ON THE EXISTENCE OF POSITIVE SOLUTIONS
FOR PERIODIC PARABOLIC SUBLINEAR PROBLEMS

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We give necessary and sufficient conditions for the existence of positive solutions for sublinear Dirichlet periodic parabolic problems $Lu = g(x, t, u)$ in $\Omega \times \mathbb{R}$ (where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain) for a wide class of Carathéodory functions $g : \Omega \times \mathbb{R} \times [0, \infty) \to \mathbb{R}$ satisfying some integrability and positivity conditions.

1. Introduction

Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$. For $T > 0$, $1 \leq p \leq \infty$, and $1 \leq q \leq \infty$, let $L^p(L^q)$ be the Banach space of $T$-periodic functions $f$ on $\Omega \times \mathbb{R}$ (i.e., satisfying $f(x, t) = f(x, t + T)$ a.e. $(x, t) \in \Omega \times \mathbb{R}$) such that

$$\|f\|_{L^p(L^q)} := \left\|\left\|f(\cdot, t)\right\|_{L^q(\Omega)}\right\|_{L^p(0, T)} < \infty. \quad (1.1)$$

Similarly, let $L^p_T$ be the Banach space of $T$-periodic functions $f$ such that $f|_{\Omega \times (0, T)} \in L^p(\Omega \times (0, T))$, equipped with the norm $\|f\|_{L^p_T} := \|f|_{\Omega \times (0, T)}\|_{L^p(\Omega \times (0, T))}$. Finally, let $C_T$ be the space of continuous and $T$-periodic functions on $\Omega \times \mathbb{R}$ provided with the $L^\infty$-norm.

For the whole paper, we fix $\nu, s \in (1, \infty]$ such that $N/2\nu + 1/s < 1$, $s > 2$. Let $\{a_{ij}\}$ and $\{b_j\}$, $1 \leq i, j \leq N$, be two families of functions satisfying $a_{ij}, b_j \in L^\infty_T$ and $a_{ij} = a_{ji}$. Assume that $\sum a_{ij}(x, t)\xi_i\xi_j \geq \alpha_0|\xi|^2$ for some $\alpha_0 > 0$ and all $(x, t) \in \Omega \times \mathbb{R}$, $\xi \in \mathbb{R}^N$. Let $A$ be the $N \times N$ matrix whose $i, j$ entry is $a_{ij}$, let $b = (b_1, \ldots, b_N)$, let $0 \leq c_0 \in L^s(L^\nu)$, and let $L$ be the parabolic operator given by

$$Lu = u_t - \text{div}(A\nabla u) + \langle b, \nabla u \rangle + c_0u. \quad (1.2)$$

Let $W = \{u \in L^2((0, T), H^1_0(\Omega)): u_t \in L^2((0, T), H^{-1}(\Omega))\}$. Given $f \in L^1_{T,\text{loc}}(\Omega \times \mathbb{R})$, we say that $u$ is a (weak) solution of the Dirichlet periodic problem $Lu = f$.
in $\Omega \times \mathbb{R}, u = 0$ on $\partial \Omega \times \mathbb{R}$, if $u$ is $T$-periodic, $u|_{\Omega \times (0, T)} \in W$, and
\[
\int_{\Omega \times (0, T)} \left[ -u \frac{\partial h}{\partial t} + \langle A \nabla u, \nabla h \rangle + \langle b, \nabla u \rangle h + c_0 u h \right] = \int_{\Omega \times (0, T)} fh
\]
for all $h \in C^\infty_c(\Omega \times \mathbb{R})$ (and so for all $h \in L^\infty_T$ such that $h|_{\Omega \times (0, T)} \in V_0$, where $V_0 := L^2((0, T), H^{-1}_0(\Omega)))$. For $u \in W$, the inequality $Lu \geq f$ (resp., $\leq f$) will be understood in the same sense.

Let $\tilde{W} = \{ u \in L^2((0, T), H^1(\Omega)) : u_t \in L^2((0, T), H^{-1}(\Omega)) \}$. Following [6], we say that $v$ is a supersolution of the above problem if $v|_{\Omega \times (0, T)} \in \tilde{W}$, $vt \in L^2((0, T), H^{-1}(\Omega)) + L^1(\Omega \times (0, T))$ for $\eta > 0$ small enough, $v|_{\partial \Omega \times (0, T)} \geq 0$, $v(\cdot, 0) \geq v(\cdot, T)$ a.e. in $\Omega$, and
\[
\int_{\Omega \times (0, T)} \left[ -v \frac{\partial h}{\partial t} + \langle A \nabla v, \nabla h \rangle + \langle b, \nabla v \rangle h + c_0 v h \right] \geq \int_{\Omega \times (0, T)} fh
\]
for all $0 \leq h \in C^\infty_c(\Omega \times (0, T))$ (and so for all $h \in L^\infty_T$ such that $h|_{\Omega \times (0, T)} \in V_0$ with $V_0$ as above). A subsolution is similarly defined by reversing the above inequalities.

Let $m \in L^s(L^v)$ and let
\[
P(m) := \int_0^T \text{esssup}_{x \in \Omega} m(x, t) dt
\]
(with the value “$+\infty$” allowed). For such $m$ (cf. [8, Theorem 3.6]), $P(m) > 0$ is necessary and sufficient for the existence of a positive principal eigenvalue for the periodic parabolic Dirichlet problem with weight function $m$ (i.e., an eigenvalue with a positive $T$-periodic eigenfunction associated to the problem $Lu = \lambda m u$ in $\Omega \times \mathbb{R}, u = 0$ on $\partial \Omega \times \mathbb{R}$). Moreover, this positive principal eigenvalue denoted by $\lambda_1(L, m)$ (or $\lambda_1(m)$), if exists, is unique.

We are interested in the existence of positive solutions for the semilinear periodic parabolic problem
\[
Lu = g(x, t, u) \quad \text{in } \Omega \times \mathbb{R},
\]
\[
u = 0 \quad \text{on } \partial \Omega \times \mathbb{R},
\]
\[
u T\text{-periodic},
\]
where $g$ is a given function on $\Omega \times \mathbb{R} \times [0, \infty)$.

In [9, Theorem 3.7], it is proved that
\[
\lambda_1 \left( \sup_{\xi > 0} \frac{g(\cdot, \xi)}{\xi} \right) < 1 < \lambda_1 \left( \inf_{\xi > 0} \frac{g(\cdot, \xi)}{\xi} \right)
\]
is a necessary and sufficient condition for the existence of positive solutions in $C_T$ for (1.6) provided that $g$ satisfies $\xi - g(x, t, \xi) \in C^1[0, \infty)$, $\xi - g(x, t, \xi)/\xi$
nonincreasing in $(0, \infty)$, and some integrability and positivity conditions. In [10, Theorem 3.1], with the same monotonicity and regularity assumptions, and assuming also some integrability conditions, it is proved that if either $\inf_{\xi>0}(g(\cdot, \xi)/\xi) \in L^1(L^r)$ and $P(\inf_{\xi>0}(g(\cdot, \xi)/\xi)) \leq 0$ or $\inf_{\xi>0}(g(\cdot, \xi)/\xi) \leq 0$, then

$$\lambda_1 \left( \sup_{\xi>0} \frac{g(\cdot, \xi)}{\xi} \right) < 1 \quad (1.8)$$

is necessary and sufficient for the existence of a positive solution $u \in C_T$ of (1.6).

Our aim in this paper is to prove, following a different approach, similar results without monotonicity and $C^1$-regularity assumptions on $g$ (see Theorems 3.1, 3.2, 3.3, and 3.4). Moreover, we will also cover some cases where $\lim_{\xi \to 0^+} (g(\cdot, \xi)/\xi) = \infty$. These theorems will be obtained using the well-known sub- and supersolutions method combined with some facts concerning linear problems with weight.

In order to relate our results to others in the literature, we mention that, for the case $\xi \to g(\cdot, \xi)/\xi$ nonincreasing, similar results to Theorem 3.1 for elliptic problems have been obtained, for example, in [4, 5, 13], assuming more regularity in the function $g$. In the periodic parabolic case, there are also well-known results if $\xi \to g(\cdot, \xi)/\xi$ is concave and Hölder-continuous, and $g(\cdot, 0) = 0$ (see [2, 3, 12] and the references therein).

On the other side, necessary and sufficient conditions for the existence of positive solutions for equations of type $Lu = a(x)u - b(x)u^p$, $p > 1$, $b \geq 0$ (logistic equation), are also known (see, e.g., [11, 12]). More general equations of the form $Lu = a(x)u - b(x)f(x,u)$, with $b \geq 0$ and $f$ superlinear, were studied, for example, in [7] for $f \in C^0(\Omega \times [0, \infty))$, $f$ strictly increasing, and $b > 0$, and, for the Laplacian, the case $f = f(u)$ is treated in [1] assuming $f \in C([0, \infty))$. Theorem 3.2 generalizes the aforementioned results, while Theorems 3.3 and 3.4 also extend some well-known results, see, for example, [2, 3, 11, 12].

Some examples are also given at the end of the paper.

2. Preliminaries and auxiliary results

As usual, for $\xi \in [0, \infty)$ and $u : \Omega \times \mathbb{R} \to [0, \infty)$, we write $g(\xi)$ and $g(u)$ for the functions $(x,t) \to g(x,t,\xi)$ and $(x,t) \to g(x,t,u(x,t))$, $(x,t) \in \Omega \times \mathbb{R}$. We assume, from now on, that $g : \Omega \times \mathbb{R} \times [0, \infty) \to \mathbb{R}$ is a Carathéodory function (i.e., $(x,t) \to g(x,t,\xi)$ is measurable for all $\xi \in [0, \infty)$, and $\xi \to g(x,t,\xi)$ is continuous in $[0, \infty)$ a.e. $(x,t) \in \Omega \times \mathbb{R}$) such that $\sup_{0 \leq \xi \leq \xi}(g(\sigma)/\sigma)$ and $\inf_{0 \leq \sigma \leq \xi}(g(\sigma)/\sigma)$ are measurable functions for all $\xi > 0$, and $\inf_{\xi > 0}(g(\xi)/\xi) \neq \sup_{\xi > 0}(g(\xi)/\xi)$, that is, (1.6) is not a linear problem.

We start recalling some facts about periodic parabolic problems with weight.

**Remark 2.1.** (a) Let $D = \{m \in L^s(L^r) : P(m) > 0\}$. Then $D$ is open in $L^s(L^r)$ and the map $m \to \lambda_1(m)$ is continuous from $D$ into $\mathbb{R}$ (cf. [8, Theorem 3.9]). Also,
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the following comparison principle holds: if \( m_1, m_2 \in L^s(L^\nu) \) and \( m_1 \leq m_2 \) in \( \Omega \times \mathbb{R} \), then \( \lambda_1(m_1) \geq \lambda_1(m_2) \); and if, in addition, \( m_1 < m_2 \) in a set of positive measure, then \( \lambda_1(m_1) > \lambda_1(m_2) \) (cf. [8, Remark 3.7]).

(b) For \( \lambda \in \mathbb{R} \) and \( m \in L^s(L^\nu) \), let \( \mu_m(\lambda) \) be defined as the unique \( \mu \in \mathbb{R} \) such that the Dirichlet periodic problem \( Lu = \lambda mu + \mu_m(\lambda)u \) in \( \Omega \times \mathbb{R} \) has a positive solution \( u \). We call that \( \mu_m(\lambda) \) is well defined and that the map \((\lambda, m) \rightarrow \mu_m(\lambda)\) is continuous from \( \mathbb{R} \times L^s(L^\nu) \) into \( \mathbb{R} \) (cf. [9, Proposition 2.7]). Moreover, \( \mu_m(0) > 0 \), \( \mu_m \) is concave and continuous, and a given \( \lambda \in \mathbb{R} \) is a principal eigenvalue associated to the weight \( m \) if and only if \( \mu_m(\lambda) > 0 \) (cf. [8, Lemma 3.2]). Also, if \( \lambda_1(m) \) exists, then for \( \lambda > 0 \), \( \mu_m(\lambda) > 0 \) if and only if \( \lambda < \lambda_1(m) \), and if \( \lambda_1(m) \) does not exist, \( \mu_m(\lambda) > 0 \) for all \( \lambda > 0 \).

(c) Let \( m \in L^s(L^\nu) \) such that \( P(m) > 0 \) and let \( m_j \) be a sequence such that \( m_j \) converges to \( m \) in \( L^s(L^\nu) \). Then it follows from [9, Remark 2.5] that \( P(m_j) > 0 \) for \( j \) large enough.

Remark 2.2. If \( u \in L^\infty_T \) is a positive solution of (1.6) and

\[
\begin{align*}
\inf_{0 < \xi \leq M} \left( \frac{\mathcal{g}(\xi)}{\xi} \right) &\in L^s(L^\nu), \\
\sup_{0 < \xi \leq M} \left( \frac{\mathcal{g}(\xi)}{\xi} \right) &\in L^s(L^\nu),
\end{align*}
\]

for all \( M > 0 \), then \( u \in C_T \) and \( u(x, t) > 0 \) for all \((x, t) \in \Omega \times \mathbb{R} \). Indeed, this follows from [9, Remark 2.2 and Corollary 2.12].

We introduce some additional notation. For \((x, t, \xi) \in \Omega \times \mathbb{R} \times (0, \infty) \), let

\[
\begin{align*}
\mathcal{g}(x, t, \xi) &= \xi \sup_{0 < \xi \leq \sigma} \left( \frac{g(x, t, \sigma)}{\sigma} \right), \\
\mathcal{g}(x, t, \xi) &= \xi \inf_{0 < \sigma \leq \xi} \left( \frac{g(x, t, \sigma)}{\sigma} \right)
\end{align*}
\]

(with the values “\( \pm \infty \)” allowed). It is easy to check that if \( g(\xi) \) is finite for \( \xi \leq \xi_0 \), then \( \xi \to \mathcal{g}(\xi) \) is continuous in \((0, \xi_0)\) a.e. in \( \Omega \times \mathbb{R} \), and that if \( \overline{g}(\xi) \) is finite for \( \xi_0 \leq \xi \), then \( \xi \to \overline{g}(\xi) \) is continuous in \((\xi_0, \infty)\) a.e. in \( \Omega \times \mathbb{R} \). We also set

\[
\begin{align*}
m_{\infty}(x, t) &= \inf_{\xi > 0} \left( \frac{g(x, t, \xi)}{\xi} \right), \\
m_0(x, t) &= \sup_{\xi > 0} \left( \frac{g(x, t, \xi)}{\xi} \right), \\
m_0(x, t) &= \liminf_{\xi \to 0^+} \left( \frac{g(x, t, \xi)}{\xi} \right), \\
m_{\infty}(x, t) &= \limsup_{\xi \to \infty} \left( \frac{g(x, t, \xi)}{\xi} \right).
\end{align*}
\]
Note that
\[
\begin{align*}
\underline{m}_\infty &= \lim_{\xi \to -\infty} \left( \frac{\underline{g}(\xi)}{\xi} \right), & \overline{m}_0 &= \lim_{\xi \to 0^+} \left( \frac{\overline{g}(\xi)}{\xi} \right), \\
\underline{m}_0 &= \lim_{\xi \to 0^-} \left( \frac{\underline{g}(\xi)}{\xi} \right), & \overline{m}_\infty &= \lim_{\xi \to -\infty} \left( \frac{\overline{g}(\xi)}{\xi} \right). 
\end{align*}
\] (2.4)

**Lemma 2.3.** Let \( \xi_0 > 0 \). Assume that \( \overline{g}(\xi) \in L^s(L^r) \) for all \( \xi \geq \xi_0 \) and that either \( \overline{m}_\infty \in L^s(L^r) \) with \( \lambda_1(\overline{m}_\infty) > 1 \) (if \( \lambda_1(\overline{m}_\infty) \) exists) or \( \overline{m}_\infty \leq 0 \). Then, for all \( c > 0 \), there exists a supersolution \( w \in C_T \) of (1.6) such that \( w \geq c \).

**Proof.** We first study the case \( \overline{m}_\infty \in L^s(L^r) \). Let \( c > 0 \). We claim that there exists \( \xi \geq c \) such that \( \mu_{\overline{g}(\xi)}/\xi \geq \overline{w} \). Indeed, for \( \xi \geq \xi_0 \), we have \( \overline{m}_\infty \leq \overline{g}(\xi)/\xi \leq \overline{g}(\xi_0)/\xi_0 \) and also \( \lim_{\xi \to \infty} (\overline{g}(\xi)/\xi) = \overline{m}_\infty \) with convergence a.e. Thus, by dominated convergence, \( \lim_{\xi \to \infty} (\overline{g}(\xi)/\xi) = \overline{m}_\infty \) in \( L^s(L^r) \) and then Remark 2.1(b) implies \( \lim_{\xi \to \infty} \mu_{\overline{g}(\xi)}/\xi (\xi) = \mu_{\overline{m}_\infty}(\lambda) \) for all \( \lambda \). Moreover, either if \( P(\overline{m}_\infty) > 0 \) and \( \lambda_1(\overline{m}_\infty) > 1 \) or if \( P(\overline{m}_\infty) \leq 0 \), the last statement in Remark 2.1(b) also gives \( \mu_{\overline{m}_\infty}(1) > 0 \). Thus, it follows that \( \mu_{\overline{g}(\xi)}/\xi (1) > 0 \) for \( \xi \) large enough.

We fix \( \xi^* \geq \max(\xi_0, c) \) such that \( \mu_{\overline{g}(\xi^*)}/\xi^* (1) > 0 \). Let \( k \) be a function defined by \( k(x, t) = \sup_{\xi \geq \xi^*} |\overline{g}(\xi)/\xi| \). Since \( \overline{m}_\infty \leq k \leq \overline{g}(\xi^*)/\xi^* \), we get \( k \in L^r(L^s) \). For \( \xi \in [0, \infty) \), let \( g^*(x, t, \xi) = \overline{g}(\xi_0)/\xi_0 + k(x, t) \xi \). Then \( g^*(x, t, \xi) \geq 0 \) and \( g^*(\xi)/\xi \in L^s(L^r) \) for \( \xi \geq \xi^* \). Also, \( \mu_{L+\lambda_kg^*(\xi^*)}/\xi^* (\lambda) = \mu_{L\overline{g}(\xi^*)}/\xi^* (\lambda) \) for all \( \lambda \). In particular, \( \mu_{L+\lambda_kg^*(\xi^*)}/\xi^* (1) = \mu_{L\overline{g}(\xi^*)}/\xi^* (1) > 0 \). Thus, Lemma 2.9 in [9] says that the Dirichlet periodic problem \( (L + k - g^*(\xi^*)/\xi^*)\Phi = g^*(\xi^*) \) in \( \Omega \times \mathbb{R} \) has a solution \( \Phi \in C_T \) satisfying \( \Phi(x, t) > 0 \) a.e. \( (x, t) \in \Omega \times \mathbb{R} \). Now,
\[
g(\xi^* + \Phi) \leq \overline{g}(\xi^* + \Phi) \\
\leq \frac{\overline{g}(\xi^*)}{\xi^*}(\xi^* + \Phi) \\
\leq \overline{g}(\xi^*) + k\xi^* + \frac{\overline{g}(\xi^*)}{\xi^*} \Phi \\
= g^*(\xi^*) + \frac{g^*(\xi^*)}{\xi^*} \Phi - k\Phi \\
= L\Phi \leq L(\xi^* + \Phi),
\] (2.5)
and therefore \( \xi^* + \Phi \) is a supersolution for (1.6).

Consider now the case \( \overline{m}_\infty \leq 0 \). In this case, we have \( \lim_{\xi \to -\infty} (\overline{g}^*(\xi)/\xi) = 0 \) a.e. in \( \Omega \times \mathbb{R} \), where, as usual, we write \( f = f^+ - f^- \). Also, \( 0 \leq \overline{g}^+(\xi)/\xi \leq \overline{g}^+(\xi_0)/\xi_0 \) for all \( \xi \geq \xi_0 \), and thus \( \lim_{\xi \to -\infty} (\overline{g}^+(\xi)/\xi) = 0 \) in \( L^s(L^r) \). So, \( \lim_{\xi \to -\infty} \mu_{\overline{g}^+(\xi)}/\xi (\lambda) = \lambda_1 \) for all \( \lambda \), where \( \lambda_1 \) is the (positive) principal eigenvalue for \( L \) associated to the weight 1 (because for \( m \equiv 1, m \equiv \lambda_1 \)). Thus, we can choose \( \xi^* \geq \max(\xi_0, c) \) such that \( \mu_{\overline{g}^*(\xi^*)}/\xi^* > 0 \), and then, as above, the Dirichlet periodic problem \( (L - \overline{g}^+(\xi^*)/\xi^*)\Phi = \overline{g}^+(\xi^*) \) in \( \Omega \times \mathbb{R} \) has a solution \( \Phi \in C_T \) satisfying \( \Phi(x, t) > 0 \) a.e.
\( (x, t) \) in \( \Omega \times \mathbb{R} \). Also,

\[
g(\xi^* + \Phi) \leq \tilde{g}^+(\xi^* + \Phi)
\leq \frac{\tilde{g}^+(\xi^*)}{\xi^*} (\xi^* + \Phi)
= \tilde{g}^+(\xi^*) + \frac{\tilde{g}^+(\xi^*)}{\xi^*} \Phi
= L\Phi \leq L(\Phi + \xi^*),
\]

and this concludes the proof. \( \square \)

**Lemma 2.4.** Let \( \xi_0 > 0 \). Assume that \( g(\xi_0) \in L'(L^2) \), \( P(g(\xi_0)/\xi_0) > 0 \), and \( \lambda_1(g(\xi_0)/\xi_0) \leq 1 \). Then there exists a subsolution \( v \in C_T \) of (1.6) such that \( v(x, t) > 0 \) for all \((x, t) \in \Omega \times \mathbb{R} \).

**Proof.** Let \( \Phi \) be the positive eigenfunction of

\[
\left( L + \frac{g^-(\xi_0)}{\xi_0} \right) \Phi = \lambda_1 \left( \frac{g^+(\xi_0)}{\xi_0} \right) \left( \frac{g^+(\xi_0)}{\xi_0} \right) \Phi \quad \text{in} \; \Omega \times \mathbb{R},
\]

\[
\Phi = 0 \quad \text{on} \; \partial \Omega \times \mathbb{R},
\]

\( \Phi \) \( T \)-periodic.

Then \( \Phi \in C_T \) and \( \Phi(x, t) > 0 \) for all \((x, t) \in \Omega \times \mathbb{R} \). Now, \( \lambda_1(L, g(\xi_0)/\xi_0) < 1 \) implies \( \mu(L_1, g(\xi_0)/\xi_0) \leq 0 \). Thus, since \( \mu(L, g(\xi_0)/\xi_0) = \mu(L_1, g^-(\xi_0)/\xi_0)g^+(\xi_0)/\xi_0(1) \), we get \( \lambda_1(g^+(\xi_0)/\xi_0) \leq 1 \).

Let \( \varepsilon > 0 \) be such that \( \varepsilon < \xi_0/\|\Phi\|_\infty \). Taking into account the above-mentioned facts and that \( \xi \to g(\xi)/\xi \) is nonincreasing, we have

\[
L(\varepsilon \Phi) + g^-(\varepsilon \Phi) \leq \left( L + \frac{g^-(\varepsilon \|\Phi\|)}{\varepsilon \|\Phi\|} \right) \varepsilon \Phi
\leq \left( L + \frac{g^-(\xi_0)}{\xi_0} \right) \varepsilon \Phi
\leq \left( \frac{g^+(\xi_0)}{\xi_0} \right) \varepsilon \Phi
\leq \left( \frac{g^+(\varepsilon \|\Phi\|)}{\varepsilon \|\Phi\|} \right) \varepsilon \Phi
\leq g^+(\varepsilon \Phi),
\]

and the lemma follows. \( \square \)
3. The main results

**Theorem 3.1.** (a) Assume that

1. \( m_0, \overline{m} \in L^1(L^r) \), \( P(m_0) > 0 \), and \( P(\overline{m}) > 0 \),
2. \( \overline{g}(\xi_0) \in L^1(L^r) \) for some \( \xi_0 > 0 \) and \( g(\xi_1) \in L^1(L^r) \) for some \( \xi_1 > 0 \).

Then, if \( \lambda_1(m_0) < 1 < \lambda_1(\overline{m}) \), there exists a solution \( u \in L_T^\infty \) of (1.6) satisfying \( u(x, t) > 0 \) for all \( (x, t) \in \Omega \times \mathbb{R} \).

(b) Assume (1), \( \overline{m}_0 = m_0, \overline{m} = \overline{m}_\infty \), and that for all \( \xi > 0 \),

\[
\overline{m}_0 \neq \frac{\overline{g}(\xi)}{\xi}, \tag{3.1}
\]

\[
\overline{m}_\infty \neq \frac{g(\xi)}{\xi}. \tag{3.2}
\]

Then there exists a positive solution \( u \in L_T^\infty \) of (1.6) if and only if \( \lambda_1(m_0) < 1 < \lambda_1(\overline{m}_\infty) \).

**Proof.** Suppose that \( \lambda_1(m_0) < 1 < \lambda_1(\overline{m}) \). Since, for \( 0 < \xi \leq \xi_1 \), we have \( g(\xi_1)/\xi \leq \overline{g}(\xi)/\xi \leq m_0 \) and \( \lim_{\xi \rightarrow 0^+} \overline{g}(\xi)/\xi = m_0 \) a.e. in \( \Omega \times \mathbb{R} \), taking into account (1) and (2), we get \( \overline{g}(\xi)/\xi \in L^1(L^r) \) for such \( \xi \) and so \( \lim_{\xi \rightarrow 0^+} \overline{g}(\xi)/\xi = m_0 \) with convergence in \( L^1(L^r) \). Then, by Remark 2.1(c), we have \( \lim_{\xi \rightarrow 0^+} P(\overline{g}(\xi)/\xi) = P(m_0) > 0 \), and thus there exists \( \lambda_1(g(\xi)/\xi) > 0 \) small enough. Moreover, Remark 2.1(a) says that \( \lim_{\xi \rightarrow 0^+} \lambda_1(g(\xi)/\xi) = \lambda_1(m_0) < 1 \) and so \( \lambda_1(g(\xi)/\xi) < 1 \) for such \( \xi \). Hence, Lemma 2.4 can be applied to give a subsolution \( \overline{v} \in C_T \) of (1.6) with \( \overline{v}(x, t) > 0 \) for all \( (x, t) \in \Omega \times \mathbb{R} \).

On the other hand, for all \( \xi \geq \xi_0 \), we have \( \overline{m}_\infty \leq \overline{g}(\xi)/\xi \leq g(\xi_0)/\xi_0 \), and so \( \overline{g}(\xi)/\xi \in L^1(L^r) \). Therefore, taking \( c = \|v\|_\infty \) in Lemma 2.3, we obtain a supersolution \( w \in C_T \) of (1.6) with \( w \geq c \geq v \). Now, [6, Theorem 1] gives a solution \( u \in L_T^\infty \) such that \( v \leq u \leq w \) and then \( u(x, t) > 0 \) for all \( (x, t) \in \Omega \times \mathbb{R} \). Thus (a) is proved.

To prove (b), suppose that \( u \in L_T^\infty \) is a positive solution of (1.6). By Remark 2.2, we have \( u(x, t) > 0 \) for all \( (x, t) \). Let \( m_u : \Omega \times \mathbb{R} \rightarrow \mathbb{R} \) be defined by \( m_u = g(u)/u \). Since \( m_u \) is measurable and \( m_\infty \leq m_u \leq m_0 \), it follows that \( m_u \in L^1(L^r) \). Moreover, we have \( Lu = m_u u \) and so \( 1 = \lambda_1(m_u) \). Now, the comparison principle in Remark 2.1(a) gives \( 1 = \lambda_1(m_u) \geq \lambda_1(\overline{m}_0) = \lambda_1(m_0) \) and also \( 1 \leq \lambda_1(\overline{m}_\infty) = \lambda_1(\overline{m}_\infty) \). Suppose \( \lambda_1(\overline{m}_0) = 1 \). Since \( \lambda_1(m_u) = 1 \) and \( m_u \leq m_0 \), we must have \( m_u(x, t) = m_0(x, t) \) a.e. \( (x, t) \in \Omega \times \mathbb{R} \) (see Remark 2.1(a)), but \( \inf_{0 < \xi < \|u\|_\infty} (g(\xi)/\xi) \geq g(u)/u = m_0 \) in \( \Omega \times \mathbb{R} \) contradicting (3.1). Then \( \lambda_1(m_0) < 1 \). Suppose now that \( \lambda_1(m_\infty) = 1 \). Reasoning as above, we get \( 1 = \lambda_1(m_u) \leq \lambda_1(m_\infty) = 1 \) and so \( m_u = m_\infty \). Thus, \( \inf_{0 < \xi < \|u\|_\infty} (g(\xi)/\xi) \leq g(u)/u = \inf_{\xi > 0} (g(\xi)/\xi) \) a.e., which is again a contradiction. Then \( \lambda_1(m_\infty) > 1 \) .

**Theorem 3.2.** (a) Assume that

1. \( m_0 \in L^1(L^r), P(m_0) > 0 \),
2. \( \overline{g}(\xi_0) \in L^1(L^r) \) for some \( \xi_0 > 0 \) and \( g(\xi_1) \in L^1(L^r) \) for some \( \xi_1 > 0 \).
3. \( m_0 \neq \overline{g}(\xi)/\xi \), \( \overline{m}_0 \neq g(\xi)/\xi \).
4. \( \lim_{\xi \rightarrow 0^+} \overline{g}(\xi)/\xi = m_0 \triangleq \lambda_1(m_0) > 0 \), \( \lim_{\xi \rightarrow 0^+} g(\xi)/\xi = m_\infty \triangleq \lambda_1(m_\infty) > 0 \).
(4) $g(\xi_0) \in L^1(L^r)$ for some $\xi_0 > 0$ and $g(\xi) \in L^1(L^r)$ for all $\xi > 0$,
(5) either $m_\infty \in L^1(L^r)$ and $P(m_\infty) \leq 0$ or $m_\infty \leq 0$.

Then, if $\lambda_1(m_0) < 1$, there exists a solution $u \in L_T^\infty$ of (1.6) satisfying $u(x,t) > 0$
for all $(x,t) \in \Omega \times \mathbb{R}$.

(b) Assume, in addition, (3.1) and $m_0 = m_\infty$. Then there exists a positive solution
$u \in L_T^\infty$ of (1.6) if and only if $\lambda_1(m_\infty) < 1$.

Proof. As in the above theorem, we have $g(\xi)/\xi \in L^1(L^r)$ and $\lambda_1(g(\xi)/\xi) < 1$
for $\xi > 0$ small enough, and so Lemma 2.4 gives a subsolution $v \in C_T$ satisfying
$v(x,t) > 0$ for all $(x,t)$. On the other side, since $g(\xi)/\xi \leq g(\xi)/\xi \leq g(\xi)/\xi_0$ for
$\xi \geq \xi_0$, from (4), we have $g(\xi)/\xi \in L^1(L^r)$ for such $\xi$. Therefore, (a) follows as in
Theorem 3.1 taking $c = \|v\|_\infty$ in Lemma 2.3, and the proof of (b) follows similarly
to part (b) of Theorem 3.1.

Theorem 3.3. (a) Assume (2) and that

(6) $m_\infty \in L^1(L^r)$ and $P(m_\infty) > 0$,
(7) $P(g(\xi)/\xi) > 0$ for $\xi > 0$ small and $\lim_{\xi \to 0^+} \lambda_1(g(\xi)/\xi) = 0$.

Then, if $\lambda_1(m_\infty) > 1$, there exists a solution $u \in L_T^\infty$ of (1.6) satisfying $u(x,t) > 0$
for all $(x,t) \in \Omega \times \mathbb{R}$.

(b) Assume, in addition, (3.2) and $m_\infty = m_\infty$. Then there exists a positive solution
$u \in L_T^\infty$ of (1.6) if and only if $\lambda_1(m_\infty) > 1$.

Proof. Reasoning as above, (a) follows from Lemmas 2.3, 2.4, and [6, Theorem
1]. Suppose now that $u \in L_T^\infty$ is a positive solution of (1.6). Let $\epsilon > 0$ such that
$\epsilon < \|u\|_\infty$. Let $g_\epsilon$ be defined by $g_\epsilon(\xi) = g(\xi)$ if $\xi \geq \epsilon$ and $g_\epsilon(\xi) = g(\epsilon)$ if $\xi < \epsilon$. We
have $Lu = g(\epsilon) \geq g_\epsilon(\epsilon) \geq g_\epsilon(\xi)$ and also $g_\epsilon(\xi)/\xi \in L^1(L^r)$. Thus, $1 \leq \lambda_1(g_\epsilon(\xi)/\xi)$. Moreover,
since $g_\epsilon(\xi)/\xi \geq m_\infty$, the comparison principle in Remark 2.1(a) gives
$1 \leq \lambda_1(m_\infty)$. Suppose $1 = \lambda_1(m_\infty)$. Then $g_\epsilon(\xi)/\xi = m_\infty$. But $g_\epsilon(\xi)/\xi \geq g_\epsilon(\xi)/\xi = g_\epsilon(\xi)/\xi = g_\epsilon(\xi)/\xi = g_\epsilon(\xi)/\xi$ and therefore $m_\infty = g_\epsilon(\xi)/\xi$ in contradiction with (3.2).

Theorem 3.4. Assume (4), (5), and (7). Then (1.6) has a positive solution $u \in L_T^\infty$
satisfying $u(x,t) > 0$ for all $(x,t) \in \Omega \times \mathbb{R}$.

Proof. The theorem follows again from Lemmas 2.3, 2.4, and [6, Theorem
1].

3.1. Examples. (a) Suppose there exist $\lim_{\xi \to 0^+} (g(\xi)/\xi)$ and $\lim_{\xi \to \infty} (g(\xi)/\xi)$
and assume $\inf_{\xi>0} g(\xi)/\xi, \sup_{\xi>0} g(\xi)/\xi \in L^1(L^r)$, with $P(\inf_{\xi>0} g(\xi)/\xi) > 0$. If
$\lim_{\xi \to 0^+} (g(\xi)/\xi) = \sup_{\xi>0} g(\xi)/\xi$ and $\lim_{\xi \to \infty} (g(\xi)/\xi) = \inf_{\xi>0} g(\xi)/\xi$, from
Theorem 3.1, we conclude that (1.6) has a positive solution $u \in L_T^\infty$ if and only if
$\lambda_1(\lim_{\xi \to 0^+} (g(\xi)/\xi)) < 1 < \lambda_1(\lim_{\xi \to \infty} (g(\xi)/\xi))$.

(b) Consider the Dirichlet periodic problem $Lu = \sin u$ in $\Omega \times \mathbb{R}$. Theorem 3.2
says that this problem has a positive $T$-periodic solution if and only if $\lambda_1 < 1$,
where $\lambda_1$ is the positive principal eigenvalue corresponding to the weight 1.
(c1) Consider the problem

\[
Lu = a(x, t)u^\gamma - f(x, t, u) \quad \text{in } \Omega \times \mathbb{R},
\]

\[
u = 0 \quad \text{on } \partial \Omega \times \mathbb{R},
\]

\[u_T \text{-periodic},
\]

where \(0 < \gamma \leq 1\) and \(f\) is a Carathéodory function such that \(f(\xi) \in L^s(L^r)\) for all \(\xi > 0\) and \(f(0) = 0\). Assume that \(\gamma = 1\), \(a \in L^s(L^r)\), \(P(a) > 0\), \(a \leq \lim_{\xi \to \infty} f(\xi) \leq \infty\), \(\inf_{0 < \xi \leq \xi_0} f(\xi) \in L^r(L^s)\) for some \(\xi_0 > 0\), and \(\inf_{0 < \xi \leq \xi_0} f(\xi) \in L^r(L^s)\) for all \(\xi_0 > 0\).

From Theorem 3.2, it follows that (3.3) has a positive solution \(u \in L^\infty_T\) if and only if \(\lambda_1(a) < 1\).

(c2) Consider now the case \(0 < \gamma < 1\) and \(a(x, t) \geq 0\) a.e. \((x, t) \in \Omega \times \mathbb{R}\). If \(f(\xi) = -b\) with \(b \in L^s(L^r)\) and \(P(b) > 0\), then Theorem 3.3 says that (3.3) has a positive solution \(u \in L^\infty_T\) if and only if \(1 < \lambda_1(b)\). On the other hand, suppose \(\lim_{\xi \to \infty} f(\xi) = \infty\), \(\inf_{\xi_0 < \xi \leq \xi_0} f(\xi) \in L^r(L^s)\) for some \(\xi_0 > 0\), and \(\sup_{0 < \xi \leq \xi_0} f(\xi) \in L^s(L^r)\) for all \(\xi_0 > 0\). Then Theorem 3.4 gives a positive solution \(u \in L^\infty_T\) for (3.3).

We note that in all the cases, the positive solution \(u\) satisfies \(u(x, t) > 0\) for all \((x, t)\). Moreover, recalling Remark 2.2, we also have that in (a), (b), and (c1) \(u \in C_T\).

Remark 3.5. An inspection of the proofs shows that all the above results remain true for the corresponding elliptic problem, replacing \(L^s(L^r)\) by \(L^r(\Omega)\) with \(r > N/2\), and \(P(m)\) by \(\text{esssup}_{x \in \Omega} m(x)\).

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References


Periodic parabolic sublinear problems


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