P(\theta(t) > a) = P(|\theta(t)| > a)$.

Proof. By reflection principle [5], (Theorem 2.21) it holds that for any $t > 0$

$$\max_{0 \leq l \leq t} B(l) = d |B(t)|.$$

By Skew-product representation $B(t)$ is independent of $|W(t)|$, hence since $B(l)$ is independent of $H(t) = \int_0^t \frac{dm}{|W(m)|}$, it holds

$$\max_{0 \leq l \leq t} B(H(l)) = d |B(H(t))|,$$

showing the assertion of the lemma.

Lemma 2.2. $\{N(t) - \theta(t) : t \geq 0\} = d \{|\theta(t)| : t \geq 0\}$.

Proof. According to Lévy’s representation of the reflecting Brownian motion [5], (Theorem 2.34) we have

$$\{\max_{0 \leq l \leq t} B(l) - B(t) : t \geq 0\} = d \{|B(t)| : t \geq 0\}.$$

Hence as in the preceding proof,

$$\{\max_{0 \leq l \leq t} B(H(l)) - B(H(t)) : t \geq 0\} = d \{|B(H(t))| : t \geq 0\},$$

as desired.
Proof of Theorem 1.1. Lemma 2.2 together with Lemma 2.1 show that the process \( \{M_s : s \geq 0\} \) has the same law as \( \{L_s : s \geq 0\} \), being nothing but the last zero of the process \( \{N(t) - \theta(t) : 0 \leq t \leq s\} \) for any \( s \). So it remains to prove part (a). Fix \( \epsilon > 0 \). Set \( T_\epsilon = \inf\{t \geq 0 : |W(t)| = c\} \), for which we sometimes write \( T(c) \) for typographical reasons. We first prove the upper bound. By (1.1) it holds that

\[
P(M_t < t^\alpha) = P\left( \max_{0 \leq u \leq t^\alpha} B(H(u)) > \max_{t^\alpha \leq u \leq t} B(H(u)) \right) = P\left( \max_{0 \leq u \leq t^\alpha} B(H(u)) - B(H(t^\alpha)) > \max_{t^\alpha \leq u \leq t} B(H(u)) - B(H(t^\alpha)) \right)
\]

where \( \tilde{B} \) is a linear Brownian motion started at zero which is independent of \( W \). Corresponding to (1.1) we can write \( \bar{W}(0) = (1, 0) \), arg \( W(l) = \bar{B}(\bar{H}(l)) \), \( \bar{H}(l) = \int_0^l \frac{dm}{|W(m)|^2} \) with \( \bar{W} \) independent of \( W \), and put \( \bar{T}_\epsilon = \inf\{l \geq 0 : |ar{W}(l)| = c\} \). By Lemma 2.1 and Lemma 2.2 we have \( \max_{0 \leq u \leq t^\alpha} B(H(u)) - B(H(t^\alpha)) = \max_{0 \leq u \leq t^\alpha} B(H(u)) \), and therefore

\[
P\left( \max_{0 \leq u \leq t^\alpha} B(H(u)) - B(H(t^\alpha)) > \max_{t^\alpha \leq u \leq t} B(H(u)) - \tilde{B}(H(t^\alpha)) \right) = P\left( \max_{0 \leq u \leq t^\alpha} B(H(u)) > \max_{t^\alpha \leq u \leq t} \tilde{B}(H(u)) - B(H(t^\alpha)) \right).
\]

By standard large deviation result (cf. e.g., [4], (11) and (12)), given \( \epsilon > 0 \), it holds that for all sufficiently large \( t \)

\[
P(t^\alpha \leq T_{\frac{a+b}{a},t} \leq T_{\frac{a+b}{b},t} \leq t) \geq 1 - \epsilon.
\]

Therefore, we get

\[
P\left( \max_{0 \leq u \leq t^\alpha} B(H(u)) > \max_{t^\alpha \leq u \leq t} \tilde{B}(H(u)) - B(H(t^\alpha)) \right) \leq P\left( \max_{0 \leq u \leq T(\frac{a+b}{a})} B(H(u)) > \max_{T(\frac{a+b}{a}) \leq u \leq T(\frac{a+b}{b})} \tilde{B}(H(u)) - B(H(T_{\frac{a+b}{a},t})) \right) + \epsilon.
\]

Also, strong Markov property tells us

\[
\int_{T_{\frac{a+b}{a},t}}^{T_{\frac{a+b}{b},t}} \frac{dm}{|W(m)|^2} = d \int_0^{T_{\frac{a+b}{b},t}} \frac{dm}{|W(m)|^2},
\]

and \( H(T_{\frac{a+b}{a},t}) \) is independent of \( H(T_{\frac{a+b}{b},t}) \).

So, if we set for \( a, b < \infty \)

\[
Q(a, b) = P\left( \max_{0 \leq u \leq T_{\epsilon}} B(H(u)) > \max_{0 \leq u \leq T(\epsilon)} \tilde{B}(H(u)) \right),
\]

it holds that

\[
P\left( \max_{0 \leq u \leq T(\frac{a+b}{a})} B(H(u)) > \max_{T(\frac{a+b}{a}) \leq u \leq T(\frac{a+b}{b})} \tilde{B}(H(u)) - B(H(T_{\frac{a+b}{a},t})) \right) = Q(\frac{a+b}{a}, T(\frac{a+b}{a})),
\]

Note that by Skew-product representation \( B(l)( \text{resp. } \tilde{B}(l)) \) is independent of \( H(T_{\frac{a+b}{a},t}) \) (resp. \( \tilde{H}(T_{\frac{a+b}{a},t}) \)). Then, if \( \tilde{\theta}(l) = \tilde{B}(\tilde{H}(l)) \), by reflection principle we get

\[
Q(\frac{a+b}{a}, T(\frac{a+b}{a})) = P\left( |B(H(T_{\frac{a+b}{a},t}))| > |\tilde{B}(\tilde{H}(T_{\frac{a+b}{a},t}))| \right)
\]

\[
= P\left( |\theta(T_{\frac{a+b}{a},t})| > |\tilde{\theta}(T_{\frac{a+b}{a},t})| \right).
\]

(2.5)
Moreover, since $\theta(T_n)$ follows the Cauchy distribution with parameter $|\log r|$ (cf. e.g., [6], Section 5, Exercise 2.16, [11], Proposition 2.3, and [12]), we get

$$Q(t^{\frac{1}{\alpha}}, t^{\frac{1-\epsilon}{\alpha}}) = P(|\theta(T_n)| > |\theta(T_n)|^{\frac{1}{\alpha}}) = V\left(\frac{a+\epsilon}{1-a-2\epsilon}\right). \tag{2.6}$$

Therefore, since $\epsilon$ is arbitrary, this gives the desired upper bound.

Next, we prove the lower bound. By standard large deviation result (cf. e.g., [4], (11) and (12)), given $\epsilon > 0$, it holds that for all sufficiently large $t$

$$P(T_n^{\frac{1}{\alpha}} < t, t < T_n^{\frac{1}{\alpha}}) \geq 1 - \epsilon. \tag{2.7}$$

Moreover, by repeating the argument in (2.1) and (2.4), we get

$$P\left(\max_{0 \leq u \leq t} B(H(u)) > \max_{t^2 \leq u \leq t} \tilde{B}(H(u)) - \tilde{B}(H(t^2))\right) \geq Q(t^{\frac{a}{\alpha}}, t^{\frac{1-\epsilon}{\alpha}}) - \epsilon.$$ \(\square\)

Moreover, since $\theta(T_n)$ follows the Cauchy distribution with parameter $|\log r|$ (cf. e.g., [6], Section 5, Exercise 2.16, [11], Proposition 2.3, and [12]), we get

$$Q(t^{\frac{1}{\alpha}}, t^{\frac{1-\epsilon}{\alpha}}) = P(|\theta(T_n)| > |\theta(T_n)|^{\frac{1}{\alpha}}) = V\left(\frac{a+\epsilon}{1-a-2\epsilon}\right). \tag{2.6}$$

Therefore, since $\epsilon$ is arbitrary, this gives the desired upper bound.

Next, we prove the lower bound. By standard large deviation result (cf. e.g., [4], (11) and (12)), given $\epsilon > 0$, it holds that for all sufficiently large $t$

$$P(T_n^{\frac{1}{\alpha}} < t, t < T_n^{\frac{1}{\alpha}}) \geq 1 - \epsilon. \tag{2.7}$$

Moreover, by repeating the argument in (2.1) and (2.4), we get

$$P\left(\max_{0 \leq u \leq t} B(H(u)) > \max_{t^2 \leq u \leq t} \tilde{B}(H(u)) - \tilde{B}(H(t^2))\right) \geq Q(t^{\frac{a}{\alpha}}, t^{\frac{1-\epsilon}{\alpha}}) - \epsilon.$$ \(\square\)

2.2 Proof of Theorem 1.2

Proof of Theorem 1.2. We first prove $\lim \inf_{t \to \infty} M_t / t^{\alpha(t)} = \infty$ if $I\{\alpha\} < \infty$. We may replace $\alpha(t)$ by $\alpha(t) \vee (\log \log t)^{-2}$. Indeed, if we set

$$\tilde{\alpha}(t) = \alpha(t) 1\{\alpha(t) > (\log \log t)^{-2}\} + (\log \log t)^{-2} 1\{\alpha(t) \leq (\log \log t)^{-2}\},$$

$I\{\tilde{\alpha}\} < \infty$. By standard large deviation result (cf. e.g., [4], (11) and (12)) for any $q < \infty$ there exist $0 < c_1, c_2 < \infty$ such that

$$P(qt^{\tilde{\alpha}(t)}) \leq T(t^{\tilde{\alpha}(t)}), T(t^{-\tilde{\alpha}(t)}) \leq \tilde{t} \geq 1 - c_1 \exp(-t^{c_2\tilde{\alpha}(t)}). \tag{2.8}$$

Therefore, by the same arguments as made for (2.1), (2.2), (2.3), (2.4), (2.5) and (2.6) we infer that for any $q < \infty$

$$P(M_t < qt^{\alpha(t)}) = P\left(\max_{0 \leq u \leq qt^{\alpha(t)}} B(H(u)) - B(H(qt^{\alpha(t)}))) > \max_{qt^{\alpha(t)} \leq u \leq t} \tilde{B}(H(u)) - \tilde{B}(H(qt^{\alpha(t)})))\right) \leq Q(t^{\alpha(t)}, t^{\frac{1}{2} - 5\alpha(t)}) + c_1 \exp(-t^{c_2\alpha(t)})$$

$$= V\left(\frac{4\alpha(t)}{2 - 5\alpha(t)}\right) + c_1 \exp(-t^{c_2\alpha(t)}).$$

We set $t_n = \exp(e^n)$. Then, noting that $V(\alpha(n)) = O(\alpha(n) |\log \alpha(n)|$, we deduce from (2.8) that for some $C < \infty$

$$P(M_{t_n} < t_n^{\alpha(t_n)}) \leq C\alpha(t_n) |\log \alpha(t_n)| + c_1 \exp(-t_n^{c_2\alpha(t_n)}).$$

The sum of the right-hand side over $n$ is finite since $\sum_{n=1}^{\infty} \alpha(t_n) |\log \alpha(t_n)| < \infty$ if $I\{\alpha\} < \infty$, and $\alpha(t) \geq (\log \log t)^{-2}$ according to our assumption. Thus, by Borel-Cantelli lemma for any $q < \infty$, with probability one

$$\frac{M_{t_n}}{t_n^{\alpha(t_n)}} > q \quad \text{for almost all } n. \tag{2.9}$$
Note that if we choose \( t \) such that \( t_n < t \leq t_{n+1} \), then \( t_n^{\alpha(t_n)} > t^{\alpha(t)} \) and from (2.9) it follows that \( M_t > M_{t_n} > q t^{\alpha(t)} \) for all sufficiently large \( n \). Hence,

\[
\liminf_{t \to \infty} \frac{M_t}{t^{\alpha(t)}} > q \quad \text{a.s.}
\]

Since \( q < \infty \) is arbitrary, this concludes the proof.

Next, we prove \( \lim \inf_{t \to \infty} M_t/t^{\alpha(t)} = 0 \) assuming that \( I\{\alpha\} = \infty \). For any \( a < b < c \), we set

\[
\theta^*[a,b] = \max\{\theta(t) : T_a \leq t \leq T_b\},
\]

and define \( \overline{M}[a,b] \) via

\[
\theta(\overline{M}[a,b]) = \theta^*[a,b] \quad \text{and} \quad T_a \leq \overline{M}[a,b] \leq T_b.
\]

Recall we have set \( t_n = \exp(e^n) \). For \( q > 0 \), denote by \( A_n \) the event

\[
\overline{M}[qt^{\alpha(t_n)}, t_n] < T(q t^{2\alpha(t_n)}).
\]

Bringing in the set \( D = \{ n \in \mathbb{N} : \alpha(t_n) > \frac{1}{\log \log t_n}\} \), we shall prove \( \sum_{n=1}^{\infty} P(A_n) = \infty \) and

\[
\liminf_{n \in D, n \to \infty} \frac{\sum_{j=1}^{n} \sum_{i=1}^{k} P(A_j \cap A_k)}{\left( \sum_{j=1}^{n} P(A_j) \right)^2} < \infty,
\]

(2.10)

which together imply \( P(\limsup_{n \in D, n \to \infty} A_n) = 1 \) according to the Borel-Cantelli lemma (cf. [10], p.319 or [3]) and Kolmogorov’s 0–1 law. First we prove \( \sum_{n=1}^{\infty} P(A_n) = \infty \). Note that it holds that for \( 0 < a < b < c \)

\[
P(\theta^*[a,b] > \theta^*[b,c]) = P(\theta^*[1,\frac{b}{a}] > \theta^*[\frac{b}{a},\frac{c}{a}]).
\]

Thus,

\[
P(\theta^*[qt^{\alpha(t)}, q t^{2\alpha(t)}] > \theta^*[q t^{\alpha(t)}, t]) = P(\theta^*[1, t^{\alpha(t)}] > \theta^*[t^{\alpha(t)}, \frac{1}{q} t^{1-\alpha(t)}]).
\]

Therefore, we get by the same argument as employed for (2.1), (2.2), (2.3), (2.4), (2.5) and (2.6)

\[
P(\overline{M}[qt^{\alpha(t)}, t] \in T(q t^{2\alpha(t)}))
\]

\[
= P(\theta^*[1, t^{\alpha(t)}] > \theta^*[t^{\alpha(t)}, \frac{1}{q} t^{1-\alpha(t)}])
\]

\[
= P\left( \max_{u \leq T(t^{\alpha(t)})} B(H(u)) - B(H(t^{\alpha(t)})) > \max_{T(t^{\alpha(t)}) \leq u \leq T(\frac{1}{q} t^{1-\alpha(t)})} \tilde{B}(H(u)) - \tilde{B}(H(t^{\alpha(t)})) \right)
\]

\[
= Q\left( t^{\alpha(t)}, \frac{1}{q} t^{1-2\alpha(t)} \right)
\]

\[
= V\left( \frac{\alpha(t)}{1-2\alpha(t)} - (\log t \log q)^{-1} \right).
\]

(2.11)

Moreover, using \( V(\alpha(n)) \asymp \alpha(n) \log \alpha(n) \) again, we get for some \( C > 0 \)

\[
P(A_n) \geq C \alpha(t_n) |\log \alpha(t_n)|.
\]

It holds that \( \sum_{n \in D} \alpha(t_n) |\log \alpha(t_n)| = \infty \) if \( I\{\alpha\} = \infty \), since \( \sum_{n \notin D} \alpha(t_n) |\log \alpha(t_n)| < \infty \). So we get \( \sum_{n \in D} P(A_n) = \infty \).
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Next we prove (2.10). We only need to consider \( \sum_{j=1,j \in D} \sum_{k<j,k \in D} P(A_j \cap A_k) \). First we consider \( \sum_{j=1,j \in D} \sum_{k < j,k \in D} P(A_j \cap A_k) \) where \( R_{k,j} = \{ k : q^\alpha(t_k) \geq t_k \} \). Note that for \( a < b \leq c < d < \infty \)
\[
\mathcal{M}[a,b] - T_a \text{ is independent of } \mathcal{M}[c,d] - T_c. \tag{2.12}
\]
Then, since \( q_k^\alpha(t_k) < t_k \leq q_j^\alpha(t_j) < t_j \) when \( k \) is satisfied with \( q_j^\alpha(t_j) \geq t_k \), it holds that
\[
P(A_j \cap A_k) = P(A_j) P(A_k). \tag{2.13}
\]
So, next we consider the case \( q_j^\alpha(t_j) < t_k \). We denote by \( A'_{k,j} \) the event \( \mathcal{M}[q_k^\alpha(t_k), q_j^\alpha(t_j)] < T(q_k^\alpha(t_k)) \). Note that when \( k \) is satisfied with \( q_j^\alpha(t_j) < t_k \), we have \( A_k \subset A'_k \), and by (2.12) \( P(A_j \cap A'_{k,j}) = P(A_j) P(A'_k) \). Then, since by the same argument for (2.11)
\[
P(A'_{k,j}) = V(e^{\alpha(t_k)}), \]
we get
\[
P(A_j \cap A_k) \leq P(A_j \cap A'_{k,j}) = P(A_j) P(A'_k) = P(A_j) V(e^{\alpha(t_k)}), \tag{2.14}
\]
Furthermore, since \( \alpha(t_k) \leq 2 \alpha(t_{k+1}) \) due to the assumption (1.4), we get
\[
\sum_{k \in R_{c}^c, k < j, k \in D} P(A'_{k,j}) = \sum_{k \in R_{c}^c, k < j, k \in D} V(e^{\alpha(t_k)}),
\]
\[
\leq \sum_{k=1}^{\infty} V(e^{\alpha(t_k)}),
\]
where \( R_{c}^c = \{ k : q_j^\alpha(t_j) < t_k \} \). So, by (2.14) and (2.15) we get
\[
\sum_{j=1,j \in D} \sum_{k \leq j,k \in D} P(A_j \cap A_k) \leq \sum_{j=1,j \in D} \sum_{k \leq j,k \in D} P(A_j) P(A_k) + C' \sum_{j=1,j \in D} P(A_j),
\]
completing the proof of (2.10). Therefore, we can conclude that with probability one
\[
\mathcal{M}[q_n^{\alpha(t_n)}, t_n] < T(q_n^{2\alpha(t_n)}) \text{ infinitely often for } n \in D. \tag{2.16}
\]
On the other hand, by standard large deviation result (cf. e.g., [4], (11) and (12)) there exist \( 0 < c_3, c_4 < \infty \) such that
\[
P(T(q_n^{2\alpha(t)} > q_n^{\alpha(t)}, t_n < T_n)) \geq 1 - c_3 \exp(-c_4 t_n^{\alpha(t)}).
\]
Moreover, \( \sum_{n \in D} c_3 \exp(-c_4 t_n^{\alpha(t)}) < \infty \). Then, by Borel-Cantelli lemma it holds that with probability one
\[
T(q_n^{2\alpha(t_n)}) \leq q_n^{\alpha(t_n)}, \quad M_{\frac{t_n}{2}} \leq \mathcal{M}[q_n^{\alpha(t_n)}, t_n], \text{ for almost all } n \in D. \tag{2.17}
\]
So, by (2.16) and (2.17) it holds that
\[
\liminf_{t \to \infty} \frac{M_t}{q^{2\alpha(t)}} \leq \liminf_{n \in D, n \to \infty} \frac{M_n}{q^{2\alpha(t_n)}} \leq \liminf_{n \in D, n \to \infty} \frac{M_{\frac{t_n}{2}}}{q^{\alpha(t_n)}} \leq \liminf_{n \in D, n \to \infty} \frac{\mathcal{M}[q_n^{\alpha(t_n)}, t_n]}{T(q_n^{2\alpha(t_n)})} < 1 \text{ a.s.}
\]
The proof finishes since \( q > 0 \) is arbitrary by replacing \( \alpha(t) \) by \( \frac{\alpha(t)}{2\alpha} \).
\[\square\]
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References


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