GLOBAL EXISTENCE AND ENERGY DECAY OF SOLUTIONS TO THE CAUCHY PROBLEM FOR A WAVE EQUATION WITH A WEAKLY NONLINEAR DISSIPATION

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We prove the global existence and study decay properties of the solutions to the wave equation with a weak nonlinear dissipative term by constructing a stable set in $H^1(\mathbb{R}^n)$.

1. Introduction

We consider the Cauchy problem for the nonlinear wave equation with a weak nonlinear dissipation and source terms of the type

$$u'' - \Delta u + \lambda^2(x)u + \sigma(t)g(u') = |u|^{p-1}u \quad \text{in } \mathbb{R}^n \times [0, +\infty],$$

$$u(x,0) = u_0(x), \quad u'(x,0) = u_1(x) \quad \text{in } \mathbb{R}^n,$$  

(1.1)

where $g : \mathbb{R} \to \mathbb{R}$ is a continuous nondecreasing function and $\lambda$ and $\sigma$ are positive functions.

When we have a bounded domain instead of $\mathbb{R}^n$, and for the case $g(x) = \delta x$ ($\delta > 0$) (without the term $\lambda^2(x)u$), Ikehata and Suzuki [8] investigated the dynamics, they have shown that for sufficiently small initial data $(u_0, u_1)$, the trajectory $(u(t), u'(t))$ tends to $(0, 0)$ in $H^1_0(\Omega) \times L^2(\Omega)$ as $t \to +\infty$. When $g(x) = \delta |x|^{m-1}x$ ($m \geq 1$, $\delta \equiv 0$, $\sigma \equiv 1$), Georgiev and Todorova [4] introduced a new method and determined suitable relations between $m$ and $p$, for which there is global existence or alternatively finite-time blow up. Precisely they showed that the solutions continue to exist globally in time if $m \geq p$ and blow up in finite time if $m < p$ and the initial energy is sufficiently negative. This result was later generalized to an abstract setting by Levine and Serrin [12] and Levine et al. [11]. In these papers, the authors showed that no solution with negative initial energy can be extended on $[0, \infty[$, if the source term dominates over the damping term ($p > m$). This generalization allowed them also to apply their result to quasilinear wave equations (see [1, 17]). Quite recently, Ikehata [7] proved that a global solution exists with no relation between $p$ and $m$ by the use of a stable set method due to Sattinger [18].

For the Cauchy problem (1.1) with $\lambda \equiv 1$ and $\sigma \equiv 1$, when $g(x) = \delta |x|^{m-1}x$ ($m \geq 1$) Todorova [21] (see [16]) proved that the energy decay rate is $E(t) \leq (1 + t)^{-2-n(m-1)/(m-1)}$.
for $t \geq 0$. She used a general method introduced by Nakao [14] on condition that the data have compact support. Unfortunately, this method does not seem to be applicable in the case of more general functions $\lambda$ and $\sigma$.

Our purpose in this paper is to give a global solvability in the class $H^1$ and energy decay estimates of the solutions to the Cauchy problem (1.1) for a weak linear perturbation and a weak nonlinear dissipation.

We use a new method recently introduced by Martinez [13] (see also [2]) to study the decay rate of solutions to the wave equation

$$u'' - \Delta x u + g(u') = 0 \quad \text{in } \Omega \times \mathbb{R}^+,$$

where $\Omega$ is a bounded domain of $\mathbb{R}^n$. This method is based on a new integral inequality that generalizes a result of Haraux [6]. So we proceed with the argument combining the method in [13] with the concept of modified stable set on $H^1(\mathbb{R}^n)$. Here the modified stable set is the extended $\mathbb{R}^n$ version of Sattinger’s stable set.

2. Preliminaries and main results

$\lambda(x)$, $\sigma(t)$, and $g$ satisfy the following hypotheses.

(i) $\lambda(x)$ is a locally bounded measurable function defined on $\mathbb{R}^n$ and satisfies

$$\lambda(x) \geq d(|x|), \quad \text{if } |x| \leq 1,$$

where $d$ is a decreasing function such that $\lim_{y \to \infty} d(y) = 0$.

(ii) $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ is a nonincreasing function of class $C^1$ on $\mathbb{R}_+$.

Consider $g : \mathbb{R} \to \mathbb{R}$ a nondecreasing $C^0$ function and suppose that there exist $C_i > 0$, $i = 1, 2, 3, 4$, such that

$$c_1 |v|^m \leq |g(v)| \leq c_2 |v|^r \quad \forall |v| \geq 1,$$

where $m \geq 1$ and $1 \leq r \leq (n+2)/(n-2)^+$. We first state two well-known lemmas, and then we state and prove two other lemmas that will be needed later.

**Lemma 2.1.** Let $q$ be a number with $2 \leq q < +\infty$ ($n = 1, 2$) or $2 \leq q \leq 2n/(n-2)(n \geq 3)$. Then there is a constant $c_* = c(q)$ such that

$$\|u\|_q \leq c_* \|u\|_{H^1(\mathbb{R}^n)} \quad \text{for } u \in H^1(\mathbb{R}^n).$$

**Lemma 2.2 (Gagliardo-Nirenberg).** Let $1 \leq r < q \leq +\infty$ and $p \geq 2$. Then, the inequality

$$\|u\|_p \leq C \|\nabla_x^m u\|_2^\theta \|u\|_r^{1-\theta} \quad \text{for } u \in \mathcal{D}((-\Delta)^{m/2})L^r$$

holds with some constant $C > 0$ and

$$\theta = \left(\frac{1}{r} - \frac{1}{p}\right) \left(\frac{m}{n} + \frac{1}{r} - \frac{1}{2}\right)^{-1} \quad \text{provided that } 0 < \theta \leq 1 \quad \text{(assuming that } 0 < \theta < 1 \text{ if } m - n/2 \text{ is a nonnegative integer).}$$
**Lemma 2.3** [10]. Let \( E : \mathbb{R}_+ \to \mathbb{R}_+ \) be a nonincreasing function and assume that there are two constants \( p \geq 1 \) and \( A > 0 \) such that
\[
\int_S^{+\infty} E^{(p+1)/2}(t) \, dt \leq AE(S), \quad 0 \leq S < +\infty,
\] (2.7)
then
\[
E(t) \leq cE(0)(1 + t)^{-2/(p-1)} \quad \forall \, t \geq 0, \text{ if } p > 1,
\]
\[
E(t) \leq cE(0)e^{-\omega t} \quad \forall \, t \geq 0, \text{ if } p = 1,
\] (2.8)
where \( c \) and \( \omega \) are positive constants independent of the initial energy \( E(0) \).

**Lemma 2.4** [13]. Let \( E : \mathbb{R}_+ \to \mathbb{R}_+ \) be a nonincreasing function and \( \phi : \mathbb{R}_+ \to \mathbb{R}_+ \) an increasing \( C^2 \) function such that
\[
\phi(0) = 0, \quad \phi(t) \to +\infty \quad \text{as } t \to +\infty.
\] (2.9)
Assume that there exist \( p \geq 1 \) and \( A > 0 \) such that
\[
\int_S^{+\infty} E^{(p+1)/2}(t) \phi'(t) \, dt \leq AE(S), \quad 0 \leq S < +\infty,
\] (2.10)
then
\[
E(t) \leq cE(0)(1 + \phi(t))^{-2/(p-1)} \quad \forall \, t \geq 0, \text{ if } p > 1,
\]
\[
E(t) \leq cE(0)e^{-\omega \phi(t)} \quad \forall \, t \geq 0, \text{ if } p = 1,
\] (2.11)
where \( c \) and \( \omega \) are positive constants independent of the initial energy \( E(0) \).

**Proof of Lemma 2.4.** Let \( f : \mathbb{R}_+ \to \mathbb{R}_+ \) be defined by \( f(x) := E(\phi^{-1}(x)) \), (we remark that \( \phi^{-1} \) has a sense by the hypotheses assumed on \( \phi \), \( f \) is nonincreasing, \( f(0) = E(0) \), and if we set \( x := \phi(t) \), we obtain
\[
\int_{\phi(S)}^{\phi(T)} f(x)^{(p+1)/2} \, dx = \int_{\phi(S)}^{\phi(T)} E(\phi^{-1}(x))^{(p+1)/2} \, dx = \int_S^T E(t)^{(p+1)/2} \phi'(t) \, dt
\]
\[
\leq AE(S) = Af(\phi(S)), \quad 0 \leq S < T < +\infty.
\] (2.12)
Setting \( s := \phi(S) \) and letting \( T \to +\infty \), we deduce that
\[
\int_s^{+\infty} f(x)^{(p+1)/2} \, dx \leq Af(s), \quad 0 \leq s < +\infty.
\] (2.13)
Thanks to **Lemma 2.3**, we deduced the desired results. \( \square \)

Before stating the global existence theorem and decay property of problem (1.1), we will introduce the notion of the modified stable set. Let
\[
K(u) = \|\nabla_x u\|^2_2 + \|u\|^2_2 - \|u\|_{p+1}^2 \quad \text{if } \lambda \equiv 1,
\]
\[
I(u) = \|\nabla_x u\|^2_2 - \|u\|_{p+1}^2 \quad \text{if } \lambda \neq \text{const},
\] (2.14)
for \( u \in H^1(\mathbb{R}^n) \). Then we define the modified stable set \( \widetilde{W}^* \) and \( \widetilde{W}^{**} \) by
\[
\widetilde{W}^* \equiv \{ u \in H^1(\mathbb{R}^n) \setminus K(u) > 0 \} \cup \{ 0 \} \quad \text{if } \lambda \equiv 1,
\]
\[
\widetilde{W}^{**} \equiv \{ u \in H^1(\mathbb{R}^n) \setminus I(u) > 0 \} \cup \{ 0 \} \quad \text{if } \lambda \neq \text{const}.
\]

Next, let \( J(u) \) and \( E(t) \) be the potential and energy associated with problem (1.1), respectively:
\[
J(u) = \frac{1}{2} \| \nabla_x u \|_2^2 + \frac{1}{2} \| \lambda(x) u \|_2^2 - \frac{1}{p+1} \| u \|_{p+1}^{p+1} \quad \text{for } u \in H^1(\mathbb{R}^n),
\]
\[
E(t) = \frac{1}{2} \| u' \|_2^2 + J(u).
\]

We get the local existence solution.

**Theorem 2.5.** Let \( 1 < p \leq (n+2)/(n-2) \) \((1 < p < \infty \text{ if } n = 1,2)\) and assume that \((u_0,u_1)\) \(\in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)\) and \(u_0\) belong to the modified stable set \( \widetilde{W}^* \). Then there exists \( T > 0 \) such that the Cauchy problem (1.1) has a unique solution \( u(t) \) on \( \mathbb{R}^n \times [0,T) \) in the class

\[
u(t,x) \in C([0,T);H^1(\mathbb{R}^n)) \cap C^1([0,T);L^2(\mathbb{R}^n)),
\]

satisfying
\[
u(t) \in \widetilde{W}^*,
\]

and this solution can be continued in time as long as \( u(t) \in \widetilde{W}^* \).

When \( \lambda \neq \text{const} \), we use the following theorem of local existence in the space \( H^2 \times H^1 \), and the decay property of the energy \( E(t) \) is necessarily required for the local solution to remain in \( \widetilde{W}^{**} \) as \( t \to \infty \); this fact of course guarantees the global existence in \( H^2 \times H^1 \) and by approximation, we obtain global existence in \( H^1 \times L^1 \).

**Theorem 2.6 [15].** Let \((u_0,u_1)\) \(\in H^2 \times H^1\). Suppose that
\[
1 \leq p \leq \frac{n}{n-4} \quad (1 \leq \infty \text{ if } N \leq 4).
\]

Then under the hypotheses (2.1), (2.2), and (2.3), problem (1.1) admits a unique local solution \( u(t) \) on some interval \([0,T]\), \( T \equiv T(u_0,u_1) > 0 \), in the class \( W^{2,\infty}([0,T];L^2) \cap W^{1,\infty}([0,T];H^1) \cap L^\infty([0,T];H^2) \), satisfying the finite propagation property.

**Proof of Theorem 2.5 (see [15, 18]).** Since the argument is standard, we only sketch the main idea of the proof. Let \((u_0,u_1)\) \(\in H^1 \times L^2\) and \(u_0 \in \widetilde{W}^*\). Then we have a unique local solution \( u(t) \) for some \( T > 0 \). Indeed, taking suitable approximate functions \( f_j \) such that (see [20])
\[
f_j(u) = f(u) \quad \text{if } |u| \leq j, \quad |f_j(u)| \leq |f(u)|, \quad |f_j(u)| \leq c_j |u|,
\]
problem (1.1) with \( f(u) \equiv |u|^{p-1}u \) replaced by \( f_j(u) \) admits a unique solution \( u_j(t) \) \(\in C([0,T);H^1(\mathbb{R}^n)) \cap C^1([0,T);L^2(\mathbb{R}^n))\). Further, we can prove that \( u_j(t) \in \widetilde{W}^* \), \( 0 < t < T \),
for sufficiently large $j$, and there exists a subsequence of $\{u_j(t)\}$ which converges to a function $\tilde{u}(t)$ in certain senses. $\tilde{u}(t)$ is, in fact, a weak solution in $C([0,T];H^1(\mathbb{R}^n)) \cap C^1([0,T];L^2(\mathbb{R}^n))$ (see [19, 20]) and such a solution is unique by Ginibre and Velo [5] and Brenner [3]. We can also construct such a solution which meets moreover the finite propagation property, if we assume that the initial data $u_0(x)$ and $u_1(x)$ are of compact support:

$$\text{supp } u_0 \cup \text{supp } u_1 \subset \{x \in \mathbb{R}^n, |x| < L\}, \text{ for some } L > 0. \quad (2.21)$$

Applying [9, Appendix 1] of John, then the solution is also of compact support: $\text{supp } u_j(t) \subset \{x \in \mathbb{R}^n, |x| < L + t\}$. So, we have $\text{supp } \tilde{u}(t) \subset \{x \in \mathbb{R}^n, |x| < L + t\}$. □

We denote the life span of the solution $u(t,x)$ of the Cauchy problem (1.1) by $T_{\text{max}}$. First we consider the case $\lambda(x) \equiv \text{const}$ ($\lambda(x) \equiv 1$ without loss of generality). And construct a stable set in $H^1(\mathbb{R}^n)$.

Setting

$$C_0 \equiv K\left\{ \frac{2(p + 1)}{(p - 1)} \right\}^{(p-1)/2}, \quad (2.22)$$

$$\int_0^\infty \sigma(\tau) d\tau = +\infty \text{ if } m = 1, \quad (2.23)$$

$$\int_0^\infty (1 + \tau)^{-n(m-1)/2} \sigma(\tau) d\tau = +\infty \text{ if } m > 1. \quad (2.24)$$

Theorem 2.7. Let $u(t,x)$ be a local solution of problem (1.1) on $[0, T_{\text{max}})$ with initial data $u_0 \in \tilde{W}^*$, $u_1 \in L^2(\mathbb{R}^n)$ with sufficiently small initial energy $E(0)$ so that

$$C_0 E(0)^{(p-1)/2} < 1. \quad (2.25)$$

Then $T_{\text{max}} = \infty$. Furthermore, the global solution of the Cauchy problem (1.1) has the following energy decay property. Under (2.22), (2.3), and (2.23),

$$E(t) \leq E(0) \exp \left( 1 - \omega \int_0^t \sigma(\tau) d\tau \right) \quad \forall t > 0. \quad (2.26)$$

Under (2.2), (2.3), and (2.24),

$$E(t) \leq \left( \frac{C(E(0))}{\int_0^t (1 + \tau)^{-n(m-1)/2} \sigma(\tau) d\tau} \right)^{2/(m-1)} \quad \forall t > 0. \quad (2.27)$$

Secondly, we consider the case $\lambda(x) \not\equiv \text{const}$ and we assume that

$$\frac{n + 4}{n} \leq p \leq \frac{n}{n - 2}. \quad (2.28)$$

(1) If $\sigma(t) = \mathcal{O}(\tilde{d}(t))$, where $\tilde{d}(t) = d(L + t)$. 
Cauchy problem for nonlinear wave equation

If $m = 1$, we suppose that

$$\int_0^\infty \sigma(\tau) \, d\tau = +\infty$$

(2.29)

with

$$(\tilde{d}(t))^{-(4-(n-2)(p-1))/2} \exp \left( 1 - \omega \int_0^t \sigma(\tau) \, d\tau \right)^{(p-1)/2} < \infty,$$

(2.30)

$$(\tilde{d}(t))^{-1} \exp \left( \frac{1}{2} - \frac{\omega}{2} \int_0^t \sigma(\tau) \, d\tau \right) < \infty.$$

If $m > 1$, we suppose that

$$\int_0^\infty (1 + \tau)^{-n(m-1)/2} \sigma(\tau) \, d\tau = \infty$$

(2.31)

with

$$(\tilde{d}(t))^{-(4-(n-2)(p-1))/2} \left( \int_0^t (1 + \tau)^{-n(m-1)/2} \sigma(\tau) \, d\tau \right)^{(p-1)/(m-1)} < \infty,$$

(2.32)

$$(\tilde{d}(t))^{-1} \left( \int_0^t (1 + \tau)^{-n(m-1)/2} \sigma(\tau) \, d\tau \right)^{1/(m-1)} < \infty.$$

(2) If $\tilde{d}(t) = \mathcal{O}(\sigma(\tau))$.

If $m = 1$, we suppose that for some $0 \leq \alpha < 1$,

$$\int_0^\infty \frac{\tilde{d}^2(\tau)}{\sigma^\alpha(\tau)} \, d\tau = +\infty$$

(2.33)

with

$$(\tilde{d}(t))^{-(4-(n-2)(p-1))/2} \exp \left( 1 - \omega \int_0^t \frac{\tilde{d}^2(\tau)}{\sigma^\alpha(\tau)} \, d\tau \right)^{(p-1)/2} < \infty,$$

(2.34)

$$(\tilde{d}(t))^{-1} \exp \left( \frac{1}{2} - \frac{\omega}{2} \int_0^t \frac{\tilde{d}^2(\tau)}{\sigma^\alpha(\tau)} \, d\tau \right) < \infty.$$

If $m > 1$, we suppose that for some $0 \leq \alpha < 1$,

$$\int_0^\infty (1 + \tau)^{-n(m-1)/2} \sigma^{-((1+\alpha)(1+m)-2)/2} \tilde{d}^{m+1}(\tau) \, d\tau = \infty$$

(2.35)
with
\[
\left(\ddot{d}(t)\right)^{-\frac{(4-(n-2)(p-1))/2}{\left(\int_0^t (1+\tau)^{-\frac{n(m-1)/2}{\sigma^{-\frac{(1+\alpha)(1+m)-2}{\sigma^{-\frac{(1+\alpha)(1+m)-2}{2}}(\tau)\ddot{d}^{m+1}(\tau) d\tau}^2\right)^{\frac{p-1/(m-1)}{}} < \infty, \right.
\]
\[
\left(\int_0^t (1+\tau)^{-\frac{n(m-1)/2}{\sigma^{-\frac{(1+\alpha)(1+m)-2}{2}}(\tau)\ddot{d}^{m+1}(\tau) d\tau}^2\right)^{\frac{1/(m-1)}{}} < \infty. \right.
\]

We have the following theorem.

**Theorem 2.8.** Let \((u_0, u_1) \in H^1 \times L^2\), \(u_0 \in \widetilde{W}^\ast\), and let the initial energy \(E(0)\) be sufficiently small. The following cases are considered.

(i) \(\sigma(t) = \mathcal{O}(\ddot{d}(t))\).

Suppose (2.2), (2.3), (2.29), and (2.30) or (2.2), (2.3), (2.31), and (2.32). Then problem (1.1) admits a unique solution \(u(t) \in C([0, \infty); H^1) \cap C^1([0, \infty); L^2)\) and has the same decay property as Theorem 2.7.

(ii) \(\ddot{d}(t) = \mathcal{O}(\sigma(t))\).

Suppose (2.2), (2.3), (2.33), and (2.34) or (2.2), (2.3), (2.35), and (2.36). Then problem (1.1) admits a unique solution \(u(t) \in C([0, \infty); H^1) \cap C^1([0, \infty); L^2)\). Furthermore, the global solution of the Cauchy problem (1.1) has the following energy decay property:

\[
E(t) \leq E(0) \exp\left(1 - \omega \int_0^t \frac{d^2(\tau)}{\sigma^2(\tau)} d\tau\right) \quad \forall t > 0 \text{ if } m = 1, \tag{2.37}
\]

\[
E(t) \leq \left(\frac{C(E(0))}{\int_0^t (1+\tau)^{-(n(m-1)/2-\sigma^{-\frac{(1+\alpha)(1+m)-2}{2}}(\tau)\ddot{d}^{m+1}(\tau) d\tau}^2\right)^{\frac{1/(m-1)}{}} < \infty. \right. \tag{2.38}
\]

**Remark 2.9.** In Theorem 2.7, the global existence and energy decay are independent, but in Theorem 2.8, we need the estimation of the energy decay for a local solution to prove global existence.

**Examples 2.10.** (1) If \(\sigma(t) = 1/t^\theta\), by applying Theorem 2.7 we obtain

\[
E(t) \leq E(0)e^{1-\omega t^{1-\theta}} \quad \text{if } m = 1, \tag{2.39}
\]

\[
E(t) \leq C(E(0))(1+t)^{-\frac{(2-\theta)(m-1)-2}{(m-1)}} \quad \text{if } 1 < m < 1 + \frac{2-2\theta}{n}, 0 < \theta < 1, \tag{2.39}
\]

\[
E(t) \leq C(E(0))(\ln t)^{-\frac{2}{(m-1)}} \quad \text{if } m = 1 + \frac{2-2\theta}{n}, 0 < \theta < 1. \tag{2.39}
\]

(2) If \(\sigma(t) = 1/t^\theta \ln \ln t \ln t \cdots \ln t^p\), by applying Theorem 2.7, we obtain

\[
E(t) \leq E(0)(\ln t^p t)^{-\omega} \quad \text{if } m = 1, \theta = 1. \tag{2.40}
\]

For example, if \(n(m-1)/2 + \theta = 1\), that is, \(1 < m < 1 + 2/n\),

\[
E(t) \leq C(E(0))(\ln t^p t)^{-\frac{2}{(m-1)}}. \tag{2.41}
\]
(3) If \( \sigma(t) = 1/t^\theta \) and \( d(r) = 1/r^\gamma \) with \( \theta \geq \gamma \) by applying Theorem 2.8, we obtain

\[
E(t) \leq C(E(0)) (1 + t)^{-2/2(m - 1) - 2\theta/2(m - 1)} \quad \text{if } 1 < m < 1 + \frac{2 - 2\theta}{2\gamma + n}, \quad 0 < \theta < 1,
\]

\[
E(t) \leq C(E(0)) (\ln t)^{-2/2(m - 1)} \quad \text{if } m = 1 + \frac{2 - 2\theta}{2\gamma + n}, \quad 0 < \theta < 1.
\]

In order to show the global existence, it suffices to obtain the a priori estimates for \( E(t) \) and \( \|u(t)\|_2 \) in the interval of existence.

To prove Theorem 2.7 we first have the following energy identity to problem (1.1).

**Lemma 2.11 (energy identity).** Let \( u(t, x) \) be a local solution to problem (1.1) on \([0, T_{\text{max}})\) as in Theorem 2.5. Then

\[
E(t) + \int_{\mathbb{R}^n} \int_0^t \sigma(s) u'(s) g(u'(s)) \, ds \, dx = E(0)
\]

for all \( t \in [0, T_{\text{max}}) \).

Next we state several facts about the modified stable set \( \tilde{W}^* \).

**Lemma 2.12.** Suppose that

\[
1 < p \leq \frac{n + 2}{n - 2}.
\]

Then

(i) \( \tilde{W}^* \) is a neighborhood of 0 in \( H^1(\mathbb{R}^n) \),

(ii) for \( u \in \tilde{W}^* \),

\[
J(u) \geq \frac{p - 1}{2(p + 1)} (\|\nabla_x u\|_2^2 + \|u\|_2^2).
\]

**Proof of Lemma 2.12.** (i) From Lemma 2.1 we have

\[
\|u\|_{p+1}^{p+1} \leq K \|u\|_{p+1}^{p+1} \leq K \|u\|_{H^1}^{p-1} (\|u\|_2^2 + \|\nabla_x u\|_2^2).
\]

Let

\[
U(0) \equiv \left\{ u \in H^1(\mathbb{R}^N) \setminus \|u\|_{H^1}^{p-1} < \frac{1}{K} \right\}.
\]

Then, for any \( u \in U(0) \setminus \{0\} \), we deduce from (2.46) that

\[
\|u\|_{p+1}^{p+1} < \|u\|_2^2 + \|\nabla_x u\|_2^2,
\]

that is, \( K(u) > 0 \). This implies \( U(0) \subset \tilde{W}^* \).

(ii) By the definition of \( K(u) \) and \( J(u) \), we have the identity

\[
(p + 1)J(u) = K(u) + \frac{(p - 1)}{2} (\|\nabla_x u\|_2^2 + \|u\|_2^2).
\]
Since \( u \in \tilde{W}^* \), we have \( K(u) \geq 0 \). Therefore from (2.44) we get the desired in-equality (2.45). \( \square \)

**Lemma 2.13.** Let \( u(t) \) be a solution to problem (1.1) on \([0, T_{\text{max}}]\). Suppose (2.44) holds. If \( u_0 \in \tilde{W}^* \) and \( u_1 \in L^2(\mathbb{R}^n) \) satisfy

\[
C_0 E(0)^{(p-1)/2} < 1,
\]

then

(i) \( u(t) \in \tilde{W}^* \) on \([0, T_{\text{max}}]\),
(ii) \( \|u(t)\|_2 \leq I_0 \) on \([0, T_{\text{max}}]\).

**Proof of Lemma 2.13.** Suppose that there exists a number \( t^* \in [0, T_{\text{max}}] \) such that \( u(t) \in \tilde{W}^* \) on \([0, t^*]\) and \( u(t^*) \notin \tilde{W}^* \). Then we have

\[
K(u(t^*)) = 0, \quad u(t^*) \neq 0.
\]

Since \( u(t) \in \tilde{W}^* \) on \([0, t^*]\), it holds that

\[
\frac{p-1}{2(p+1)} (\|\nabla_x u\|_2^2 + \|u\|_2^2) \leq J(u) \leq E(t);
\]

it follows from the nonincreasing of the energy that

\[
\|\nabla_x u\|_2^2 + \|u\|_2^2 \leq \frac{2(p+1)}{p-1} E(0) = I^2_0.
\]

Hence, we obtain

\[
\|u\|_2^2 \leq \frac{2(p+1)}{p-1} E(0) = I^2_0 \quad \text{on} \quad [0, t^*].
\]

Next, from Lemma 2.1 and (2.54) we have

\[
\|u\|_{p+1} \leq K \|u(t)\|^{p+1}_{H^1(\mathbb{R}^n)} \leq K \|u(t)\|^{p+1}_{H^1(\mathbb{R}^n)} (\|\nabla_x u\|_2^2 + \|u\|_2^2) \leq K I^p_0 (\|\nabla_x u\|_2^2 + \|u\|_2^2) \leq C_0 E(0)^{(p-1)/2} (\|u(t)\|_2^2 + \|\nabla_x u(t)\|_2^2)
\]

for all \( t \in [0, t^*] \), where \( C_0 \) is the constant defined by (2.22). Note that from (2.55) and our hypothesis

\[
\eta_0 \equiv C_0 E(0)^{(p-1)/2} < 1,
\]

it follows that

\[
\|u(t)\|_{p+1} \leq (1 - \eta_0) (\|u(t)\|_2^2 + \|\nabla_x u(t)\|_2^2).
\]
Therefore, we obtain
\[
K(u(t^*)) \geq \eta_0(\|u(t^*)\|_2^2 + \|\nabla_x u(t^*)\|_2^2)
\]
which contradicts (2.51). Thus, we conclude that \(u(t) \in \tilde{W}^*\) on \([0, T_{\text{max}}]\). The assertion (ii) can be obtained by the same argument as for (2.54). This completes the proof of Lemma 2.13.

Lemma 2.14. Under the same assumptions as in Lemma 2.13, there exists a constant \(M_2\) depending on \(\|u_0\|_{H^1}\) and \(\|u_1\|_2\) such that
\[
\|u(t)\|_{H^1}^2 + \|u'(t)\|_2^2 \leq M_2^2
\]
for all \(t \in [0, T_{\text{max}}]\).

Proof of Lemma 2.14. It follows from Lemma 2.13 that \(u(t) \in \tilde{W}^*\) on \([0, T_{\text{max}}]\). So Lemma 2.12(ii) implies that
\[
J(u) \geq \frac{p - 1}{2(p + 1)}(\|u(t)\|_2^2 + \|\nabla_x u(t)\|_2^2)
\]
on \([0, T_{\text{max}}]\).

Hence, from Lemma 2.11 and (2.60) we get
\[
\frac{1}{2} \|u'(t)\|_2^2 + \frac{p - 1}{2(p + 1)}(\|u\|_2^2 + \|\nabla_x u(t)\|_2^2) \leq E(t) \leq E(0).
\]
So we get
\[
\|u(t)\|_{H^1}^2 + \|u'(t)\|_2^2 \leq M_2^2,
\]
for some \(M_2 > 0\).

The above inequality and the continuation principle lead to the global existence of the solution, that is, \(T_{\text{max}} = \infty\). \(\square\)

Proof of the energy decay. From now on, we denote by \(c\) various positive constants which may be different at different occurrences. We multiply the first equation of (1.1) by \(E^q u\), where \(\phi\) is a function satisfying all the hypotheses of Lemma 2.4. We obtain
\[
0 = \int_S^T E^q \phi' \int_{\mathbb{R}^n} uu'' - \Delta u + u + \sigma(t)g(u') - |u|^{p-1} u \ dx \ dt
\]
\[
\quad = \left[ E^q \phi' \int_{\mathbb{R}^n} uu' \ dx \right]_S^T - \int_S^T (qE \phi_{E^{-1}}' + E^q \phi'') \int_{\mathbb{R}^n} uu' \ dx \ dt - 2 \int_S^T E^q \phi' \int_{\mathbb{R}^n} u^2 \ dx \ dt
\]
\[
\quad + \int_S^T E^q \phi' \int_{\mathbb{R}^n} u^2 + |u|^2 + |\nabla u|^2 - \frac{2}{p + 1} |u|^{p+1} \ dx \ dt + \int_S^T E^q \phi' \int_{\mathbb{R}^n} \sigma(t)g(u') \ dx \ dt
\]
\[
\quad \int_S^T E^q \phi' \int_{\mathbb{R}^n} \left( \frac{2}{p + 1} - 1 \right) |u|^{p+1} \ dx \ dt.
\]
Since

\[
(1 - \frac{2}{p + 1}) \int_{\mathbb{R}^n} |u|^{p+1} \, dx \leq (1 - \eta_0) \frac{p - 1}{p + 1} ||u(t)||_{H^1(\mathbb{R}^n)}^2 \, dx \\
\leq (1 - \eta_0) \frac{p - 1}{p + 1} (p + 1) E(t) \\
= 2(1 - \eta_0) E(t),
\]

we deduce that

\[
2\eta_0 \int_{T_s}^{T} E^{q+1} \phi' \, dt \leq - \left[ E^q \phi' \int_{\mathbb{R}^n} uu' \, dx \right]_{T_s}^{T} + \int_{T_s}^{T} (qE' E^{q-1} \phi' + E^q \phi'') \int_{\mathbb{R}^n} uu' \, dx \, dt \\
+ 2 \int_{T_s}^{T} E^q \phi' \int_{\mathbb{R}^n} u'^2 \, dx \, dt - \int_{T_s}^{T} E^q \phi' \int_{\mathbb{R}^n} \sigma(t) u^2 g(u') \, dx \, dt \\
\leq - \left[ E^q \phi' \int_{\mathbb{R}^n} uu' \, dx \right]_{T_s}^{T} + \int_{T_s}^{T} (qE' E^{q-1} \phi' + E^q \phi'') \int_{\mathbb{R}^n} uu' \, dx \, dt \\
+ 2 \int_{T_s}^{T} E^q \phi' \int_{\mathbb{R}^n} u'^2 \, dx \, dt + c(\epsilon) \int_{T_s}^{T} E^q \phi' \int_{|u'| \leq 1} g(u')^2 \, dx \, dt \\
+ \epsilon \int_{T_s}^{T} E^q \phi' \int_{\mathbb{R}^n} u'^2 \, dx \, dt + \int_{T_s}^{T} E^q \phi' \int_{|u'| \geq 1} \sigma(t) u^2 g(u') \, dx \, dt
\]

for every \( \epsilon > 0 \). Also, applying Hölder’s and Young’s inequalities, we have

\[
\int_{T_s}^{T} E^q \phi' \int_{|u'| > 1} \sigma(t) u^2 g(u') \, dx \, dt \\
\leq \int_{T_s}^{T} E^q \phi' \sigma(t) \left( \int_{\Omega} |u|^{r+1} \, dx \right)^{1/(r+1)} \left( \int_{|u'| > 1} |g(u')| \right)^{(r+1)/r} \, dx \, dt \\
\leq c \int_{T_s}^{T} E^{(2q+1)/2} \phi' \sigma^{1/(r+1)}(t) \left( \int_{|u'| > 1} \sigma(t) u^2 g(u') \, dx \right)^{r/(r+1)} \, dt \\
\leq \int_{T_s}^{T} \phi' \sigma^{1/(r+1)}(t) E^{(2q+1)/2} (-E')^{r/(r+1)} \, dt \\
\leq c \int_{T_s}^{T} \phi' \left( \sigma^{1/(r+1)}(t) E^{(2q+1)/2 - r/(r+1)} \right) \left( (-E')^{r/(r+1)} E^{r/(r+1)} \right) \, dt \\
\leq c(\epsilon') \int_{T_s}^{T} \phi' (-E') \, dt + \epsilon' \int_{T_s}^{T} \phi' \sigma(t) E^{r/(r+1) ((2q+1)/2 - r/(r+1))} \, dt \\
\leq c(\epsilon') E(S)^2 + \epsilon' \sigma(0) E(0)^{(2q-r-1)/2} \int_{T_s}^{T} \phi' E^{q+1} \, dt
\]
for every $\varepsilon' > 0$. Choosing $\varepsilon$ and $\varepsilon'$ small enough, we obtain

\[ \int_S^{T_S} \left( E^q \phi' \right) dt \leq \left[ E^q \phi' \int_{\mathbb{R}^n} uu' dx \right]_S^T + \int_S^{T_S} \left( qE^{q-2} \phi' + E^q \phi'' \right) \int_{\mathbb{R}^n} uu' dx dt \]

\[ + \int_{|u'| \geq 1} \sigma(t) u g(u') dx dt + c \int_S^{T_S} E \phi' \int_{\mathbb{R}^n} u'^2 dx dt \]

\[ \leq cE(S) + c \int_S^{T_S} E \phi' \int_{\mathbb{R}^n} u'^2 dx dt. \]  

(2.67)

Since $xg(x) \geq 0$ for all $x \in \mathbb{R}$, it follows that the energy is nonincreasing, locally absolutely continuous and

\[ E'(t) = -\int_{\mathbb{R}^n} \sigma(t) u' g(u') dx \text{ a.c. in } \mathbb{R}_+. \]  

□

Proof of (2.26). We consider the case $m = 1$, that is,

\[ c_3 |v| \leq |g(v)| \leq c_4 |v| \quad \text{for all } |v| \leq 1. \]  

(2.68)

Then we have

\[ u'^2 \leq \frac{c_{13}}{\sigma(t)} u' \rho(t, u') \quad \forall t \in \mathbb{R}, \forall x \in \mathbb{R}^n, \]  

(2.69)

where $\rho(t, s) = \sigma(t) g(s)$ for all $s \in \mathbb{R}$. Therefore we deduce from (2.67) (applied with $q = 0$) that

\[ \int_S^{T_S} E(t) \phi'(t) dt \leq CE(S) + 2C \int_S^{T_S} \phi'(t) \int_{\mathbb{R}^n} \frac{1}{\sigma(t)} u' \rho(t, u') dx dt. \]  

(2.70)

Define

\[ \phi(t) = \int_0^t \sigma(\tau) d\tau. \]  

(2.71)

It is clear that $\phi$ is a nondecreasing function of class $C^2$ on $\mathbb{R}_+$. The hypothesis (2.23) ensures that

\[ \phi(t) \rightarrow +\infty \quad \text{as } t \rightarrow +\infty. \]  

(2.72)

Then we deduce from (2.70) that

\[ \int_S^{T_S} E(t) \phi'(t) dt \leq CE(S) + 2C \int_S^{T_S} u' \rho(t, u') dx dt \leq 3CE(S), \]  

(2.73)

and thanks to Lemma 2.4 we obtain

\[ E(t) \leq E(0) e^{(1 - \phi(t))/3C}. \]  

(2.74)

□

Proof of (2.27). Now we assume that $m > 1$ in (2.2). Define $\phi$ by (2.71). We apply Lemma 2.4 with $q = (m - 1)/2$. 

□
We need to estimate
\[ \int_{S}^{T} E^q \phi' \int_{\mathbb{R}^n} u'^2 \, dx \, dt. \] (2.75)

For \( t \geq 0 \), consider
\[ \Omega_1 = \{ x \in \mathbb{R}^n, \ |u'| \leq 1 \}, \quad \Omega_2 = \{ x \in \mathbb{R}^n, \ |u'| > 1 \}. \] (2.76)

First we note that for every \( t \geq 0 \),
\[ \Omega_1 \cup \Omega_2 = \mathbb{R}^n. \] (2.77)

Next we deduce from (2.2) and (2.3) that for every \( t \geq 0 \),
(i) if \( x \in \Omega_1 \), then \( u'^2 \leq ((1/\sigma(t))u' \rho(t,u'))^{2/(m+1)} \),
(ii) if \( x \in \Omega_2 \), then \( u'^2 \leq (1/\sigma(t))u' \rho(t,u') \).

Hence, using Hölder’s inequality, we get that
\[
\int_{S}^{T} E^q \phi' \int_{\mathbb{R}^n} u'^2 \, dx \, dt \\
\leq 2 \int_{S}^{T} E^q \phi' \int_{\mathbb{R}^n} \frac{1}{\sigma(t)} u' \rho(t,u') \, dx \, dt \\
+ 2 \int_{S}^{T} E^q \phi' \int_{\mathbb{R}^n} \frac{1}{\sigma(t)} u' \rho(t,u') \, dx \, dt \\
\leq 2 \int_{S}^{T} E^q \phi' \int_{\mathbb{R}^n} u' \rho(t,u') \, dx \, dt \\
+ 2 \int_{S}^{T} E^q \phi' \int_{\mathbb{R}^n} u' \rho(t,u') \, dx \, dt \\
\leq 2E(S)^{1+q} + c' \int_{S}^{T} E^q \phi' (1 + t)^{n(m-1)/(m+1)} \left( \int_{\mathbb{R}^n} u' g(u') \, dx \right)^{2/(m+1)} \, dt \\
\leq cE(S)^{1+q} + c' \int_{S}^{T} E^q \phi' (1 + t)^{n(m-1)/(m+1)} \sigma^{-2/(m+1)} t(-E')^{2/(m+1)} \, dt. \] (2.78)

Set \( \varepsilon > 0 \); thanks to Young’s inequality and to our definitions of \( p \) and \( \phi \), we obtain
\[
\int_{S}^{T} E^q \phi' \int_{\mathbb{R}^n} u'^2 \, dx \, dt \\
\leq cE(S)^{1+q} + 2 \frac{m-1}{m+1} e^{(m+1)/(m-1)} \int_{S}^{T} E^{1+q} (\phi')^{(m+1)/(m-1)} (1 + t)^n \sigma^{-2/(m-1)} \, dt \\
+ \frac{4}{m+1} \frac{1}{\varepsilon^{(m+1)/2}} E(S). \] (2.79)

We choose \( \phi' \) such that
\[ \phi'^{2/(m-1)} (1 + t)^n \sigma^{-2/(m-1)} = 1, \] (2.80)
\[ \phi(t) = \int_0^t (1 + s)^{-\frac{n(m-1)}{2}} \sigma(s) ds. \] (2.81)

Then we deduce from (2.79) that
\[ \int_S^{T} E^{1+q} \phi' \, dt \leq 2CE(S), \] (2.82)
and thanks to Lemma 2.4 (applied with \( c = 0 \)) we obtain
\[ E(t) \leq \frac{C}{\phi(t)^{2/(m-1)}}. \] (2.83)

**Proof of Theorem 2.8.** First, we see that if \( u \in \tilde{W}^{1,2} \), then
\[ \| \nabla_x u \|_2^2 + 2 \int_{\mathbb{R}^n} F(u(t)) \, dx \geq \| \nabla_x u \|_2^2 - \frac{2}{p+1} \| u(t) \|_{p+1}^{p+1} \geq \frac{p-1}{p+1} \| \nabla_x u \|_2^2. \] (2.84)

In the proof, we often use the following inequality:
\[ \| u(t) \|_2 \leq \frac{1}{\tilde{d}(t)} \| \lambda(x) u(t) \|_2. \] (2.85)

Now, we assume that \( I(u_0) > (1/2) \| \nabla_x u_0 \|_2^2 \). Then
\[ I(u(t)) \geq \frac{1}{2} \| \nabla_x u(t) \|_2^2 \] (2.86)
for some interval near \( t = 0 \). As long as (2.86) holds, we have \( J(t) \equiv I(u(t)) \). Thus
\[
2\eta_0 \int_S^{T} E^{q+1} \phi' \, dt \leq - \left[ E^q \phi' \int_{\mathbb{R}^n} u'' \, dx \right]_S^{T} + \int_S^{T} \left( qE^{q-1} \phi' + E^q \phi'' \right) \int_{\mathbb{R}^n} u'' \, dx \, dt
+ 2 \int_S^{T} E^q \phi' \int_{\mathbb{R}^n} u'^2 \, dx \, dt - \int_S^{T} E^q \phi' \int_{\mathbb{R}^n} \sigma(t) u(g(u')) \, dx \, dt
\leq - \left[ E^q \phi' \int_{\mathbb{R}^n} u'' \, dx \right]_S^{T} + \int_S^{T} \left( qE^{q-1} \phi' + E^q \phi'' \right) \int_{\mathbb{R}^n} u'' \, dx \, dt
+ 2 \int_S^{T} E^q \phi' \int_{\mathbb{R}^n} u'^2 \, dx \, dt + c(\varepsilon) \int_S^{T} E^q \phi' \int_{|u'| \leq 1} \left( \frac{\sigma(t)}{\lambda(x)} \right)^2 g(u')^2 \, dx \, dt
+ \varepsilon \int_S^{T} E^q \phi' \int_{\mathbb{R}^n} \lambda^2(x) u'^2 \, dx \, dt + \int_S^{T} E^q \phi' \int_{|u'| \geq 1} \sigma(t) u(g(u')) \, dx \, dt. \] (2.87)

If \( \sigma(t) = \mathcal{O}(\tilde{d}(t)) \), that is, \( \sigma(t) \to 0 \) as \( t \to \infty \) more rapidly than \( \tilde{d}(t) \), we find the same results of asymptotic behaviour as in Theorem 2.7.
If $\tilde{d}(t) = \mathcal{O}(\sigma(t))$, so, we obtain

$$\int_{S}^{T} E^q \phi' dt \leq - \left[ E^q \phi' \int_{\mathbb{R}^n} uu' dx \right]_{S}^{T} + \int_{S}^{T} \left( q E' E^{q-1} \phi' + E^q \phi'' \right) \int_{\mathbb{R}^n} uu' dx dt$$

(2.88)

$$\int_{S}^{T} E^q \phi' \int_{|u'| \geq 1} u'^2 dx dt \leq \int_{S}^{T} E^q \phi' \int_{|u'| \leq 1} \left( \frac{\sigma(t)}{\lambda(x)} \right)^2 |u'|^2 dx dt.$$  

We consider the case $m = 1$. Thus under (2.2) and (2.3), we have

$$\int_{S}^{T} E^q \phi' \int_{\mathbb{R}^n} \left( \frac{\sigma(t)}{\lambda(x)} \right)^2 u'^2 dx dt \leq C \int_{S}^{T} E^q \phi' \int_{\mathbb{R}^n} \left( \frac{\sigma^\alpha(t)}{d^2(t)} \right) \sigma(t) u' g(u') dx dt$$

(2.89)

for all $0 \leq \alpha < 1$. We choose

$$\phi(t) = \int_{t}^{T} \frac{\tilde{d}^2(s)}{\sigma^\alpha(s)} ds.$$  

(2.90)

It is clear that $\phi$ is a nondecreasing function of class $C^2$ on $\mathbb{R}_+$. Hypothesis (2.33) ensures that

$$\phi(t) \rightarrow +\infty \quad \text{as} \quad t \rightarrow +\infty.$$  

(2.91)

By (2.85), the definition of $E$, and the Cauchy-Schwartz inequality, we have

$$- \left[ E^q \phi' \int_{\mathbb{R}^n} uu' dx \right]_{S}^{T} = E^q(S) \phi'(S) \int_{\mathbb{R}^n} u(S) u'(S) dx - E^q(T) \phi'(T) \int_{\mathbb{R}^n} u(T) u'(T) dx$$

$$\leq CE^{q+1}(S) \left[ \frac{\phi'(S)}{d(S)} + \frac{\phi'(T)}{d(T)} \right] \leq CE^{q+1}(S),$$

(2.92)

$$\int_{S}^{T} \left( q E' E^{q-1} \phi' + E^q \phi'' \right) \int_{\mathbb{R}^n} uu' dx dt \leq \int_{S}^{T} q |E'| E^q \frac{\phi'(t)}{d(t)} dt + \int_{S}^{T} E^{q+1} \frac{|\phi''(t)|}{d(t)} dt$$

when we have (in the case $m = 1$)

$$\phi' = \frac{\tilde{d}^2(t)}{\sigma^\alpha(t)}, \quad \phi''(t) = \frac{2 \tilde{d}(t) \tilde{d}(t)}{\sigma^\alpha(t)} - \alpha \frac{\sigma'(t) \tilde{d}^2(t)}{\sigma^{\alpha+1}(t)}.$$

(2.93)

So

$$\frac{\phi''(t)}{d(t)} \leq - \frac{2 \tilde{d}'(t) \tilde{d}(t)}{d^\alpha(t)} - \alpha \frac{\tilde{d}(t) \sigma'(t)}{\sigma(t) \sigma^\alpha(t)},$$

(2.94)
\[ \tilde{d}(t)/\sigma(t) \text{ is bounded, so we obtain} \]
\[ \int_{T}^{S} E_{t+1} \left| \frac{\phi''(t)}{d(t)} \right| \ dt \leq -E_{t+1}(S) \left[ \tilde{d}^{1-a}(t) + \sigma^{1-a}(t) \right]^{T}_{S} \]
\[ \leq E_{t+1}(S)(\tilde{d}^{1-a}(t) + \sigma^{1-a}(t)) \]
\[ \leq CE_{t+1}(S). \] (2.95)

Then we deduce from (2.88) that
\[ \int_{S}^{T} E\phi' \ dt \leq CE(S) + 2C \int_{S}^{T} \int_{\mathbb{R}^n} \sigma(t)u'g(u') \ dx \ dt \leq 3CE(S), \] (2.96)
and thanks to Lemma 2.4 we obtain
\[ E(t) \leq E(0) e^{(1-\phi(t))/3C}. \] (2.97)

Using the condition that \( \tilde{d}(t) = O(\sigma(t)) \) and using H"{o}lder's inequality, we get that
\[ \int_{S}^{T} E^{-1} \phi' \int_{|u'| \leq 1} \frac{\sigma^2(t)}{\lambda^2(x)} u'^2 \ dx \ dt \]
\[ \leq 2 \int_{S}^{T} E^{-1} \phi' \int_{\mathbb{R}^n} \frac{\sigma^2(t)}{\lambda^2(x)} \left( \frac{1}{\sigma(t)} \sigma(t)u'g(u') \right) \ dx \ dt \]
\[ \leq 2 \int_{S}^{T} E^{-1} \phi' \int_{|x| \leq L+1} \frac{\sigma^2(t)}{\lambda^2(x)} \left( \frac{1}{\sigma(t)} \sigma(t)u'g(u') \right) \ dx \ dt \]
\[ \leq 2 \int_{S}^{T} E^{-1} \phi' \left( \frac{\sigma(t)}{d(t)} \right)^2 (1 + t)^{n(m-1)/(m+1)} \left( \int_{\mathbb{R}^n} u'g(u') \ dx \right)^{2/(m+1)} \ dt \]
\[ \leq c' \int_{S}^{T} E^{-1} \phi' \left( \frac{\sigma(t)}{d(t)} \right)^2 (1 + t)^{n(m-1)/(m+1)} \left( -\frac{E}{\sigma(t)} \right)^{2/(m+1)} \ dt \]
\[ \leq c' \int_{S}^{T} E^{-1} \phi' \left( \frac{\sigma(t)}{d(t)} \right)^2 (1 + t)^{n(m-1)/(m+1)} \sigma^{-2/(m+1)}(t)(-E')^{2/(m+1)} \ dt \]
\[ \leq c' \int_{S}^{T} E^{-1} \phi' \frac{\sigma^{a+1}(t)}{d^2(t)} (1 + t)^{n(m-1)/(m+1)} \sigma^{-2/(m+1)}(t)(-E')^{2/(m+1)} \ dt. \]

Set \( \varepsilon > 0 \); thanks to Young’s inequality and to our definitions of \( p \) and \( \phi \), we obtain
\[ \int_{S}^{T} E^{-1} \phi' \int_{\mathbb{R}^n} \left( \frac{\sigma(t)}{\lambda(x)} \right)^2 u'^2 \ dx \ dt \]
\[ \leq cE(S)^{1/q + 2m - 1 \over m + 1} \varepsilon^{(m+1)/(m-1)} \]
\[ \times \int_{S}^{T} E^{-1} \phi' (\phi'')^{(m+1)/(m-1)} \frac{\sigma^{(a+1)(m+1)/(m-1)}(t)}{d^{2(m+1)/(m-1)}(t)} (1 + t)^n \sigma^{-2/(m-1)}(t) \ dt \]
\[ + \frac{4}{m + 1} \varepsilon^{(m+1)/2} E(S). \] (2.99)
We choose \( \phi \) such that

\[
\phi(t) = \int_0^t (1 + \tau)^{-n(m-1)/2} \sigma^{-(1+\alpha)(m+1)-2}(\tau) \bar{d}^{m+1}(\tau) d\tau. \tag{2.100}
\]

It is clear that \( \phi \) is a nondecreasing function of class \( C^2 \) on \( \mathbb{R}_+ \). The hypothesis (2.35) ensures that \( \phi(t) \to +\infty \) as \( t \to +\infty \). By (2.85), the definition of \( E \), and the Cauchy-Schwartz inequality we have (2.92) when we have (the case \( m > 1 \))

\[
\phi'(t) = (1 + t)^{-n(m-1)/2} \sigma^{-(1+\alpha)(m+1)-2}(t) \bar{d}^{m+1}(t), \\
\phi''(t) = -\frac{n(m-1)}{2}(1 + t)^{-n(m-1)/2-1} \sigma^{-(1+\alpha)(m+1)-2}(t) \bar{d}^{m+1}(t) \\
+ (1 + t)^{-n(m-1)/2} \left( -\frac{(1 + \alpha)(m + 1) - 2}{2} \sigma^{-(1+\alpha)(m+1)/2}(t) \phi'(t) \bar{d}^{m+1}(t) \\
+ (m + 1) \bar{d}^m(t) \phi'(t) \sigma^{-(1+\alpha)(m+1)-2}(t) \right). \tag{2.101}
\]

Thus

\[
\frac{\phi''(t)}{\bar{d}(t)} \leq C \frac{\bar{d}^m(t)}{\sigma^m(t)} \sigma^{(1-\alpha)(m+1)/2}(t) - C' \frac{\bar{d}^m(t)}{\sigma^m(t)} \sigma^{(1+\alpha)(m+1)-2m/2}(t) \\
- C'' \frac{\bar{d}^m(t)}{\sigma^{(1+\alpha)(m+1)-2}(t)} \frac{\sigma'(t)}{\bar{d}^{(1+\alpha)(m+1)-2m/2}(t)}, \tag{2.102}
\]

\( \bar{d}(t)/\sigma(t) \) is bounded, so we obtain

\[
\int_S^{T_q+1} \frac{\phi''(t)}{\bar{d}(t)} dt \leq -E^{q+1}(s) \left[ \bar{d}^{(1-\alpha)(m+1)/2}(t) + \sigma^{(1-\alpha)(m+1)/2}(t) \right] \bigg|_S^T \leq E^{q+1}(s) (\bar{d}^{(1-\alpha)(m+1)/2}(t) + \sigma^{(1-\alpha)(m+1)/2}(t)) \tag{2.103}
\]

\( \leq CE^{q+1}(S) \).

We deduce from this choice

\[
\int_S^{T} E^{q+1}(t) \phi' dt \leq 2CE(S), \tag{2.104}
\]

and thanks to Lemma 2.4 (applied with \( c = 0 \), we obtain

\[
E(t) \leq \frac{C(E(0))}{\phi^{2/(m-1)}}. \tag{2.105}
\]

Since \( u_0 \in \tilde{W}^{**} \) and \( \tilde{W}^{**} \) is an open set, putting

\[
T_1 = \sup \{ t \in [0, +\infty) : u(s) \in \tilde{W}^{**} \text{ for } 0 \leq s \leq t \}, \tag{2.106}
\]

we see that \( T_1 > 0 \) and \( u(t) \in \tilde{W}^{**} \) for \( 0 \leq t < T_1 \). If \( T_1 < T_{\max} \), where \( T_{\max} \) is the lifespan of the solution, then \( u(T_1) \in \partial \tilde{W}^{**} \); that is,

\[
I(u(T_1)) = 0, \quad u(T_1) \neq 0. \tag{2.107}
\]
We see from Lemma 2.2 and (2.85) that
\[ \|u(t)\|_{p+1}^2 \leq C\|u(t)\|_2^{4-(n-2)(p-1)/2} \|\nabla_x u(t)\|_2^{(p-1)/2} \leq (\tilde{d}(t))^{-4-(n-2)(p-1)/2} E^{(p-1)/2} \|\nabla_x u(t)\|_2^2 \] (2.108)
for \(0 \leq t \leq T_1\), where we have used the assumption \(p \geq (n+4)/n\) and
\[ B(t) = \frac{C(E(0)) \tilde{d}^{-(4-(n-2)(p-1)/2)}(t)}{\left( \int_0^t (1+\tau)^{-n(m-1)/2} \sigma^{-(1+\alpha)(m+1)-2}(\tau) \tilde{d}^{m+1}(\tau) d\tau \right)^{(p-1)/(m-1)}}. \] (2.109)

Next, we put
\[ T_2 \equiv \sup \left\{ t \in [0,\infty) : B(s) < \frac{1}{2} \text{ for } 0 \leq s < t \right\}, \] (2.110)
and then we see that \(T_2 > 0\) and \(T_2 = T_1\) because \(B(t) < 1/2\) by the condition that \(E(0)\) is small. Then
\[ I(u(t)) = \|\nabla_x u(t)\|_2^2 - B(t) \|\nabla_x u(t)\|_2^2 \geq \frac{1}{2} \|\nabla_x u(t)\|_2^2 \] (2.111)
for \(0 \leq t \leq T_1\). Moreover, (2.107) and (2.111) imply
\[ K(u(T_1)) \geq \frac{1}{2} \|\nabla_x u(T_1)\|_2^2 > 0, \] (2.112)
which is a contradiction, and hence, it might be \(T_1 = T_{\max}\). Therefore, (2.105) holds true for \(0 \leq T \leq T_{\max}\). To prove global existence in \(H^2 \cap H^1\), we need to derive the estimates for second derivatives of \(u(t)\) on the basis of the energy estimate of \(E(t)\), we utilize the differentiated equation
\[ u_{ttt} - \Delta_x u' + \lambda^2(x) u' + \sigma(t) g(u') + \sigma(t) g(u') u'' + f'(u) u' = 0, \] (2.113)
where \(f(u) = |u|^{p-1} u\). Multiplying (2.113) by \(u''\), we have
\[ \frac{d}{dt} E_2(t) + 2\sigma(t) \int_{\mathbb{R}^n} g'(u') |u''(t)|^2 dx \leq 2 \int_{\mathbb{R}^n} |f'(u)| |u'(t)| |u''(t)| dx + 2 |\sigma'(t)| \int_{\mathbb{R}^n} |g(u')| |u''(t)| dx, \] (2.114)
where we set
\[ E_2(t) = \|u''(t)\|_2^2 + \|\nabla_x u(t)\|_2^2 + \|\lambda u'(t)\|_2^2. \] (2.115)
By (2.2) and (2.3), we have
\[ \int_{\mathbb{R}^n} |g(u')|^2 dx \leq C \int_{|u'| \leq 1} |u'|^{2/m} dx + C' \int_{|u'| \geq 1} |u'|^{2r} dx \leq C(L+t)^{n(m-1)/m} E^{1/m} + C'E^{(2-(n-2)(r-1)/2)E_2^{n(r-1)/2}}. \] (2.116)
and by Lemma 2.2

\[
\begin{aligned}
2 \int_{\mathbb{R}^n} \left| f'(u) \right| \left| u'(t) \right| \left| u''(t) \right| \, dx & \leq C \| u(t) \|^{p-1}_{m(p-1)} \| u'(t) \|_{2n/(n-2)} \| u''(t) \|_2 \\
& \leq c \| u(t) \|^{p-1}_{m(p-1)} \| \nabla_x u'(t) \|_2 \| u''(t) \|_2 \\
& \leq c \| u(t) \|^{p-1}_{m(p-1)} E_2(t).
\end{aligned}
\]  

(2.117)

Since \((n+2)/n \leq p \leq n/(n-2)\), then

\[
\| u(t) \|^{p-1}_{m(p-1)} \leq \tilde{d}^{-(2-(n-2)(p-1)/2)} E^{(p-1)/2}.
\]

(2.118)

Thus, we have

\[
\frac{d}{dt} E_2 \leq \tilde{d}^{-(2-(n-2)(p-1)/2)} E^{(p-1)/2} E_2(t) \\
+ 2 \left| \sigma'(t) \right| \left( (L + t)^{n(m-1)/m} E^{1/m} + C E^{(2-(n-2)(r-1)/2)} E_2^{r(r-1)/2} \right)^{1/2} E_2^{1/2}.
\]

(2.119)

We have also, applying Young inequality,

\[E^{(2-(n-2)(r-1)/2)} E_2^{n(r-1)/2} \leq C E^{(2-(n-2)(r-1))/(2-n(r-1))} (t) + E_2(t),\]

(2.120)

hence, we deduce that

\[\frac{d}{dt} E_2(t) \leq C(t) (1 + E_2(t)).\]

(2.121)

So, we obtain

\[E_2(t) \leq C e^{\int_0^t C(s) \, ds}.\]

(2.122)

From (2.122) and the first equation of problem (1.1), we also prove easily that

\[\| \Delta_x u(t) \|_2 \leq C' < \infty\]

(2.123)

for all \(t \geq 0\). Indeed, we have

\[\| \Delta_x u(t) \|_2 \leq \| u''(t) \|_2 + \| \lambda^2 u(t) \|_2 + \sigma(t) \| g(u'(t)) \|_2 + \| f(u) \|_2\]

(2.124)

and also we have

\[\| f(u) \|_2 \leq C \| u(t) \|^{p-1}_{2p} \leq C \| u(t) \|^{2(p-1)}_{n(p-1)/2} \| \Delta_x u(t) \|_2^2.\]

(2.125)

Here, to check the last inequality of (2.125), we note that if \((n+4)/n \leq p \leq (n+2)/(n-2)\), then

\[C \| u(t) \|^{2(p-1)}_{n(p-1)/2} \leq C \| u(t) \|^{4-(n-2)(p-1)}_{2n/(n-2)} \| \nabla_x u(t) \|_2^{n(p-1)-4} \leq (\tilde{d}(t))^{4-(n-2)(p-1)} E^{p-1} < \frac{1}{2}.\]

(2.126)
Thus the solution in the sense of Theorem 2.6 exists globally in time $t$ under the assumption $(u_0, u_1) \in H^2 \times H^1$.

When $(u_0, u_1) \in H^1 \times L^2$ we approximate $(u_0, u_1)$ by $(u^k_0, u^k_1) \in H^2 \times H^1$, $k = 1, 2, \ldots$, in the topologies of $H^1 \times L^2$, which satisfy

$$\text{supp } u^k_0 \cap \text{supp } u^k_1 \subset \{ x \in \mathbb{R}^n \mid |x| \leq L \}. \quad (2.127)$$

Since $\lim_{k \to \infty} (u^k_0, u^k_1) = (u_0, u_1)$ in $H^1 \times L^2$, then for these initial data problem (1.1) has global solutions $u^k \in W^{2, \infty}_{\text{loc}}([0, \infty); L^2) \cap W^{1, \infty}_{\text{loc}}([0, \infty); H^1) \cap L^\infty_{\text{loc}}([0, \infty); H^2)$, which satisfy (2.122) and (2.123). We can easily see that $\{u^k(t)\}$ converges uniformly on each compact interval $[0, T]$, $T > 0$. Uniqueness follow from a standard argument. The proof of Theorem 2.8 is now completed. $\square$

References


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This subject has been extensively studied in the past years for one-, two-, and three-dimensional space. Additionally, such dynamical systems can exhibit a very important and still unexplained phenomenon, called as the Fermi acceleration phenomenon. Basically, the phenomenon of Fermi acceleration (FA) is a process in which a classical particle can acquire unbounded energy from collisions with a heavy moving wall. This phenomenon was originally proposed by Enrico Fermi in 1949 as a possible explanation of the origin of the large energies of the cosmic particles. His original model was then modified and considered under different approaches and using many versions. Moreover, applications of FA have been of a large broad interest in many different fields of science including plasma physics, astrophysics, atomic physics, optics, and time-dependent billiard problems and they are useful for controlling chaos in Engineering and dynamical systems exhibiting chaos (both conservative and dissipative chaos).

We intend to publish in this special issue papers reporting research on time-dependent billiards. The topic includes both conservative and dissipative dynamics. Papers discussing dynamical properties, statistical and mathematical results, stability investigation of the phase space structure, the phenomenon of Fermi acceleration, conditions for having suppression of Fermi acceleration, and computational and numerical methods for exploring these structures and applications are welcome.

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</tr>
</thead>
<tbody>
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</tr>
<tr>
<td>Publication Date</td>
<td>June 1, 2009</td>
</tr>
</tbody>
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