Weighted $L^2$–cohomology of Coxeter groups based on barycentric subdivisons

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Abstract

Associated to any finite flag complex $L$ there is a right-angled Coxeter group $W_L$ and a contractible cubical complex $\Sigma_L$ (the Davis complex) on which $W_L$ acts properly and cocompactly, and such that the link of each vertex is $L$. It follows that if $L$ is a generalized homology sphere, then $\Sigma_L$ is a contractible homology manifold. We prove a generalized version of the Singer Conjecture (on the vanishing of the reduced weighted $L^2_q$–cohomology above the middle dimension) for the right-angled Coxeter groups based on barycentric subdivisions in even dimensions. We also prove this conjecture for the groups based on the barycentric subdivision of the boundary complex of a simplex.

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1 Introduction

A construction of Davis ([1], [2], [4]), associates to any finite flag complex $L$, a “right-angled” Coxeter group $W_L$ and a contractible cubical cell complex $\Sigma_L$ on which $W_L$ acts properly and cocompactly. $W_L$ has the following presentation: the generators are the vertices of $L$, each generator has order 2, and two generators commute if the span an edge in $L$. The most important feature of this construction is that the link of each vertex of $\Sigma_L$ is isomorphic to $L$. A simplicial complex $L$ is a generalized homology $m$-sphere (for short, a GHS$m$) if it is a homology $m$-manifold having the same homology as a standard sphere $S^m$ (the homology is with real coefficients.) It follows, that if $L$ is a GHS$n-1$, then $\Sigma_L$ is a homology $n$-manifold.

If $L$ is a simplicial complex, $bL$ will denote the barycentric subdivision of $L$. $bL$ is a flag simplicial complex. Let $\partial \Delta^n$ denote the boundary complex of the standard $n$-dimensional simplex.

We study a certain weighted $L^2$-cohomology theory $L^2_qH^*$, described in [7], [5]. Suppose, for each vertex of $v \in L$ we are given a positive real number $q_v$, and let $q$ denote the vector with components $q_v$. Given a minimal word $w = v_1 \ldots v_n \in W_L$, let $q^w = q_{v_1} \ldots q_{v_n}$. For each $W_L$-orbit of cubes pick a representative $w_0$ and let $w(\sigma) = w$ if $\sigma = w_0$. (The ambiguity in the choices will not matter in our discussion.) Let $L^2_qC^i(\Sigma_L) = \{ \Sigma c_\sigma | \Sigma c_\sigma^2q^w(\sigma) < \infty \}$ be the Hilbert space of infinite $i$-cochains, which are square-summable with respect to the weight $q^w$. The usual coboundary operator $d$ is then a bounded operator, and we define the reduced weighted $L^2_q$-cohomology to be $L^2_qH^i(\Sigma_L) = \text{Ker}(d^i)/\text{Im}(d^{i-1})$. Similarly, one can define the reduced weighted $L^2_q$-homology, except, instead of the usual boundary operator one uses the adjoint of $d$. It follows from the Hodge decomposition that the resulting homology and cohomology spaces are naturally isomorphic. These spaces are Hilbert modules over the Hecke–von Neumann algebra $N_q$ (an appropriately completed Hecke algebra of $W_L$.) This allows us to introduce the weighted $L^2_q$ Betti numbers — the dimension of $L^2_qH^i$ over $N_q$. If $q = 1 = (1, \ldots, 1)$, we obtain the usual reduced $L^2$–cohomology, and we omit the index $q$. We write $q \leq 1$, if each component of $q$ is $\leq 1$.

The following conjecture, attributed to Singer, goes back to 1970’s.

**The Singer Conjecture** If $M^n$ is a closed aspherical manifold, then

$$L^2_qH^i(\tilde{M}^n) = 0 \text{ for all } i \neq n/2.$$

As explained in [5, Section 14], the appropriate generalization of the Singer Conjecture to the weighted case is the following conjecture:
The Generalized Singer Conjecture  Suppose $L$ is a flag $GHS^{n-1}$. Then $L^2_q H^i(\Sigma_L) = 0$ for $i > n/2$ and $q \leq 1$.

(Poincaré duality shows that for $q = 1$ this conjecture implies the Singer Conjecture for $\Sigma$.)

This conjecture holds true for $n \leq 4$ by [5]. One of the main results of this paper is a proof of this conjecture for barycentric subdivisions in even dimensions. The proof uses a reduction to a very special case $L = b_d \Delta^{2k-1}$.

We prove this case as Theorem 5.2. It turns out (Theorem 5.3), that this result implies the vanishing of the $L^2_q$–cohomology in a certain range for arbitrary right-angled Coxeter groups based on barycentric subdivisions. (For $q = 1$, this implication is proved in [6].) In particular, it follows that the Generalized Singer Conjecture is true for all barycentric subdivisions in even dimensions (Theorem 5.4), and for $b_d \Delta^n$ in all dimensions (Theorem 5.6).

This paper relies very heavily on [5]. In the inductive proofs we mostly omit the first steps, they are easy exercises using [5].

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2 Vanishing conjectures

We will follow the notation from [5]. Given a flag complex $L$ and a full subcomplex $A$, set:

$$h^q_i(L) = L^2 H_i(\Sigma_L)$$
$$h^q_i(A) = L^2 H_i(W_L \Sigma_A)$$
$$h^q_i(L, A) = L^2 H_i(\Sigma_L, W_L \Sigma_A)$$
$$b^i_q(L) = \dim_{\mathbb{F}_q}(h^q_i(L))$$
$$b^i_q(L, A) = \dim_{\mathbb{F}_q}(h^q_i(L, A))$$

The dimension of $\Sigma_L$ is one greater than the dimension of $L$. Hence, $b^i_q(L) = 0$ for $i > \dim L + 1$.

We will use the following three properties of $L^2_q$–homology.

Proposition 2.1  (See [5, Section 15])
The Mayer–Vietoris sequence  If \( L = L_1 \cup L_2 \) and \( A = L_1 \cap L_2 \), where \( L_1 \) and \( L_2 \) (and therefore, \( A \)) are full subcomplexes of \( L \), then
\[
\to h^q_i(A) \to h^q_i(L_1) \oplus h^q_i(L_2) \to h^q_i(L) \to
\]
is weakly exact.

The Künneth Formula  The Betti numbers of the join of two complexes are given by:
\[
b^k_q(L_1 * L_2) = \sum_{i+j=k} b^i_q(L_1) b^j_q(L_2).
\]

Poincaré Duality  If \( L \) is a flag GHS\(^{n-1} \), then \( b^i_q(L) = b^{n-i-1}_q(L) \).

If \( \sigma \) is a simplex in \( L \), let \( L_\sigma \) denote the link of \( \sigma \) in \( L \). To simplify notation we will write \( bL_v \) instead of \( (bL)_v \), to denote the link of the vertex \( v \) in \( bL \). Let \( \mathcal{C} \) be a class of GHS’s closed under the operation of taking link of vertices, i.e. if \( S \in \mathcal{C} \) and \( v \) is a vertex of \( S \) then \( S_v \in \mathcal{C} \). Following Section 15 of [5] we consider several variations of the Generalized Singer Conjecture for the class \( \mathcal{C} \).

I(\( n \))  If \( S \in \mathcal{C} \) and \( \dim S = n - 1 \), then \( b^i_q(S) = 0 \) for \( i > n/2 \) and \( q \leq 1 \).

III(\( 2k + 1 \))  Let \( S \in \mathcal{C} \) and \( \dim S = 2k \). Let \( v \) be a vertex of \( S \). Then the map \( i_*: h^q_k(S_v) \to h^q_k(S) \), induced by the inclusion, is the zero homomorphism for \( q \geq 1 \).

V(\( n \))  Let \( S \in \mathcal{C} \) and \( \dim S = n - 1 \). Let \( A \) be a full subcomplex of \( S \).
\begin{itemize}
  \item If \( n = 2k \) is even, then \( b^i_q(S, A) = 0 \) for all \( i > k \) and \( q \leq 1 \).
  \item If \( n = 2k + 1 \) is odd, then \( b^i_q(A) = 0 \) for all \( i > k \) and \( q \leq 1 \).
\end{itemize}

The argument in Section 16 of [5] goes through without changes if we consider only GHS’s from a class \( \mathcal{C} \) to give the following:

Theorem 2.2  (Compare [5, Section 16])  If we only consider GHS’s from a class \( \mathcal{C} \), then the following implications hold.
\[
\begin{align*}
(1) & \quad I(2k + 1) \implies III'(2k + 1). \\
(2) & \quad V(n) \implies I(n). \\
(3) & \quad V(2k - 1) \implies V(2k). \\
(4) & \quad [V(2k) \text{ and } III'(2k + 1)] \implies V(2k + 1).
\end{align*}
\]
Let $\mathcal{JD}$ denote the class of finite joins of the barycentric subdivisions of the boundary complexes of standard simplices:

$$\mathcal{JD} = \{b\partial\Delta^{n_1} \ast \cdots \ast b\partial\Delta^{n_i}\}.$$

**Lemma 2.3** The class $\mathcal{JD}$ is closed under the operation of taking link of vertices.

**Proof** Let $S = b\partial\Delta^{n_1} \ast \cdots \ast b\partial\Delta^{n_j}$ and $v \in S$. We can assume that $v \in b\partial\Delta^{n_1}$. Then $S_v = b\partial\Delta^{n_1}_v \ast b\partial\Delta^{n_2}_v \ast \cdots \ast b\partial\Delta^{n_j}_v = b\partial\Delta^{\dim(v)} \ast b\partial\Delta^{n_1-\dim(v)-1} \ast b\partial\Delta^{n_2} \ast \cdots \ast b\partial\Delta^{n_j}$, and therefore $S \in \mathcal{JD}$.

Next, consider the following statement:

**III'** $(2k+1)$ Let $v$ be a vertex of $b\partial\Delta^{2k+1}$. Then the map $i_* : h^q_k(b\partial\Delta^{2k+1}) \to h^q_k(b\partial\Delta^{2k+1})$, induced by the inclusion, is the zero homomorphism for $q \geq 1$.

**Lemma 2.4** $\text{III}'(2k+1) \implies \text{III}(2k+1)$ for the class $\mathcal{JD}$.

**Proof** By induction, we can assume that the lemma holds for all odd numbers $< 2k + 1$. It then follows from the Theorem 2.2 that $V(m)$ and therefore $I(m)$ hold for all $m < 2k + 1$.

Let $S = b\partial\Delta^{n_1} \ast \cdots \ast b\partial\Delta^{n_j}$ with $n_1 + \cdots + n_j = 2k + 1$ and $v \in S$. We assume that $v \in b\partial\Delta^{n_1}$. Then $S_v = b\partial\Delta^{n_1}_v \ast b\partial\Delta^{n_2}_v \ast \cdots \ast b\partial\Delta^{n_j}_v$ and, by the Künneth formula, the map in question decomposes as the direct sum of maps of the form

$$(h^q_k(\partial\Delta^{n_1}_v) \to h^q_k(\partial\Delta^{n_1})) \otimes \bigotimes_{i=2}^j (h^q_k(\partial\Delta^{n_i}_v) \to h^q_k(\partial\Delta^{n_i}))$$

where $k_1 + \cdots + k_j = k$. Since $n_1 + \cdots + n_j = 2k + 1$ it follows that $k_i < n_i/2$ for some index $i$. If $n_i < 2k + 1$, then the range of the corresponding map in the above tensor product is 0 by $I(n_i)$ and Poincaré duality, and therefore the tensor product map is 0. If $n_i = 2k + 1$ then, in fact, $i = 1$ (the join is a trivial join) and the result follows from $\text{III}'(2k + 1)$.

Thus, it follows from Theorem 2.2, Lemmas 2.3 and 2.4, and induction on dimension, that in order to prove the Generalized Singer Conjecture for the class $\mathcal{JD}$ all we need is to prove $\text{III}'(2k+1)$. 

3 Removal of an odd-dimensional vertex

Let $L$ be a simplicial complex and $bL$ be its barycentric subdivision. The vertices of $bL$ are naturally graded by “dimension”: each vertex $v$ of $bL$ is the barycenter of a unique cell (which we still denote $v$) of the complex $L$, and we call the dimension of this cell the dimension of the vertex $v$. Let $E_L$ denote the subcomplex of $bL$ spanned by the even dimensional vertices. Let $A_L$ denote the set of full subcomplexes $A$ of $L$ containing $E_L$, which have the following property: if $A$ contains a vertex of odd dimension $2j+1$, then $A$ contains all vertices of $bL$ of dimensions $\leq 2j$. In other words, any such $A$ can be obtained inductively from $bL$ by repeated removal of an odd-dimensional vertex of the highest dimension.

If $L = \partial \Delta^n$ we will use the notation $E_n = E_L$ and $A_n = A_L$.

Lemma 3.1 Assume $\Pi''(2m + 1)$ holds for $2m + 1 < n$. Then for any $(n - 1)$–dimensional simplicial complex $L$ and any complex $A \in A_L$ we have:

$$b_q^i(A) = b_q^i(bL) = 0 \text{ for } i > (n + 1)/2 \text{ and } q \leq 1.$$

Proof By induction, we can assume that the lemma holds for all $m < n$. First, we claim that removal of odd-dimensional vertices does not change the homology above $(n+1)/2$. Let $A \in A_L$ and let $B = A - v$ where $v$ is a vertex of the highest odd dimension of $A$. We let $\dim(v) = 2d - 1$, $1 \leq d \leq k$. We want to prove that $b_q^i(A) = b_q^i(B)$ for $i > (n + 1)/2$. Consider the Mayer–Vietoris sequence of the union $A = B \cup_{A_v} CA_v$:

$$\rightarrow b_q^i(A_v) \rightarrow b_q^i(B) \oplus b_q^i(CA_v) \rightarrow b_q^i(A) \rightarrow b_{q-1}^i(A_v)$$

Suppose $i > (n + 1)/2$. Since $A_v = B \cap bL_v = B \cap (b\partial \Delta^{2d-1} * b(L_v))$, and since $B \in A_L$, it follows, by construction, that $A_v$ splits as the join:

$$A_v = b\partial \Delta^{2d-1} * A_1,$$

with $A_1 \in A(L_v)$. By inductive assumption the lemma holds for $L_v$, i.e. $b_q^i(A_1) = b_q^i(b(L_v)) = 0$ for $i > (n + 1)/2 - d$.

Since $\Pi''(2d-1)$ holds by hypothesis, by Lemma 2.4 and Theorem 2.2, $\Pi(2d-1)$ holds for the class $J \mathcal{D}$, and, thus, $b_q^i(b\partial \Delta^{2d-1}) = 0$ for $i \geq d$.

Then, by the Künneth formula, $b_q^{i-1}(A_v) = 0$ for $i - 1 \geq (n + 1)/2$, i.e. for $i > (n + 1)/2$. By [5, Proposition 15.2(d)], $b_q^i(CA_v) = \frac{1}{q+1} b_q^i(A_v)$. Therefore in the above sequence the terms corresponding to $A_v$ and $CA_v$ are $0$, and
the claim follows. Then it follows by induction, that \( b_i^q(A) = b_i^q(bL) \) for all \( A \in A_L \) and \( i > (n + 1)/2 \).

To prove the vanishing we note that, in particular, \( b_i^q(E_L) = b_i^q(bL) \) for \( i > (n + 1)/2 \). Since \( E_L \) is spanned by the even-dimensional vertices of \( bL \) and since a simplex in \( bL \) has vertices of pairwise different dimensions, we have \( \dim(E_L) = \lceil (n + 1)/2 \rceil - 1 \). Therefore, \( b_i^q(E_L) = 0 \) for \( i > (n + 1)/2 \) and we have proved the lemma.

In the special case \( L = \Delta^{2k+1} \) this lemma admits the following strengthening:

\begin{lemma}
Let \( n = 2k + 1 \). Assume \( \text{III}''(2m + 1) \) holds for \( 2m + 1 < n \). Then for any complex \( A \in A_n \), \( A \subset b\partial \Delta^n \), we have:
\[ b_i^q(A) = b_i^q(b\partial \Delta^n) \text{ for } i > k \text{ and } q \leq 1. \]
\end{lemma}

\textbf{Proof} We proceed as in the previous proof. As before, we have \( B = A - v \), \( \dim(v) = 2d - 1 \) and \( A_v = b\partial \Delta^{2d-1} * A_1 \), where now \( A_1 \subset b\partial \Delta^{2k+1-2d} \). Therefore, the inductive assumption and the hypothesis on \( \text{III}''(2d - 1) \) imply that \( b_i(A_1) = 0 \) for \( i > k + d \). The lemma follows as before.

As explained in [6], when \( q = 1 \), the removal of the odd-dimensional vertex does not change homology in all dimensions. We record this result below.

\begin{lemma}
Assume \( \text{III}''(2m + 1) \) holds for \( 2m + 1 < n \) and \( q = 1 \). Then for any \( (n-1)\)–dimensional simplicial complex \( L \) and for any complex \( A \in A_L \), obtained by the repeated removal of highest odd-dimensional vertices, we have:
\[ b^*(A) = b^*(bL). \]
\end{lemma}

\textbf{Proof} Again we repeat the proof of Lemma 3.1. As before, we have the splitting \( A_v = b\partial \Delta^{2d-1} * A_1 \). The point now is that for \( q = 1 \), \( b\partial \Delta^{2d-1} \) and Poincaré duality imply \( b^*(b\partial \Delta^{2d-1}) = 0 \) and therefore \( b^*(A_v) = 0 \) by the Künneth formula.

\section{Intersection form}

\begin{lemma}
Let \( L \) be a \( \text{GHS}^{2k} \) and let \( v \) be a vertex of \( L \). Then the image of the restriction map on \( L^2 \)–cohomology \( i^* \): \( L^2 \mathcal{H}^k(\Sigma_L) \to L^2 \mathcal{H}^k(\Sigma_{L_v}) \) is an isotropic subspace of the intersection form of \( \Sigma_{L_v} \).
\end{lemma}
Proof Note that the cup product of two $L^2$–cocycles is an $L^1$–cocycle. The intersection form is the result of evaluation of the cup product of two middle-dimensional cocycles on the fundamental class, which is $L^\infty$. Since $\Sigma_{L_v}$ bounds a half-space in $\Sigma_L$, $i_\ast([\Sigma_{L_v}]) = 0$ in $L^\infty$–homology of $\Sigma_L$. Thus, if $a,b \in L^2\mathcal{H}^0(\Sigma_L)$, then $\langle i^\ast(a) \cup i^\ast(b), [\Sigma_{L_v}] \rangle = \langle a \cup b, i_\ast([\Sigma_{L_v}]) \rangle = 0$.

Lemma 4.2 Let $G$ be a group and let $A$ be a bounded $G$–invariant (with respect to the diagonal action) non-degenerate bilinear form on a Hilbert submodule $M \subset \ell^2(G)$. Then $A$ has no nontrivial $G$–invariant isotropic subspaces.

Proof Let us consider the case $M = \ell^2(G)$ first. $G$–invariance and continuity of the form $A$ implies that $A$ is completely determined by it values $a_g = (g A 1)$, $g \in G$. It is convenient to think of the form as given by $(x A y) = \langle x, Ay \rangle$, where $\langle \cdot, \cdot \rangle$ is the inner product and $A = \Sigma_{g \in G} a_g g$ is a bounded $G$–equivariant operator on $\ell^2(G)$. Non-degeneracy of $A$ means that $Ax = 0$ only if $x = 0$. $A$ is the limit of the group ring elements, and $Ax$ is the limit of the corresponding linear combinations of $G$–translates of $x$, i.e. $Ax = \lim \Sigma_{g \in G_n} a_g (gx)$, where $G_n$ is some exhaustion of $G$ by finite sets. It follows that if $x$ belongs to $G$–invariant isotropic subspace, then $Ax$ belongs to the closure of this subspace. Thus, we have $\langle Ax, Ax \rangle = (Ax A x) = 0$ by isotropy and continuity, therefore $x = 0$.

The case of general submodule $M \subset \ell^2(G)$ reduces to the above, since the bilinear form $A$ can be extended to $\ell^2(G)$, for example, by taking the orthogonal sum $A \oplus \langle \cdot, \cdot \rangle$ of $A$ on $M$ and the inner product on the orthogonal complement of $M$.

5 Vanishing theorems

Our main technical results are the following two theorems.

Theorem 5.1 $\text{III}'(2k + 1)$ is true for all $k > 0$ and $q = 1$.

Proof The proof is by induction on $k$. Suppose the theorem is true for all $m < k$.

Let $v$ be a vertex of $b\partial \Delta^{2k+1}$. We need to show that the restriction map $i^\ast: \mathfrak{h}^k(b\partial \Delta^{2k+1}) \to \mathfrak{h}^k(b\partial \Delta^{2k+1})$ is the 0–map.

First let us suppose that $v$ is a vertex of dimension 0, i.e. a vertex of $\Delta^{2k+1}$.
Consider the action of the symmetric group $S_{2k+1}$ on $\Delta^{2k+1}$ which fixes the vertex $v$ and permutes other vertices. This action gives a simplicial action of $S_{2k+1}$ on $b\partial\Delta^{2k+1}$ and therefore, after choosing a base point, lifts to a cubical action of $S_{2k+1}$ on $\Sigma_{b\partial\Delta^{2k+1}}$ stabilizing $\Sigma_{b\partial\Delta^{2k+1}}$. Let $G'$ be the group of cubical automorphisms of $\Sigma_{b\partial\Delta^{2k+1}}$ generated by this action and the standard action of $W_{b\partial\Delta^{2k+1}}$, and let $G$ be the orientation-preserving subgroup of $G'$. Similarly, let $G''_v$ be the group of cubical automorphisms of $\Sigma_{b\partial\Delta^{2k+1}}$ generated by this action and the standard action of $W_{b\partial\Delta^{2k+1}}$, and let $G'_v$ be the orientation-preserving subgroup of $G''_v$.

We claim that, as a Hilbert $G_v$--module $L^2H^k(\Sigma_{b\partial\Delta^{2k+1}})$, is a submodule of $\ell^2(G_v)$. Note that $b\partial\Delta^{2k+1}$ is naturally isomorphic to $b\partial\Delta^{2k}$.

Using the inductive assumption and Lemma 3.3, we can remove from $b\partial\Delta^{2k}$ all odd-dimensional vertices without changing the $L^2$-cohomology: $\mathfrak{h}^*(E_{2k}) = \mathfrak{h}^*(b\partial\Delta^{2k})$. Since the action of $S_{2k+1}$ on $b\partial\Delta^{2k+1}$ preserves the dimension of the vertices, we have isomorphism $L^2H^k(G_v\Sigma_{E_{2k}}) = L^2H^k(\Sigma_{b\partial\Delta^{2k}})$ as Hilbert $G_v$--modules.

The complex $E_{2k}$ is spanned by the even-dimensional vertices of $b\partial\Delta^{2k}$, which correspond to the proper subsets of vertices of $\Delta^{2k}$ of odd cardinality. Thus, the dimension of $E_{2k}$ is $k-1$, and its top-dimensional simplices are chains $v_0 < v_0v_1v_2 < \cdots < v_0\cdots v_{2k-2}$ of length $k$ of distinct vertices of $\Delta^{2k}$. Therefore $S_{2k+1}$ acts transitively on $(k-1)$--dimensional simplices of $E_{2k}$ and it follows that $G_v$ acts transitively on $k$--dimensional cubes of $G_v\Sigma_{E_{2k}}$. Therefore the space of $k$--cochains is a Hilbert $G_v$--submodule of $\ell^2(G_v)$, and the claim follows from the Hodge decomposition.

We have, by construction, $G_v = \text{Stab}_G(\Sigma_{b\partial\Delta^{2k+1}})$. Then the restriction map $i^*: L^2H^k(\Sigma_{b\partial\Delta^{2k+1}}) \to L^2H^k(\Sigma_{b\partial\Delta^{2k+1}})$ is $G_v$--equivariant and therefore its image is a $G_v$--invariant subspace of $L^2H^k(\Sigma_{b\partial\Delta^{2k+1}})$. Since $G_v$ acts preserving orientation, the intersection form is $G_v$--invariant. By Lemma 4.1 the image is isotropic, thus by Lemma 4.2 it is 0. Thus, the map $i^*: \mathfrak{h}^k(b\partial\Delta^{2k+1}) \to \mathfrak{h}^k(b\partial\Delta^{2k+1}) = L^2H^k(W_{b\partial\Delta^{2k+1}}\Sigma_{b\partial\Delta^{2k+1}})$ is the 0--map.

For vertices of the other even dimensions the argument is similar. If $\dim(v) = 2d$, then its link is $b\partial\Delta^{2d} \ast b\partial\Delta^{2k-2d}$. Again, using Lemma 3.3, we remove, without changing the $L^2$--cohomology, all odd-dimensional vertices from each factor to obtain $E_{2d} \ast E_{2k-2d}$. The group $S_{2d+1} \times S_{2k-2d+1}$ acts naturally on $b\partial\Delta^{2k+1}$ fixing the vertex $v$ and stabilizing both the link and $E_{2d} \ast E_{2k-2d}$. This action is again transitive on the top-dimensional simplices of $E_{2d} \ast E_{2k-2d}$, and the rest of the argument goes through.
Finally, if \( v \) is an odd-dimensional vertex, \( \dim(v) = 2d + 1 \), then we have \( b\partial\Delta^{2k+1} = b\partial\Delta^{2d+1} \ast b\partial\Delta^{2k-2d-1} \). The hypothesis on \( \text{III}^b \) and Theorem 2.2 and Lemma 2.4 imply that both \( I(2d + 1) \) and \( I(2k - 2d - 1) \) hold. Therefore, by the Künneth formula \( h^k(b\partial\Delta^{2k+1}) = 0 \) in this case.

**Theorem 5.2** The Generalized Singer Conjecture holds true for \( b\partial\Delta^{2k+1} \):

\[
b_i^q(b\partial\Delta^{2k+1}) = 0 \quad \text{for} \quad i > k \quad \text{and} \quad q \leq 1.
\]

**Proof** We proceed by induction on \( k \). Using the inductive assumption and Lemma 3.2, we can remove all odd-dimensional vertices without changing the weighted \( L^q_2 \)-homology above \( k \). Thus, since the remaining part \( E_{2k+1} \) is \( k \)-dimensional, the problem reduces to showing that \( h_i^q(E_{2k+1}) = 0 \) for \( q \leq 1 \). Since \( E_{2k+1} \) is \( k \)-dimensional, the natural map \( h_{k+1}(E_{2k+1}) \to h_k^q(E_{2k+1}) \) is injective and the result follows from the Theorem 5.1.

Next, we list some consequences. Lemma 3.1 implies:

**Theorem 5.3** Let \( bL \) be the barycentric subdivision of an \((n-1)\)-dimensional simplicial complex \( L \). Then

\[
b_i^q(bL) = 0 \quad \text{for} \quad i > (n+1)/2 \quad \text{and} \quad q \leq 1.
\]

Taking \( L \) to be a \( \text{GHS}^{2k-1} \), we obtain:

**Theorem 5.4** The Generalized Singer Conjecture holds true for the barycentric subdivision of a \( \text{GHS}^{n-1} \) for all even \( n \).

For odd \( n \) we obtain a weaker statement:

**Theorem 5.5** Let \( bL \) be the barycentric subdivision of a \( \text{GHS}^{2k} \). Then

\[
b_i^q(bL) = 0 \quad \text{for} \quad i > k + 1 \quad \text{and} \quad q \leq 1.
\]

In particular,

\[
b_i^q(bL) = 0 \quad \text{for} \quad i \neq k, \ k + 1.
\]

Specializing further, and combining with Theorem 5.2, we obtain:

**Theorem 5.6** The Generalized Singer Conjecture holds true for \( b\partial\Delta^n \):

\[
b_i^q(b\partial\Delta^n) = 0 \quad \text{for} \quad i > n/2 \quad \text{and} \quad q \leq 1,
\]

and, therefore, for the class \( JD \).
Finally, let us mention an application of the above result to a more analytic object. Let $T_n$ denote the space of all symmetric tridiagonal $(n + 1) \times (n + 1)$-matrices with fixed generic eigenvalues, the so-called Tomei manifold. It is proved in [8] that $T_n$ is an $n$-dimensional closed aspherical manifold.

**Theorem 5.7** The Singer Conjecture holds true for Tomei manifolds $T_n$.

**Proof** The space $T_n$ can be identified with a natural finite index orbifoldal cover of $\Sigma_{b;\Delta^n}/W_{b;\Delta^n}$ [3]. Thus $\Sigma_{b;\Delta^n}$ is the universal cover of $T^n$, and the claim follows from the previous theorem.

### References


